Today's assignment is simple. Just go out and get on the bus. It doesn't matter which bus. Whichever bus comes next. Get on, and just go. You could ride that bus to the very end, thank the driver, and then walk into the woods and just die. Just lay down right there and wait and wait until you were dead. Who is going to miss you?

2011-2012

1* A comet approaches an isolated planet of mass M and radius a. The speed of the comet when far from the planet is v

a. Calculate the total cross section σ for the comet to strike the planet

$E = \frac{l^2}{2mr^2} - \frac{GMm}{r} \qquad l = b\sqrt{2mE}$

Want to know how far away a particle can be and still collide with the planet

$$E = \frac{b^2 \cdot 2mE}{2mr^2} - \frac{GMm}{r}$$

$$Er^2 = Eb^2 - GMmr$$

$$Eb^2 = Ea^2 + GMma$$

$$\tfrac{1}{2} mv^2 \cdot b^2 = \tfrac{1}{2} mv^2 a^2 + GMma$$

$$v^2 b^2 = v^2 a^2 + 2GMa$$

$$b^2 = a^2 + \frac{2GMa}{v^2}$$

$$b = \sqrt{a^2 + 2GMa/v^2}$$

Now, the total cross section is given by $\sigma = \pi b^2$

$$\sigma = \pi\left(a^2 + \frac{2GMa}{v^2}\right)$$

b. Does σ increase or decrease with M? Explain the behavior physically

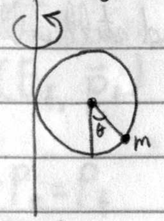
σ should increase since the gravitational force increases.

c. Does σ increase or decrease with v? Explain the behavior physically

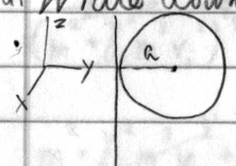
σ should decrease since there is less time for gravity to act on the comet.

Really think about it. If you went out to the middle of nowhere and just sat down in a ditch and cried by yourself until you were dead, who would be the first person to wonder where you'd gone? Call them up! Maybe they want to get ice cream?
— Joey Comeau

2. A particle of mass m is constrained to move on a circular ring of radius a which is welded on its edge to a vertical axis. The axis rotates with a constant angular frequency Ω. There is a downward gravitational acceleration g.



a. Write down a Lagrangian for the system in terms of the variable θ



$x_c = -a\sin(\Omega t)$
$y_c = a\cos(\Omega t)$

$x = -(a + a\sin\theta)\sin(\Omega t)$
$y = (a + a\sin\theta)\cos(\Omega t)$
$z = -a\cos\theta$
$\dot{x} = -(a\dot{\theta}\cos\theta)\sin(\Omega t) - (a + a\sin\theta)\Omega\cos(\Omega t)$
$\dot{y} = (a\dot{\theta}\cos\theta)\cos(\Omega t) - (a + a\sin\theta)\Omega\sin(\Omega t)$
$\dot{z} = a\dot{\theta}\sin\theta$
$\mathcal{L} = \frac{m}{2}\left(a^2\dot{\theta}^2 + (a + a\sin\theta)^2\Omega^2\right) + mga\cos\theta$

b. Find an equation that the equilibrium value θ_0 of the angle θ must satisfy

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

$\frac{d}{dt}(ma^2\dot{\theta}) - (m(a + a\sin\theta)\Omega^2 \cdot a\cos\theta - mga\sin\theta) = 0$

$ma^2\cos\theta\,\Omega^2 + ma^2\Omega^2\sin\theta\cos\theta = mga\sin\theta$

$a\cos\theta\,\Omega^2 + a\Omega^2\sin\theta\cos\theta = g\sin\theta$

since at equilibrium $\dot{\theta} = \ddot{\theta} = 0$

c. Find the value of the frequency Ω for which $\theta_0 = \pi/6$

$\cos(\pi/6) = \sqrt{3}/2$
$\sin(\pi/6) = 1/2$

$a\frac{\sqrt{3}}{2}\Omega^2 + a\Omega^2 \cdot \frac{\sqrt{3}}{4} = g/2$

$a\Omega^2(2\sqrt{3} + \sqrt{3}) = 2g$

$\Omega^2 = \frac{2g}{a} \cdot \frac{1}{3\sqrt{3}}$

$= \frac{2g}{a} \cdot \frac{\sqrt{3}}{9}$

$\Omega = \sqrt{\frac{2g\sqrt{3}}{9a}}$

d. Find the frequency ω of small oscillation about this θ_0

$ma^2\ddot{\theta} = ma^2\Omega^2\cos\theta + ma^2\Omega^2\sin\theta\cos\theta - mga\sin\theta$

$\cos(\theta_0 + \varepsilon) = \cos\theta_0 - \varepsilon\sin\theta_0$

$\sin(\theta_0 + \varepsilon) = \sin\theta_0 + \varepsilon\cos\theta_0$

$ma^2\ddot{\varepsilon} = ma^2\Omega^2(\cos\theta_0 - \varepsilon\sin\theta_0) + ma^2\Omega^2(\cos\theta_0\sin\theta_0 - \varepsilon\sin^2\theta_0 + \varepsilon\cos^2\theta_0)$

$= -mga(\sin\theta_0 + \varepsilon\cos\theta_0)$

$ma^2\ddot{\varepsilon} = -\varepsilon\,\frac{1}{2}\,ma^2\Omega^2 + ma^2\Omega^2(-\varepsilon/4 + \varepsilon\cdot 3/4) - \varepsilon mga\cdot\sqrt{3}/2$

$\ddot{\varepsilon} = ma^2\Omega^2(-\varepsilon/2 - \varepsilon/4 + 3\varepsilon/4) - \varepsilon mga\cdot\sqrt{3}/2$

$\ddot{\varepsilon} = -\frac{g\sqrt{3}}{2a}\varepsilon$

$\omega = \left(\frac{g\sqrt{3}}{2a}\right)^{1/2}$

3. Consider a point mass m moving in a central potential $V(r) = -Kr^n$ where n is nonzero but can be any positive or negative integer, and the sign of K is such that the potential is attractive. The mass is moving in a circular orbit with energy E and angular momentum L.

a. What is the radius of an orbit for a given n

· $E = \frac{L^2}{2mr^2} - Kr^n$

$\frac{\partial}{\partial r}$ gives $0 = \frac{-L^2}{mr^3} - nKr^{n-1}$

$$0 = -L^2 - mnKr^{n+2}$$

$$r^{n+2} = \frac{L^2}{mnK}$$

$$r = \left(\frac{L^2}{mnK}\right)^{1/n+2}$$

b. What is the period of revolution for this orbit?

· $L = mr^2 \dot{\theta}$

$\dot{\theta} = L/mr^2$

$T = \frac{2\pi}{\dot{\theta}} = \frac{2\pi \cdot mr_o^2}{L}$

c. Which values of n allow stable orbits

· $\frac{\partial^2}{\partial r^2}$ gives $0 = \frac{3L^2}{mr^4} - n(n-1)Kr^{n-2}$

$$0 \leq \frac{3L^2}{m} - n(n-1)Kr^{n+2}$$

$$n(n-1)Kr^{n+2} \leq \frac{3L^2}{m}$$

$$\cancel{n}(n-1)\cancel{K} \cdot \frac{\cancel{L^2}}{m\cancel{n}\cancel{K}} \leq \frac{3\cancel{L^2}}{m} \qquad n \leq 4$$

$$n-1 \leq 3$$

$$n \leq 4$$

d. Consider small deviations from a stable circular orbit, such that the mass oscillates radially around a circular orbit. What is the frequency ω of such oscillations?

· $F_{eff} = -\frac{\partial V}{\partial r} = -\left(\frac{-L^2}{mr^3} - nKr^{n-1}\right)$

$$m\ddot{r} = \frac{L^2}{mr^3} + nKr^{n-1}$$

$$m\ddot{\varepsilon} = \frac{L^2}{m(r+\varepsilon)^3} + nK(r+\varepsilon)^{n-1}$$

$$= \frac{L^2}{mr^3}(1+\varepsilon/r)^{-3} + nKr^{n-1}(1+\varepsilon/r)^{n-1}$$

$$= \cancel{\frac{L^2}{mr^3}} - \frac{3\varepsilon L^2}{mr} + \cancel{nKr^{n-1}} + n(n-1)Kr^{n-1} \cdot \frac{\varepsilon}{r}$$

$$= \left(\frac{-3L^2}{mr^4} + \frac{n(n-1)Kr^{n-1}}{r}\right)\varepsilon$$

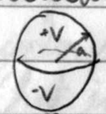
$$= \left(\frac{-2L^2}{mr^4} + \frac{2n(n-1)Kr^{n-1}}{r}\right)\varepsilon$$

$$\ddot{\varepsilon} = \left(\frac{-2L^2}{m^2r^4} + \frac{2n(n-1)Kr^{n-2}}{m}\right)\varepsilon$$

$$\omega = \left(\frac{-2L^2}{m^2r^4} + \frac{2n(n-1)Kr^{n-2}}{m}\right)^{1/2}$$

4*. A thin conducting spherical shell is composed of two hemispheres, each of radius a. The upper hemisphere is kept at potential +V while the lower hemisphere is at potential −V.

a. Calculate the electrostatic potential Φ inside the shell.



$$\Phi = \sum_{\ell=0} (A_\ell r^\ell + B_\ell r^{-(\ell+1)}) P_\ell(\cos\theta)$$

$$\Phi = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

V; $r=a$ and $0 \le \theta \le \pi/2$

$-V$; $r=a$ and $\pi/2 \le \theta \le \pi$

$$\int_0^\pi V(\theta)\cdot P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \int_0^\pi \sum A_\ell a^\ell \cdot P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta$$

$$\int_{-1}^{1} V(\theta) P_\ell(x)\, dx = \sum A_\ell a^\ell \cdot \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

$$A_\ell a^\ell \cdot \frac{2}{2\ell+1} = V\int_0^1 P_\ell(x)dx - V\int_{-1}^{0} P_\ell(x)dx$$

$$= 2V\int_0^1 P_\ell(x)dx$$

$\int_0^1 P_\ell(x)dx = 0$ when even

$= \frac{1}{2\ell+1}(P_{\ell-1}(0) - P_{\ell+1}(0))$ when odd

$$A_\ell = \frac{\cancel{2\ell+1}}{2a^\ell}\cdot\frac{2V}{\cancel{2\ell+1}}(P_{\ell-1}(0) - P_{\ell+1}(0))$$

$$= \frac{V}{a^\ell}(P_{\ell-1}(0) - P_{\ell+1}(0))$$

and $\int_0^1 P_0(x)dx = 1$ $\quad A_0 \cdot 2 = 2V$

$A_0 = V$

$$\Phi = V + \sum_{\ell=1}^{\infty} (r/a)^\ell \cdot V(P_{\ell-1}(0) - P_{\ell+1}(0))\cdot P_\ell(\cos\theta)$$

for ℓ odd

b. Calculate the electrostatic potential Φ outside the shell

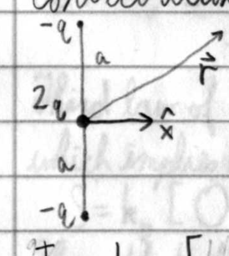
Through the same logic

$$\Phi = \sum B_\ell r^{-(\ell+1)} P_\ell(\cos\theta) \qquad B_0 a^{-1} = V$$

$$\Phi = V/a + \sum_{\ell=0}^{\infty} (a/r)^{\ell+1} V(P_{\ell-1}(0) - P_{\ell+1}(0))\cdot P_\ell(\cos\theta)$$

for ℓ odd

5.* A charge $+2q$ is located at the origin and charges $-q$ are located at $\pm a\hat{z}$, as shown. Write down the exact electrostatic potential Φ at a point $\vec{r}$ from the charge. Assuming $r \equiv |\vec{r}| >> a$, find the contribution to Φ at lowest nonzero order in a.



$$\Phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{-q}{|\vec{r}-a\hat{z}|} - \frac{q}{|\vec{r}+a\hat{z}|} + \frac{2q}{|\vec{r}|}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{-1}{(x^2+y^2+(z-a)^2)^{1/2}} - \frac{1}{(x^2+y^2+(z+a)^2)^{1/2}} + \frac{2}{r}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{-1}{(r^2-2za+a^2)^{1/2}} - \frac{1}{(r^2+2za+a^2)^{1/2}} + \frac{2}{r}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{-(1-2za/r^2+a^2/r^2)^{-1/2}}{r} - \frac{(1+2za/r^2+a^2/r^2)^{-1/2}}{r} + \frac{2}{r}\right]$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left[-\left(1+\frac{za}{r^2}-\frac{a^2}{2r^2}\right) - \left(1-\frac{za}{r^2}-\frac{a^2}{2r^2}\right) + 2\right]$$

$$= \frac{q}{4\pi\varepsilon_0 r}\cdot\frac{a^2}{r^2} = \frac{qa^2}{4\pi\varepsilon_0 r^3}$$

6.* At zero temperature, atoms in a solid form a perfect lattice. At finite temperatures, defects may appear in which atoms are missing in the lattice and can be approximated as having vanished. At low temperature, these defects are very dilute, so when n of them are present the energy of the system relative to the zero temperature state may be written as $E=n\varepsilon$, where ε is the energy to create a single defect.

a. Using the microcanonical ensemble, find the entropy of the system for n vacancies in a crystal with N atomic sites, in the limit where n and N are both large. Use this to derive an expression for the temperature of the system in terms of n and ε.

$S = k_B \ln \Omega \qquad E = n\varepsilon$

$\Omega = \binom{N}{n} = \frac{N!}{n!(N-n)!} \qquad dE = \varepsilon\, dn$

$S = k_B [N\ln N - n\ln n - (N-n)\ln(N-n)]$

$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{1}{\varepsilon}\cdot\frac{\partial S}{\partial n}$

$= \frac{k_B}{\varepsilon}\left[-\ln n - \frac{n}{n} - \frac{N}{N-n}\cdot(-1) + \ln(N-n) + \frac{n}{N-n}\ (-1)\right]$

$= \frac{k_B}{\varepsilon}\left[-\ln n + \ln(N-n) - 1 + 1\right]$

$= \frac{k_B}{\varepsilon}\ln\left(\frac{N-n}{n}\right) = \frac{1}{T}$

b. Show that your result for the entropy is consistent with the third law of thermodynamics. What is the entropy in the limit of infinite temperature?

$S = k_B\left[N\ln\left(\frac{N}{N-n}\right) + n\ln\left(\frac{N-n}{n}\right)\right]$

$= k_B\left[n\ln\left(\frac{N-n}{n}\right) - N\ln\left(\frac{N-n}{N}\right)\right]$

$\ln\left({}^{N-n}/_{n}\right) = {}^{\varepsilon}/_{k_BT} \qquad \ln(N-n) = {}^{\varepsilon}/_{k_BT} + \ln(n)$

$S = k_B\left[\frac{n\varepsilon}{k_BT} - N\left({}^{\varepsilon}/_{k_BT} + \ln n\right) + N\ln N\right]$

As $T\to\infty$, $S = -k_B N\ln n + k_B N\ln N$
$= k_B N(\ln N - \ln n)$

Third law of thermodynamics deals with absolute zero temperature, which implies $n=0$.

$S = k_B[0 - N\ln(1-0)] = 0$

c. Show that if one can arrange the system so that $n > {}^{N}/_{2}$, the temperature of the system is formally negative

$T = \frac{k_B}{\varepsilon}\frac{1}{\ln\left(\frac{N-n}{n}\right)}$

$\ln\left({}^{N-n}/_{n}\right) < 0$ if $n > {}^{N}/_{2}$ since $n > N-n$ in that case.

Thus, $T<0$ if $n > {}^{N}/_{2}$

d. Find an expression for the heat capacity of this system, $C = \left(\frac{\partial E}{\partial T}\right)_N$. How does C behave in the limits $T\to 0$ and $T\to\infty$?

$\frac{1}{T} = \frac{k_B}{\varepsilon}\ln\left(\frac{N - {}^{E}/_{\varepsilon}}{{}^{E}/_{\varepsilon}}\right) = \frac{k_B}{\varepsilon}\ln\left(\frac{\varepsilon N}{E} - 1\right)$

$\exp\left({}^{\varepsilon}/_{k_BT}\right) = \frac{\varepsilon N}{E} - 1$

$E(1 + \exp({}^{\varepsilon}/_{k_BT})) = \varepsilon N$

$E = \frac{\varepsilon N}{1+\exp({}^{\varepsilon}/_{k_BT})}$

$$E = \frac{\varepsilon N}{1+\exp(\varepsilon/k_BT)} = \varepsilon N[1+\exp(\varepsilon/k_BT)]^{-1}$$

$$\frac{\partial E}{\partial T} = \frac{-\varepsilon N}{(1+\exp(\varepsilon/k_BT))^2} \cdot \exp(\varepsilon/k_BT) \cdot -\varepsilon/k_BT^2$$

$$= \frac{\varepsilon^2 N \exp(\varepsilon/k_BT)}{k_BT^2(1+\exp(\varepsilon/k_BT))^2}$$

as $T \to \infty$, $C_V \to 0$

as $T \to 0$, $C_V = \infty$

7. The quantum energy levels of a rigid rotator have the form $\varepsilon_j = j(j+1)\varepsilon_0$, where $j = 0, 1, 2, 3$, and $\varepsilon_0 > 0$. The degeneracy of each level is $g_j = 2j+1$.

a. Find a general expression for the partition function Z of this system in the canonical ensemble

- $Z = \sum_j (2j+1)\exp(-\beta j(j+1)\varepsilon_0)$

b. Show that at high temperatures this expression may be approximated by an integral

- $Z = \int_0^\infty (2j+1)\exp(-\beta j(j+1)\varepsilon_0)\,dj$

c. Find the energy of this system and the heat capacity $C_V = \frac{\partial U}{\partial T}$ in this high-temperature limit.

- $E = A + TS$

$A = -k_BT\ln Z$

$U = -\frac{1}{Z}\cdot\frac{\partial Z}{\partial \beta}$

$$Z = \int_0^\infty (2j+1)\exp(-\beta\varepsilon_0(j^2+j))\,dj$$

$j^2 + j = u$

$(2j+1)dj = du$

$$Z = \int \exp(-\beta\varepsilon_0 u)\,du$$

$$= \frac{-1}{\beta\varepsilon_0}\exp(-\beta\varepsilon_0(j^2+j))\Big|_0^\infty = \frac{1}{\beta\varepsilon_0}$$

$$U = -\beta\varepsilon_0 \cdot -\frac{1}{\varepsilon_0\beta^2} = 1/\beta = k_BT$$

$C_V = k_B$

d. Find approximate expressions for Z, U, and C_V at low but non-zero temperature

$$Z = 1 + \sum_{j=1}^{\infty}(2j+1)\exp(-\beta(j^2+j)\varepsilon_0)$$

$$Z \approx 1 + 3\exp(-2\beta\varepsilon_0)$$

$$U = -\frac{1}{Z}\cdot\frac{\partial Z}{\partial\beta} = \frac{-1}{1+3\exp(-2\beta\varepsilon_0)} \cdot -6\varepsilon_0\exp(-2\beta\varepsilon_0)$$

$$= \frac{6\varepsilon_0\exp(-2\beta\varepsilon_0)}{1+3\exp(-2\beta\varepsilon_0)} = \frac{6\varepsilon_0}{\exp(2\beta\varepsilon_0)+3}$$

$$C_V = \frac{\partial U}{\partial T} = \frac{-6\varepsilon_0}{(\exp(2\beta\varepsilon_0)+3)^2} \cdot -2\varepsilon_0/k_BT^2 \cdot \exp(2\beta\varepsilon_0)$$

$$= \frac{12\varepsilon_0^2\exp(2\beta\varepsilon_0)}{k_BT^2[\exp(2\beta\varepsilon_0)+3]^2}$$

8. Consider a degenerate, ultrarelativistic gas of non-interacting electrons. In this limit, the energy of an electron is related to its momentum by $\varepsilon(p) = c|\vec{p}|$. Consider N electrons in a volume V

a. Evaluate the density of state $a(\varepsilon)$ for this system

$$\int a(\varepsilon)\,d\varepsilon = \frac{1}{h^3}\int d^3x\,d^3p = \frac{V}{h^3}\cdot 4\pi \int p^2\,dp$$

$$\varepsilon = cp \qquad p = \varepsilon/c$$

$$d\varepsilon = c\,dp \qquad dp = d\varepsilon/c$$

$$\cancel{a(\varepsilon) = \frac{4\pi V}{h^3}\int \frac{\varepsilon^2}{c^2}\cdot\frac{1}{c}\,d\varepsilon}$$

$$\int a(\varepsilon)\,d\varepsilon = \frac{4\pi V}{h^3}\int \frac{\varepsilon^2}{c^3}\,d\varepsilon$$

$$a(\varepsilon) = \frac{4\pi V \varepsilon^2}{h^3 c^3}$$

b. Find an expression for the grand partition function Z_G for this system. Use it to write an expression for the grand potential Φ_G of the system. Your answer should be expressed in terms of the temperature T, volume V, and chemical potential μ, and as an integral over energy. Do not evaluate the integral

$$Z_1 = \frac{1}{N}\int a(\varepsilon)\cdot \exp(-\beta\varepsilon_p)\,d\varepsilon$$

$$= \frac{4\pi V}{h^3c^3}\int_0^\infty \varepsilon^2 \exp(-\beta\varepsilon)\,d\varepsilon$$

$$Z_G = \frac{1}{N!}\left[\frac{4\pi V}{h^3c^3}\int_0^\infty \varepsilon^2 \exp(-\beta\varepsilon)\,d\varepsilon\right]^N \cdot z^N$$

$$= \exp\left(\frac{4\pi V z}{h^3c^3}\int_0^\infty \varepsilon^2 \exp(-\beta\varepsilon)\,d\varepsilon\right) \qquad z = \exp(\beta\mu)$$

$$\Phi_G = -k_B T \ln Z_G$$

$$= -k_B T\cdot\frac{4\pi V z}{h^3c^3}\int_0^\infty \varepsilon^2 \exp(-\beta\varepsilon)\,d\varepsilon$$

c. Find an expression for the total energy E of the system in terms of V, μ, and an integral over energy. Do not evaluate the integral

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z$$

$$= -\left(-\frac{4\pi V z}{h^3c^3}\int_0^\infty \varepsilon^3 \exp(-\beta\varepsilon)\,d\varepsilon\right)$$

$$= \frac{4\pi V z}{h^3c^3}\int_0^\infty \varepsilon^3 \exp(-\beta\varepsilon)\,d\varepsilon$$

d. Use these result to demonstrate that $PV = E/3$ for this system at any temperature. (Hint: An integration by parts will relate the integrals in parts b and c above).

$$PV = k_B T\,\frac{4\pi V z}{h^3c^3}\int_0^\infty \varepsilon^2 \exp(-\beta\varepsilon)\,d\varepsilon$$

$$u = \exp(-\beta\varepsilon) \qquad v = \tfrac{1}{3}\varepsilon^3$$

$$du = -\beta\exp(-\beta\varepsilon) \qquad dv = \varepsilon^2$$

$$PV = k_B T\cdot\frac{4\pi V z}{h^3c^3}\left[\cancel{\varepsilon^2\exp(-\beta\varepsilon)\Big|_0^\infty} + \frac{1}{3}\int_0^\infty \varepsilon^3\beta\exp(-\beta\varepsilon)\,d\varepsilon\right]$$

$$= \frac{4\pi V z}{3h^3c^3}\int_0^\infty \varepsilon^3 \exp(-\beta\varepsilon)\,d\varepsilon = \frac{E}{3}$$

9*. A perfectly conducting sphere of radius a is moving through a magnetic field $\vec{B}=B_0\hat{y}$ at constant velocity $v_0\hat{x}$ with $v_0 << c$. Find the electric field $\vec{E}$ for points inside and outside the sphere.

10. A thin conducting sphere of radius a is split by the $z=0$ plane into two hemispherical shells which can be kept at different potentials. Suppose the shell with $z>0$ has potential $V(t)=V_0\cos(\omega t)$, and the shell with $z<0$ has potential $V(t)=-V_0\cos(\omega t)$. The frequency ω is such that the sphere is small compared to the wavelength: $a << \lambda$ where $\lambda = 2\pi c/\omega = 2\pi/k$. Find the radiated power to leading order in ω (or equivalently in $k=\omega/c$)

- $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{x}}{r^3}$

$$p_0 = 4\pi\varepsilon_0 V_0 a^2$$

$$\vec{p}_0 = 4\pi\varepsilon_0 V_0 a^2 \cos(\omega t)\hat{z}$$

$$\vec{p} = 4\pi\varepsilon_0 V_0 a^2 \hat{z}$$

$$= 4\pi\varepsilon_0 V_0 a^2(\cos\theta\hat{\rho} - \sin\theta\hat{\theta})$$

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{32\pi^2}|\hat{n}\times\vec{p}|^2$$

$$\hat{n} = \hat{r}$$

$$\hat{n}\times\vec{p} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ 1 & 0 & 0 \\ p_r & p_\theta & 0 \end{vmatrix} = (0, -(0-0), p_\theta)$$

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{32\pi^2}\cdot \sin^2\theta$$

$$P = \frac{c^2 Z_0 k^4}{64\pi^2}$$

11*. Consider a particle of mass m which can move vertically in one dimension in the gravitational field of the Earth above a floor at $z=0$. Treat the potential as

$$V(z) = \begin{cases} \infty, & z=0 \\ mgz, & z \geq 0 \end{cases}$$

a. Solve the Schroedinger equation in momentum space for the form of the wavefunction $\psi(p)$, disregarding normalization. Write down an integral expression for the wavefunction in position space. Do not evaluate the integral.

- $$\left(\frac{p^2}{2m} + i\hbar \frac{\partial}{\partial p} \cdot mg\right)\psi = E\psi$$

$$\frac{p^2\psi(p)}{2m} + i\hbar mg \frac{\partial \psi(p)}{\partial p} = E\psi(p)$$

$$i\hbar mg \, \partial\psi(p) = \left(E\psi(p) - p^2\psi(p)/2m\right) \partial p$$

b. Sketch the three lowest energy eigenfunctions in position space based on your knowledge of how the wavefunction must behave in different regions of z

- Determined by Airy functions, and wavefunction must disappear for $z=0$

12*. Suppose an electron's state had a spatial spread a shortly after the big bang and still has a comparable spread now. In this problem, treat the time-dependent state as a wavepacket in one dimension, and ignore the expansion of the Universe.

a. What is the order of magnitude of the minimum possible value of a if the current age of the Universe is approximately 10^{10} years.

- $\Delta x \Delta p \geq \hbar/2$

$$m(\Delta x)^2 \cdot 1/\Delta t \geq \hbar/2$$

$$(\Delta x)^2 = \hbar/2m \cdot \Delta t$$

$$\Delta x \geq \sqrt{\hbar \Delta t / 2m}$$

$$\hbar \approx 10^{-34} \, Js$$

$$\Delta t \approx 10^{10} \cdot 5\times 10^5 \cdot 60 = 3\times 10^{17}$$

$$m \approx 10^{-30} \, kg$$

$$\Delta x = \sqrt{10^{13}}$$

$$\approx 3\times 10^6 \, m$$

b. How does this estimate change for an object whose mass is $m = 1 kg$.

- $\Delta x \gtrsim \sqrt{10^{-17}}$

$$\approx 3\times 10^{-8} \, m$$

13. Consider two spinless bosonic atoms of equal mass m confined in a harmonic potential in three dimensions. These atoms interact via a short-range potential $v(\vec{r})$,

$$\int d\vec{r}_1 \, d\vec{r}_2 \, \psi_1^*(\vec{r}_2) v(\vec{r}_1 - \vec{r}_2) \psi_2(\vec{r}_1) \approx \frac{4\pi\hbar^2 a}{m} \int d\vec{r} \, \psi_1^*(\vec{r}) \psi_2(\vec{r})$$

with a representing the scattering length, which can be positive (repulsive) or negative (attractive).

a. What are the energy levels of the Hamiltonian when $a = 0$?

- $(n_{1x} + n_{1y} + n_{1z} + 3/2)\hbar\omega + (n_{2x} + n_{2y} + n_{2z} + 3/2)\hbar\omega$

b. What is the ground-state wavefunction when $a = 0$

- $\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{\vec{r}_1^{\,2} m\omega}{2\hbar}\right) \exp\left(-\frac{\vec{r}_2^{\,2} m\omega}{2\hbar}\right)$

c. Using perturbation theory, determine to first order in a the shift in the ground state energy caused by the interaction. What determines the sign of this energy shift? If the angular frequency of the harmonic potential is ω, under what condition relating a and $a_0 = \sqrt{\hbar/m\omega}$ is the perturbation result expected to be accurate

- $\Delta E = \frac{4\pi\hbar^2 a}{m} \int_{-\infty}^{\infty} \frac{m\omega}{\pi\hbar} \exp\left(-\frac{\vec{r}^{\,2} m\omega}{2\hbar}\right) d\vec{r}$

$= \frac{4\pi\hbar^2 a}{m} \cdot \frac{m\omega}{\pi\hbar}\sqrt{\frac{\pi\hbar}{m\omega}} = 4\hbar a\omega\sqrt{\frac{\pi\hbar}{m\omega}}$

sign is determined by the sign of a, so repulsive or attractive

14.

a. Two non-identical spin-1/2 particles are bound to neighboring locations in space and interact with a Hamiltonian

$$\hat{\mathcal{H}} = -J(\hat{S}_{x1}\hat{S}_{x2} + \hat{S}_{y1}\hat{S}_{y2})$$

where 1 and 2 label the particles. Find the exact energy eigenvalues and eigenvectors in terms of the total spin S and the z component S_z of the total spin.

- $\vec{\mathcal{H}} = -J(\vec{S}_1 \cdot \vec{S}_2 - \hat{S}_{1z}\hat{S}_{2z})$

$\vec{S}^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2\vec{S}_1 \cdot \vec{S}_2$

$\vec{\mathcal{H}} = -J\left(\frac{\vec{S}^2 - \vec{S}_1^{\,2} - \vec{S}_2^{\,2}}{2} - \hat{S}_{1z}\hat{S}_{2z}\right)$

$\mathcal{H}|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\rangle = -J\left(\frac{2 - \frac{1}{2}\frac{3}{2} - \frac{3}{4}}{2} - \frac{1}{4}\right)\hbar^2$

$= -J\hbar^2(\tfrac{1}{4} - \tfrac{1}{4}) = 0\,|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\rangle$

$= \mathcal{H}|\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\rangle$

$\mathcal{H}|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\rangle = \mathcal{H}|\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\rangle$

$= -J\hbar^2(\tfrac{1}{4})|\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\rangle$

$\mathcal{H}|-\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\rangle = -J\hbar^2\left(\frac{0 + \frac{1}{4} - \frac{3}{4}}{2} - \frac{1}{4}\right)$

$= -J\hbar^2 \cdot (\tfrac{1}{2})|-\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\rangle$

$= -\frac{J\hbar^2}{2}|\tfrac{1}{2}, \tfrac{1}{2}; -\tfrac{1}{2}, \tfrac{1}{2}\rangle$

$\mathcal{H}|-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\rangle = -J\hbar^2(-\tfrac{1}{2} + \tfrac{1}{4})$

$= J\hbar^2(\tfrac{1}{4})|-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\rangle$

$= \frac{J\hbar^2}{4}|\tfrac{1}{2}, -\tfrac{1}{2}; -\tfrac{1}{2}, \tfrac{1}{2}\rangle$

$= \frac{J\hbar^2}{4}|-\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\rangle$

$= \frac{J\hbar^2}{4}|\tfrac{1}{2}, \tfrac{1}{2}; -\tfrac{1}{2}, -\tfrac{1}{2}\rangle$

$\mathcal{H}|-\tfrac{1}{2}, \tfrac{1}{2}; -\tfrac{1}{2}, \tfrac{1}{2}\rangle = \mathcal{H}|-\tfrac{1}{2}, -\tfrac{1}{2}; -\tfrac{1}{2}, -\tfrac{1}{2}\rangle = -J\hbar^2\left(\frac{0 + \frac{1}{4} + \frac{1}{4}}{2} - \frac{1}{4}\right)$

$= 0$

$= \mathcal{H}|-\tfrac{1}{2}, -\tfrac{1}{2}; -\tfrac{1}{2}, -\tfrac{1}{2}\rangle$

Alternatively,

$S_x = \hbar/2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad S_y = \hbar/2\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad S_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$$S_{1x} \otimes S_{2x} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$S_{1y} \otimes S_{2y} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$S_{1z} \otimes S_{2z} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

$$\mathcal{H} = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\det = -\lambda[-\lambda^3 + 4\lambda]$$
$$= \lambda^2(\lambda^2 - 4)$$
$$\lambda = 0, \pm 2$$

$|0\rangle$: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ $\quad |+-\rangle$

$|2\rangle$: $\begin{pmatrix} -2 & & & \\ & -2 & 2 & \\ & 2 & -2 & \\ & & & -2 \end{pmatrix}$

$|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad |-+\rangle$

$|-2\rangle$: $\begin{pmatrix} 2 & & & \\ & 2 & 2 & \\ & 2 & 2 & \\ & & & 2 \end{pmatrix}$

$|-2\rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$

b. Suppose a magnetic field adds a term $-\hat{\mu}\cdot\vec{B}$ to $\mathcal{H}$, where $\hat{\mu} = -\frac{e}{mc}\vec{S}$. Solve for the energy eigenvalue. Describe the restrictions on the allowed quantum numbers if the particles are identical fermions and if they are identical bosons.

Set $\vec{B} = B_z\hat{z}$

$$\Delta\mathcal{H} = \begin{pmatrix} -eB_z/mc - \lambda & & & \\ & eB_z/mc - \lambda & \hbar^2/2 & \\ & \hbar^2/2 & eB_z/mc - \lambda & \\ & & & -eB_z/mc - \lambda \end{pmatrix}$$

$$\det\mathcal{H} = (-eB_z/mc - \lambda)\det\begin{pmatrix} eB_z/mc - \lambda & -\hbar^2/2 & 0 \\ \hbar^2/2 & eB_z/mc - \lambda & 0 \\ 0 & 0 & -eB_z/mc - \lambda \end{pmatrix}$$
$$= (-eB_z/mc - \lambda)\left[(eB_z/mc - \lambda)^2(-eB_z/mc - \lambda) + \hbar^2/2\cdot\hbar^2/2\,(-eB_z/mc - \lambda)\right]$$
$$= (-eB_z/mc - \lambda)\left[(eB_z/mc - \lambda)^2 + \hbar^4/4\right]$$
$$= (-a - \lambda)\left[\lambda^2 - 2a\lambda + a^2 + \hbar^4/4\right]$$
$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2 + \hbar^4/4)}}{2},\ a$$
$$a = eB_z/mc$$

15*. Cosmic-ray protons have maximum energy E_0, which is the threshold energy for pion production $p + \gamma \to p + \pi^0$ occuring when the protons interact with the radiation from the big bang. The radiation temperature is $T = 3K$. Find E_0 in GeV. (Note: $m_p \approx 0.938\ GeV/c^2$, $m_\pi \approx 0.135\ GeV/c^2$)

Ok center of mass frame

$\xrightarrow{p} \quad \leftarrow p$

• product $\qquad E_\gamma = k_B T = 4 \times 10^{-23}\ J$

$(E_0, \vec{p}_0)$ $\qquad\qquad = 4 \times 10^{-5}\ eV$

Say final pion has no momentum. Since γ is very small, momentum of proton won't really change. Momentum of pion = 0

$$\cancel{m_p^2} + 2(E_0 E_\gamma - \vec{p}_0 E_\gamma) = \cancel{m_p^2} + m_\pi^2 + 2(E_0 E_\pi)$$

$$2E_\gamma\left(E_0 - \sqrt{E_0^2 - m_p^2}\right) = m_\pi^2 + 2E_0 m_\pi$$

$$-2E_\gamma \sqrt{E_0^2 - m_p^2} = m_\pi^2 + 2E_0 m_\pi - 2E_\gamma E_0$$

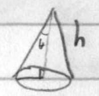
$$2E_0(E_\gamma - m_\pi) - m_\pi^2 = 2E_\gamma \sqrt{E_0^2 - m_p^2}$$

$$4E_0^2 E_\pi^2 - 8E_0^2 E_\gamma m_\pi + 4E_0^2 m_\pi^2 - m_\pi^4 = 4E_\gamma^2(E_0^2 - m_p^2)$$
$$+ 4E_0 m_\pi^2 (E_\gamma - m_\pi)$$

Solve for E_0

16.

a. The radius of a rocky exoplanet is a fraction f of the Earth's radius. If the maximum height of an Earth mountain in h_E, estimate the maximum height h_p of a mountain on the exoplanet



$$V = \tfrac{1}{3} \cdot \pi r^2 h = \tfrac{1}{3}\pi h^3 \tan^2\theta \approx \tfrac{1}{3}\pi h^3 \text{ for } \theta = 45^\circ$$

$\tan\theta = r/h$ $\qquad m = \rho \cdot V$

$$f_g = \frac{G \cdot M m}{(r + h/2)^2} \qquad M = \rho \cdot \tfrac{4}{3} \cdot \pi R^3$$

$$f_g' = \frac{G M' m'}{(r' + h_E/2)^2}$$

$$\frac{G \cdot \rho \cdot \frac{4}{3} \cdot \pi R^3 \cdot \rho \cdot \frac{1}{3} \cdot \pi h_e^3}{(R + h_e/2)^2} = \frac{G \rho \cdot \frac{4}{3} \cdot \pi (fR)^3 \rho \cdot \frac{1}{3} \cdot \pi h_p^3}{(fR + h_p/2)^2}$$

$$\frac{h_e^3}{R^2(1 + h_e/2R)^2} = \frac{f^3 h_p^3}{f^2 R^2 (1 + h_p/2fR)^2}$$

$$h_e^3(1 - h_e/R) = f h_p^3 (1 - h_p/fR)$$

$$R h_e^3 - h_e^4 = fR h_p^3 - h_p^4$$

$$h_e^3(R - h_e) = h_p^3(fR - h_p)$$

$$R h_e^3 = fR h_p^3$$

$$\left(\frac{h_p}{h_e}\right)^3 = \frac{1}{f}$$

$$h_p = f^{-1/3} h_e$$

b. $$g = \frac{G \cdot \rho \cdot \frac{4}{3}\pi R^3}{R^2}$$

$$\tfrac{2}{5} \cdot g = \frac{G \cdot \rho \cdot \frac{4}{3}\pi (fR)^3}{(fR)^2}$$

$$\frac{2}{5} \cdot \frac{\cancel{G\rho \cdot \frac{4}{3} \pi} R^{\cancel{3}}}{\cancel{R^2}} = \frac{\cancel{G\rho \cdot \frac{4}{3}\pi} (f \cancel{R})^3}{(f\cancel{R})^2}$$

$$\frac{2}{5} = f$$

$h_e = 9 \text{ km}$

$h_m = (2/5)^{-1/3} \cdot 9 = 12 \text{ km}$