6*. Consider a system of N spins on a lattice, with Hamiltonian

$$\mathcal{H} = -\varepsilon_0 \sum_{i=0}^{N} s_i$$

where $s_i = \pm 1/2$ are the two possible values for the i-th spin. In this problem, we will ignore any orbital motion and work with only the spin degrees of freedom. Assume $N >> 1$.

a. Working in microcanonical ensemble, find the temperature T when the system has an energy E.

$S = k_B \ln \Omega$

$\left(\frac{\partial S}{\partial E}\right)_{N,V} = 1/T$

$$\Omega = \binom{N}{N_\uparrow} = \frac{N!}{N_\uparrow!\,(N-N_\uparrow)!}$$

$N = N_\uparrow + N_\downarrow$

$E = -\varepsilon_0 \left[\sum_{i=0}^{N_\uparrow} (1/2) + \sum_{i=0}^{N_\downarrow} (-1/2)\right] = -\varepsilon_0/2 \left[N_\uparrow - N_\downarrow\right]$

Since we keep N constant, let's find $N_\uparrow$ in terms of E.

$N_\uparrow = N - N_\downarrow$

$-2E/\varepsilon_0 = N_\uparrow - N_\downarrow$

$N_\downarrow = N_\uparrow + 2E/\varepsilon_0$

$N_\uparrow = N - N_\uparrow - 2E/\varepsilon_0$

$N_\uparrow = N/2 - E/\varepsilon_0$

$S = k_B\left[\ln N! - \ln N_\uparrow! - \ln (N-N_\uparrow)!\right]$

Using the Stirling approximation $\ln N! = N \ln N - N$

$$S = k_B\left[N\ln N - \cancel{N} - N_\uparrow \ln N_\uparrow + \cancel{N_\uparrow} - (N-N_\uparrow)\ln(N-N_\uparrow) + \cancel{N - N_\uparrow}\right]$$

$$= k_B\left[N \ln\left(\frac{N}{N-N_\uparrow}\right) + N_\uparrow \ln\left(\frac{N-N_\uparrow}{N_\uparrow}\right)\right]$$

$$= k_B\left[N\ln\left(\frac{N}{N - N/2 + E/\varepsilon_0}\right) + \left(N/2 - E/\varepsilon_0\right)\ln\left(\frac{N - N/2 + E/\varepsilon_0}{N/2 - E/\varepsilon_0}\right)\right]$$

$$= k_B\left[N\ln N - N/2 \ln\left(N/2 + E/\varepsilon_0\right) - N/2 \ln\left(N/2 - E/\varepsilon_0\right) - E/\varepsilon_0 \ln\left(N/2 + E/\varepsilon_0\right) + E/\varepsilon_0 \ln\left(N/2 - E/\varepsilon_0\right)\right]$$

$$\frac{\partial S}{\partial E} = k_B\left(\frac{-N/2}{N/2 + E/\varepsilon_0}\cdot\frac{1}{\varepsilon_0} - \frac{N/2}{N/2 - E/\varepsilon_0}\cdot\frac{-1}{\varepsilon_0} - \frac{\ln\left(N/2 + E/\varepsilon_0\right)}{\varepsilon_0} + \frac{\ln\left(N/2 - E/\varepsilon_0\right)}{\varepsilon_0}\right.$$
$$\left. - \frac{E}{\varepsilon_0}\cdot\frac{1}{N/2 + E/\varepsilon_0}\cdot\frac{1}{\varepsilon_0} + \frac{E}{\varepsilon_0}\cdot\frac{1}{N/2 - E/\varepsilon_0}\cdot\frac{-1}{\varepsilon_0}\right)$$

$$= k_B\left(\frac{-N}{N\varepsilon_0 + 2E} + \frac{N}{N\varepsilon_0 - 2E} + \frac{1}{\varepsilon_0}\ln\left(\frac{N\varepsilon_0 - 2E}{N\varepsilon_0 + 2E}\right) - \frac{2E/\varepsilon_0}{N\varepsilon_0 + 2E} - \frac{2E/\varepsilon_0}{N\varepsilon_0 - 2E}\right)$$

$$\frac{1}{T} = \frac{k_B}{\varepsilon_0}\ln\left(\frac{N\varepsilon_0 - 2E}{N\varepsilon_0 + 2E}\right)$$

b. Working in canonical ensemble, find the average energy $\langle E\rangle$ for a given temperature T, and show the result is consistent with that of (a).

$Z_c = \sum_r \exp(-\beta E_r) = \left[\exp(\beta\varepsilon_0/2) + \exp(-\beta\varepsilon_0/2)\right]^N$

$A = -k_B T \ln Z_c$

$A = \mathcal{U} - TS$

$S = -\frac{\partial A}{\partial T}$

$= \frac{\partial A}{\partial T}\left(k_B T N \ln\left[\exp(\beta\varepsilon_0/2) + \exp(-\beta\varepsilon_0/2)\right]\right)$

$$= k_B N \ln\left[\exp(\beta\varepsilon_0/2) + \exp(-\beta\varepsilon_0/2)\right] + k_B T N \cdot \left(\frac{-\frac{\varepsilon_0}{2k_BT^2}\exp(\beta\varepsilon_0/2) + \frac{\varepsilon_0}{2k_BT^2}\exp(-\beta\varepsilon_0/2)}{\exp(\beta\varepsilon_0/2) + \exp(-\beta\varepsilon_0/2)}\right)$$

$\mathcal{U} = A + TS$

$$= \frac{k_B N\varepsilon_0}{2k_B}\cdot\frac{\exp(-\beta\varepsilon_0/2) - \exp(\beta\varepsilon_0/2)}{\exp(\beta\varepsilon_0/2) + \exp(-\beta\varepsilon_0/2)} = \frac{N\varepsilon_0}{2}\cdot\frac{1 - \exp(\beta\varepsilon_0)}{1 + \exp(\beta\varepsilon_0)}$$

$2\mathcal{U}(1 + \exp(\beta\varepsilon_0)) = N\varepsilon_0(1 - \exp(\beta\varepsilon_0))$

$\exp(\beta\varepsilon_0)(2E + N\varepsilon_0) = N\varepsilon_0 - 2E$

$$\frac{\varepsilon_0}{k_B T} = \ln\left(\frac{N\varepsilon_0 - 2E}{N\varepsilon_0 + 2E}\right)$$

$$\frac{1}{T} = \frac{k_B}{\varepsilon_0}\ln\left(\frac{N\varepsilon_0 - 2E}{N\varepsilon_0 + 2E}\right)$$

7. A chamber held at temperature T with volume V contains heteronuclear diatomic molecules of mass m and moment of inertia I. In this problem, we treat the molecules as classical rigid rotators with rotational kinetic energy

$$T_R = \frac{1}{2I}\left[p_\theta^2 + p_\phi^2/\sin^2\theta\right]$$

where p_θ and p_ϕ are canonical momenta for rotational motion. (Note the rotational kinetic energy of a diatomic molecule is in addition to the kinetic energy of its center of mass motion).

a. Treating the gas of diatomic molecules in grand canonical ensemble, and ignoring interactions among the particles, show that the partition function has the form

$$Z = \exp\left\{ z \cdot (V/\lambda^3)(2Ik_BT/\hbar^2)\right\}$$

where λ is the thermal wavelength and z is the fugacity

$\lambda = (2\pi\hbar^2/mk_BT)^{1/2}$

$$Z_G = \sum_N z^N Z_c(N)$$

$$Z_c = \sum_r \exp(-\beta E_r)$$

$$E = p^2/2m + \frac{1}{2I}\left[p_\theta^2 + p_\phi^2/\sin^2\theta\right]$$

Let's evaluate the translational part first

$$Z_{ct} = \frac{1}{N!}\left\{ \frac{1}{h^3}\iint \exp(-\beta p^2/2m)\, d^3x\, d^3p\right\}^N$$

$$= \frac{1}{N!}\left\{ \frac{V}{h^3}\left(\int \exp(-\beta p^2/2m)\, dp\right)^3\right\}^N$$

$$= \frac{1}{N!}\left\{ \frac{V}{h^3}(2\pi m/\beta)^{3/2}\right\}^N$$

$$= \frac{1}{N!}\left\{ V/\lambda^3\right\}^N$$

Now the rotational part

$$j_{rot} = \frac{1}{h^2}\int_0^{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-\beta p_\theta^2/2I)\exp(-\beta p_\phi^2/2I\sin^2\theta)\, dp_\theta\, dp_\phi\, d\theta\, d\phi$$

$$= \frac{2\pi}{h^2}\int_0^{\pi}\left(\frac{2I\pi}{\beta}\right)^{1/2}\left(\frac{2I\pi\sin^2\theta}{\beta}\right)^{1/2} d\theta$$

$$= \frac{2\pi}{h^2}\cdot\frac{2I\pi}{\beta}\int_0^{\pi}\sin\theta\, d\theta = \frac{2I}{\hbar^2\beta}$$

Putting it all together

$$Z_G = \frac{1}{N!}\cdot z^N\left[\frac{V}{\lambda^3}(2Ik_BT/\hbar^2)\right]^N$$

$$= \exp\left\{ z(V/\lambda^3)(2Ik_BT/\hbar^2)\right\}$$

since we recognize that $\exp(x) = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$

b. What is the average number of particles in terms of z and the other parameters?

$$\langle N\rangle = -\frac{\partial \ln Z_G}{\partial(-\beta\mu)} = \frac{\partial \ln Z_G}{\partial \ln z}$$

$$\ln Z_G = z(V/\lambda^3)(2Ik_BT/\hbar^2)$$

$$= \exp(\ln z)(V/\lambda^3)(2Ik_BT/\hbar^2)$$

$$\langle N\rangle = z(V/\lambda^3)(2Ik_BT/\hbar^2)$$

c. Suppose the molecules have an electric dipole moment. A separate chamber with the same volume contains the same type of molecules at temperature T, but there is an electric field which forces them to orient along the field, so that the rotational motion is effectively eliminated. Assuming any effect of the field on the center of mass motion of the molecules may be neglected, what is the number of particles in terms of the chemical potential and the other parameters?

Since we don't have rotational energy, $Z = \exp(z(V/\lambda^3))$.

$$\langle N\rangle = z\cdot V/\lambda^3$$

d. If the two chambers are brought into contact so that the particles may move between them, what will the ratio of particle numbers in the two chambers be in equilibrium?

$$\frac{\langle N\rangle_b}{\langle N\rangle_c} = \frac{2Ik_BT}{\hbar^2}$$

c. As $T \to 0$ it is observed that the ratio of particle numbers in the two chambers approaches 1. Show that the result in (d) fails to predict this, and explain why there is a discrepancy.

$$r = \frac{2Ik_BT}{\hbar^2} \to 0 \quad \text{as } T \to 0$$

The discrepancy occurs because as $T \to 0$, we start to need quantum to explain this system.

$\lambda \to \infty$ as $T \to 0$. For a classical system, we want $\lambda \ll 1$. We can easily convince ourselves that this is not true at low temperature.

8. Consider a gas of N bosons with energy-momentum relation $\varepsilon(p) = p^2/2m$ contained in a volume V. What is the pressure P at very low temperature T, assuming the system is Bose-condensed? Show that behavior of P as a function of T is different than would be expected for a classical ideal gas. (Hint: Use the grand potential). The following integral may be useful: $\int_0^\infty \frac{y^{3/2}}{\exp(y)-1}\,dy = \Gamma(5/2)\,g_{5/2}(1) \approx 1.783$

For a Bose system, $\ln Z_G = -\sum_\varepsilon \ln(1 - z\exp(-\beta\varepsilon))$

$$= -\iint_{-\infty}^{\infty} \frac{\ln(1-z\exp(-\beta\varepsilon))}{h^3}\, d^3x\, d^3p$$

$$= \frac{-4\pi V}{h^3} \int_{-\infty}^{\infty} p^2 \ln(1-z\exp(-\beta\varepsilon))\, dp$$

$$p^2 = 2m\varepsilon$$

$$2p\,dp = 2m\,d\varepsilon$$

$$\ln Z_G = \frac{-4\pi V}{h^3} \int_{-\infty}^{\infty} m\sqrt{2m\varepsilon}\, \ln(1-z\exp(-\beta\varepsilon))\, d\varepsilon$$

Using integration by parts

$$u = \ln(1 - z\exp(-\beta\varepsilon)) \qquad v = \tfrac{2}{3}\varepsilon^{3/2}$$

$$u' = \frac{1}{1 - z\exp(-\beta\varepsilon)} \cdot z\beta\exp(-\beta\varepsilon) \qquad v' = \varepsilon^{1/2}$$

$$\ln Z_G = \frac{-4\pi V}{h^3} \cdot (2m^3)^{1/2} \left[\cancelto{0}{\frac{2\varepsilon^{3/2}}{3} \ln(1-z\exp(-\beta\varepsilon))\Big|_0^\infty} - \int_0^\infty \frac{2\varepsilon^{3/2} \cdot z\beta \, d\varepsilon}{3(\exp(-\beta\varepsilon) - z)} \right]$$

$$= \frac{4\pi V}{h^3} \cdot \frac{(2m)^{3/2}}{3} \cdot \beta \int_0^\infty \frac{\varepsilon^{3/2}\, d\varepsilon}{z\exp(-\beta\varepsilon) - 1} \qquad y = -\beta\varepsilon,\quad dy = -\beta\, d\varepsilon$$

$$= \frac{-4\pi V}{h^3} \cdot \frac{(2m)^{3/2}}{3} \cdot \frac{\beta}{(-\beta)^{3/2} \cdot \beta} \int_0^\infty \frac{y^{3/2}\, dy}{\exp(y) - 1} \qquad \mu \to 0$$

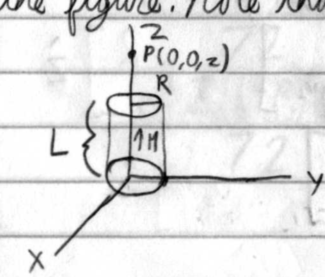
$$= \frac{4\pi V}{3} \cdot \frac{(2mk_BT)^{3/2}}{h^3} \cdot \Gamma(5/2)\, g_{5/2}(1)$$

$$= \frac{2V}{3\sqrt{\pi}} \cdot \frac{\Gamma(5/2)\, g_{5/2}(1)}{\lambda^3} = \frac{PV}{k_BT}$$

$$P = \frac{2\Gamma(5/2)\, g_{5/2}(1)}{3\sqrt{\pi} \cdot \lambda^3 \beta}$$

For a classical ideal gas, $PV = k_B NT$, $P \propto T$. In the above $P \propto T^{3/2}$

9. A cylinder of radius R and length L has a uniform magnetization M along its symmetry axis, as shown in the figure. Note that the cylinder lies in the interval $0 < z < L$.



a. Compute the magnetic field $\vec{B}$ at a point P on the symmetry axis of the cylinder. Explain why the result makes sense for $z = L/2$ when $L >> R$.

We can treat the cylinder as two plates with some charge $\sigma = \vec{M} \cdot \hat{n}$

$z=L$ $\sigma = M$

$z=0$ $\sigma = -M$

$$\Phi = \int_0^R \frac{2\pi\rho\,\sigma_{bot}}{\sqrt{\rho^2+z^2}}\,d\rho + \int_0^R \frac{2\pi\rho\,\sigma_{top}}{\sqrt{\rho^2+(z-L)^2}}\,d\rho$$

$$= -\int_0^R \frac{2\pi\rho M\,d\rho}{\sqrt{\rho^2+z^2}} + \int_0^R \frac{2\pi\rho M\,d\rho}{\sqrt{\rho^2+(z-L)^2}}$$

$\rho^2 + z^2 = u$ $\quad$ $\rho^2+(z-L)^2 = v$

$2\rho\,d\rho = du$ $\quad$ $2\rho\,d\rho = dv$

$$\Phi = \int -\pi M u^{-1/2}\,du + \int \pi M v^{-1/2}\,dv$$

$$= -2\pi M\sqrt{\rho^2+z^2}\Big|_0^R + 2\pi M\sqrt{\rho^2+(z-L)^2}\Big|_0^R$$

$$= -2\pi M\left[\sqrt{R^2+z^2} - z\right] + 2\pi M\left[\sqrt{R^2+(z-L)^2} - |z-L|\right]$$

for $0 < z < L$: $\frac{M}{2}\left[z - \sqrt{R^2+z^2} + \sqrt{R^2+(z-L)^2} - (L-z)\right]$

$$= \frac{M}{2}\left[2z - L - \sqrt{R^2+z^2} + \sqrt{R^2+(z-L)^2}\right]$$

$$\frac{d\Phi}{dz} = \frac{M}{2}\left[2 - \tfrac{1}{2}(R^2+z^2)^{-1/2}\cdot 2z + \tfrac{1}{2}(R^2+(z-L)^2)^{-1/2}\cdot 2(z-L)\right]$$

$$= \frac{M}{2}\left[\frac{z-L}{(R^2+(z-L)^2)^{1/2}} - \frac{z}{(R^2+z^2)^{1/2}} + 2\right]$$

$$\vec{B} = \frac{\mu_0 M}{2}\left[\frac{z-L}{(R^2+(z-L)^2)^{1/2}} - \frac{z}{(R^2+z^2)^{1/2}} + 4\right]\hat{z}$$

for $z > L$: $\Phi = \frac{M}{2}\left[z - \sqrt{R^2+z^2} + \sqrt{R^2+(z-L)^2} - (z-L)\right]$

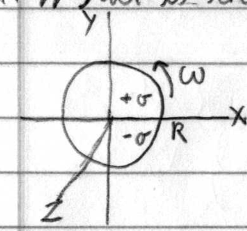
$$= \frac{M}{2}\left[L - \sqrt{R^2+z^2} + \sqrt{R^2+(z-L)^2}\right]$$

$$\frac{d\Phi}{dz} = \frac{M}{2}\left[\frac{z-L}{(R^2+(z-L)^2)^{1/2}} - \frac{z}{(R^2+z^2)^{1/2}}\right]\hat{z}$$

$$\vec{B} = \frac{\mu_0 M}{2}\left[\frac{z-L}{(R^2+(z-L)^2)^{1/2}} - \frac{z}{(R^2+z^2)^{1/2}}\right]$$

~~10.~~ An infinitely thin circular disk of radius R centered at the origin in the xy-plane is divided into two equal halves. On the top half ($y>0$) the disk has uniform charge per unit area σ, while on the bottom half ($y<0$) the uniform charge density is $-\sigma$.

a. What is the electric dipole moment of the disk?



$$\vec{p} = \int \rho\,\vec{r}\,d^3x$$

$$= \int \rho\, r \cdot r\,dr\,d\theta$$

$$= \int_0^R\int_0^{\pi} \sigma r^2\,d\theta\,dr - \int_0^R\int_0^{-\pi} \sigma r^2\,d\theta\,dr$$

$$= \pi R^3/3 + \pi R^3/3 = 2\pi R^3/3 \cdot \hat{y} \cdot \sigma$$

b. $$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{32\pi^2}\left|(\hat{n}\times\vec{p})\times\hat{n}\right|^2$$

$\vec{p}' = p_0(\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t))$ $\quad$ $p_0 = 2\pi R^3/3$

$\vec{p}' = Re(\vec{p}\exp(-i\omega t))$

$\vec{p} = p_0(\hat{x} + i\hat{y})$

$$|\hat{n}\times\vec{p}| = \frac{1}{r}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_0 & ip_0 & 0 \end{vmatrix} = (0 - izp_0, -(0 - zp_0), ixp_0 - yp_0)$$

$$= \tfrac{1}{r}(-izp_0, zp_0, (ix-y)p_0)$$

$(x, y, z) = (\cos\phi\sin\theta, \sin\theta\sin\phi, \cos\theta)$

$|(\hat{n}\times\vec{p})\times\hat{n}|^2 = |\hat{n}\times\vec{p}|^2$

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{8\cancel{32\pi^2}} \cdot \frac{\cancel{4\pi^2} R^6\sigma^2}{9} \cdot \tfrac{1}{r^2}(z^2 + z^2 + x^2 + y^2)$$

$$= \frac{c^2 Z_0 k^4 R^6 \sigma^2 (2\cos^2\theta + \sin^2\theta)}{72}$$

$$= \frac{c^2 Z_0 k^4 R^6 \sigma^2 (1 + \cos^2\theta)}{72}$$

c. What is the polarization of the radiation as a function of the polar angle θ (defined with respect to the z-axis) and the azimuthal angle ϕ?

11.* A particle of mass m, in a two-dimensional world, is trapped inside a square box with infinitely repulsive walls and dimensions $0<x<a$ and $0<y<b$, with $b>a$. Inside the box, there is a potential

$$V(\vec{r}) = \lambda \cos(\pi x/a) \cos(\pi y/b)$$

a. For $\lambda = 0$, find all of the allowed energy eigenvalues and eigenfunctions.

This is the particle in a box, so we have to remember

$$E_{n_x,n_y} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{a^2}+\frac{n_y^2}{b^2}\right)$$

$$\psi = \sqrt{\tfrac{2}{ab}}\,\sin(n_x\pi x/a)\sin(n_y\pi y/b)$$

b. For $|\lambda|$ small but non-zero, find the ground state energy correct to order λ^2

$$E_{n_x n_y}(\lambda) = E_n^0 + \langle n^{(0)}|V|n^{(0)}\rangle + \sum_{k\neq n}\frac{|\langle k^{(0)}|V|n^{(0)}\rangle|^2}{E_n^0 - E_k^0}$$

Ground state is $n_x=1, n_y=1$, $E^0 = \hbar^2\pi^2/2m\left(\frac{1}{a^2}+1/b^2\right)$

To first order

$$\langle\psi_{1,1}|V|\psi_{1,1}\rangle = 2\lambda/ab\int_0^b\int_0^a \sin^2(\pi x/a)\cos(\pi x/a)\sin^2(\pi y/b)\cos(\pi y/b)\,dx\,dy$$

$$= 2\lambda/ab\left[1/3(\pi/a)\cdot\sin^3(\pi)\cdot 1/3(\pi/b)\cdot\sin^3(\pi)\right] = 0$$

For second order,

$$\langle\psi_{n_x,n_y}|V|\psi_{1,1}\rangle = 2\lambda/ab\int_0^b\int_0^a \sin(n_x\pi x/a)\sin(\pi x/a)\cos(\pi x/a)\cdot\sin(n_y\pi y/b)\sin(\pi y/b)\cos(\pi y/b)\,dx\,dy$$

$$= 2\lambda/ab\int_0^b\int_0^a 1/2\cdot\sin(n_x\pi x/a)\sin(2\pi x/a)\,1/2\cdot\sin(n_y\pi y/b)\sin(2\pi y/b)\,dx\,dy$$

$= 0 \quad \forall n_x, n_y$ except $n_x = n_y = 2$

$$= 2\lambda/ab\cdot 1/4\left[a/2\cdot b/2\right]$$

$$= \lambda/8$$

$$E_n^2 = \frac{\lambda^2/64}{E_{22}-E_0} = \frac{\lambda^2}{64}\cdot\frac{2m}{\hbar^2\pi^2}\cdot\frac{1}{(4/a^2+4/b^2)-(1/a^2+1/b^2)}$$

$$= \frac{\lambda^2 m}{32\hbar^2\pi^2}\cdot\frac{a^2b^2}{3(a^2+b^2)}$$

$$E = \frac{\hbar^2\pi^2}{2m}\left(\frac{a^2+b^2}{a^2b^2}\right) + \frac{\lambda^2 m}{32\hbar^2\pi^2}\cdot\frac{a^2b^2}{3(a^2+b^2)}$$

12*. A particle of mass m moves in one dimension in the potential $V(x) = -V_0[\delta(x-a)+\delta(x+a)]$, with $V_0 > 0$ a constant

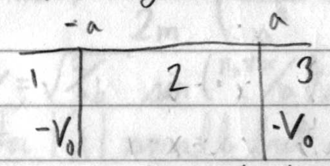
a. How does the ground state wavefunction $\psi_0(x)$ behave under parity?

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - V_0[\delta(x-a)+\delta(x+a)]\psi = E\psi$$

From the single delta potential well problem, we know

$$\psi = \begin{cases} -A\exp(\alpha(x-a)) & x<a \\ -A\exp(-\alpha(x-a)) & x>a \end{cases}$$

Thus, let's guess the solution:



$x<-a$: $-A\exp(\alpha(x+a)) - B\exp(\beta(x-a))$

$-a<x<a$: $-A\exp(\alpha(x+a)) - B\exp(\beta(x-a))$

$x>a$: $-A\exp(-\alpha(x+a)) - B\exp(-\beta(x-a))$

Boundary between 1+2

$$\frac{\partial\psi_1}{\partial x} = -A\alpha\exp(\alpha(x+a)) - \beta B\exp(\beta(x-a))$$

$$\frac{\partial^2\psi_1}{\partial x^2} = -A\alpha^2\exp(\alpha(x+a)) - B\beta^2\exp(\beta(x-a))$$

$$\frac{\partial^2\psi_2}{\partial x^2} = -A\alpha^2\exp(-\alpha(x+a)) - B\beta^2\exp(\beta(x-a))$$

$$\frac{\partial^2\psi_3}{\partial x^2} = -A\alpha^2\exp(-\alpha(x+a)) - B\beta^2\exp(-\beta(x-a))$$

1+2 $$-\frac{\hbar^2}{2m}(-A\alpha^2\exp(\alpha(x+a)) - B\beta^2\exp(\beta(x-a))) = E(-A\exp(\alpha(x+a)) - B\exp(\beta(x-a)))$$

at $x=-a$, $$-\frac{\hbar^2}{2m}(-B\beta^2\exp(-2\beta a)) - V_0(-B\exp(-2\beta a)) = -BE\exp(-2\beta a)$$

$$\frac{\hbar^2\beta^2}{2m} + V_0 = -E$$

$$\beta^2 = \tfrac{2m}{\hbar^2}(-E-V_0)$$

$$\beta = \sqrt{\tfrac{2m}{\hbar^2}(-E-V_0)}$$

$$\alpha = \sqrt{\tfrac{2m}{\hbar^2}(-E-V_0)} = \beta = i\sqrt{\tfrac{2m}{\hbar^2}(E+V_0)}$$

$$\psi(x) = \begin{cases} -\exp(i\alpha(x+a)) - \exp(i\alpha(x-a)) & x<-a \\ -\exp(-i\alpha(x+a)) - \exp(i\alpha(x-a)) & -a<x<a \\ -\exp(-i\alpha(x+a)) - \exp(-i\alpha(x-a)) & x>a \end{cases}$$

$$\Pi|\psi(x)\rangle = \begin{cases} -\exp(-i\alpha(x+a)) - \exp(-i\alpha(x-a)) & x<-a \\ -\exp(i\alpha(x+a)) - \exp(-i\alpha(x-a)) & -a<x<a \\ -\exp(i\alpha(x+a)) - \exp(i\alpha(x-a)) & x>a \end{cases}$$

$$\alpha = \sqrt{\tfrac{2m}{\hbar^2}(E+V_0)}$$

b. Writing the ground state energy in the form $E_0 = -\hbar^2\kappa^2/2m$, derive a transcendental equation that relates κ to V_0 and a. You need not solve the equation.

$$\left.\frac{\partial\psi}{\partial x}\right|_{x=a^+} - \left.\frac{\partial\psi}{\partial x}\right|_{x=a^-} = -\frac{2mV_0}{\hbar^2}\psi(a)$$

$$-i\alpha\exp(i\alpha\cdot 2a) - i\alpha\exp(0) - [-i\alpha\exp(i\alpha(2a)) + i\alpha\exp(0)] = -\tfrac{2mV_0}{\hbar^2}(-\exp(i\alpha(2a)) - 1)$$

$$\exp(i\alpha(2a)) = -1$$

$$\exp\left(2ia\sqrt{\frac{2m}{\hbar^2}\left(V_0 - \frac{\hbar^2\kappa^2}{2m}\right)}\right) = -1$$

13. Two spin-½ particles have Hamiltonian

$$\mathcal{H} = A\vec{\sigma}_1 \cdot \vec{\sigma}_2 + B(\sigma_1^z + \sigma_2^z)$$

with A and B being two positive constants.

a. Find the ground state wavefunction for $B = 0$

$\mathcal{H} = A\,\vec{\sigma}_1 \cdot \vec{\sigma}_2$

$= A(\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z})$

$\mathcal{H}|\psi\rangle = E|\psi\rangle$

$(\mathcal{H} - E)|\psi\rangle = 0$

$$\sigma_{1x} = \sigma_{1x} \otimes I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\sigma_{2x} = I \otimes \sigma_{2x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\sigma_{1y} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \sigma_{2y} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\sigma_{1z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \sigma_{2z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & -1 & 2 & \\ & 2 & -1 & \\ & & & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} A-\lambda & 0 & 0 & 0 \\ 0 & -A-\lambda & 2A & 0 \\ 0 & 2A & -A-\lambda & 0 \\ 0 & 0 & 0 & A-\lambda \end{pmatrix}$$

$$= (A-\lambda)\det \begin{bmatrix} -A-\lambda & 2A & 0 \\ 2A & -A-\lambda & 0 \\ 0 & 0 & A-\lambda \end{bmatrix}$$

$$= (A-\lambda)[(-A-\lambda)(-A-\lambda)(A-\lambda) - 2A(2A)(A-\lambda)]$$

$$= (A-\lambda)^2[\lambda^2 + 2A\lambda + A^2 - 4A^2]$$

$$= (A-\lambda)^2(\lambda + 3A)(\lambda - A)$$

$$\lambda = -3A, A$$

The ground state energy is $-3A$

$$|-3A\rangle: \begin{pmatrix} -2A & 0 & 0 & 0 \\ 0 & 2A & 2A & 0 \\ 0 & 2A & 2A & 0 \\ 0 & 0 & 0 & -2A \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|-3A\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|-+\rangle - |+-\rangle\right)$$

b. Find the ground state energy for $B \ll A$, to all orders in A and first order in B

$E^0 = E^0 + \langle\psi^0|V|\psi^0\rangle$

$V = B(\sigma_1^z + \sigma_2^z)$

$$\langle -|\sigma_1^z|-\rangle = (0\ -1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = (0\ -1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1$$

$\langle +|\sigma_2^z|+\rangle = 1$

What we find is that $E_0 = -3A$. There is no B dependance.

14*. Consider a system for which the operator $\hat{A}$ corresponding to the physical quantity A does not commute with the Hamiltonian $\hat{H}$. The operator $\hat{A}$ has eigenvalues a_1 and a_2 and corresponding eigenfunctions ϕ_1 and ϕ_2 with

$$\phi_1 = \frac{u_1 + u_2}{\sqrt{2}}, \quad \phi_2 = \frac{u_1 - u_2}{\sqrt{2}}$$

where u_1 and u_2 are the eigenfunctions of $\hat{H}$ with energies E_1 and E_2. If the system is in the state $|\psi\rangle = |\phi_1\rangle$ at $t = 0$, calculate the expectation value of $\hat{A}$ for all $t > 0$ in terms of a_1, a_2, E_1, E_2, and t.

Let's start by finding the state at any time t.

$$|\psi(t)\rangle = \mathcal{U}|\psi(0)\rangle$$
$$= \exp(-iHt/\hbar)\left(\frac{u_1+u_2}{\sqrt{2}}\right)$$
$$= \frac{\exp(-iE_1t/\hbar)|u_1\rangle + \exp(-iE_2t/\hbar)|u_2\rangle}{\sqrt{2}}$$

$$\sqrt{2}\,\phi_1 = u_1 + u_2$$
$$\sqrt{2}\,\phi_2 = u_1 - u_2$$
$$2u_1 = \sqrt{2}(\phi_1+\phi_2)$$
$$u_1 = \phi_1+\phi_2/\sqrt{2}$$
$$u_2 = \phi_1-\phi_2/\sqrt{2}$$

$$\langle A\rangle = \langle\psi(t)|A|\psi(t)\rangle$$
$$= \frac{1}{2}\left(\exp(iE_1t/\hbar)\langle u_1| + \exp(iE_2t/\hbar)\langle u_2|\right)$$
$$\cdot A\left(\exp(-iE_1t/\hbar)\cdot\frac{|\phi_1\rangle+|\phi_2\rangle}{\sqrt{2}} + \exp(-iE_2t/\hbar)\cdot\frac{|\phi_1\rangle-|\phi_2\rangle}{\sqrt{2}}\right)$$
$$= \tfrac{1}{4}\left(\exp(iE_1t/\hbar)(\langle\phi_1|+\langle\phi_2|) + \exp(iE_2t/\hbar)(\langle\phi_1|-\langle\phi_2|)\right)$$
$$\cdot A\left(\exp(-iE_1t/\hbar)(|\phi_1\rangle+|\phi_2\rangle) + \exp(-iE_2t/\hbar)(|\phi_1\rangle-|\phi_2\rangle)\right)$$
$$= \tfrac{1}{4}\left[(\exp(iE_1t/\hbar)+\exp(iE_2t/\hbar))\langle\phi_1| + (\exp(iE_1t/\hbar)-\exp(iE_2t/\hbar))\langle\phi_2|\right]$$
$$\cdot\left[a_1(\exp(-iE_1t/\hbar)+\exp(-iE_2t/\hbar))|\phi_1\rangle + a_2(\exp(-iE_1t/\hbar)-\exp(-iE_2t/\hbar))|\phi_2\rangle\right]$$
$$= \tfrac{1}{4}\left[a_1\left(2+\exp(i(E_2-E_1)t/\hbar)+\exp(i(E_1-E_2)t/\hbar)\right) + a_2\left(2-\exp(i(E_2-E_1)t/\hbar)-\exp(i(E_1-E_2)t/\hbar)\right)\right]$$
$$= \frac{a_1+a_2}{2} + \frac{a_1}{4}\left[\cos((E_2-E_1)t/\hbar)+\cos((E_1-E_2)t/\hbar)\right] - \frac{a_2}{4}\cdot 2\cos((E_2-E_1)t/\hbar)$$
$$= \frac{a_1+a_2}{2} + \frac{a_1-a_2}{2}\cdot\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)$$

~~15.~~ A beam of charged pions (π^+) of energy 10 GeV is produced in the lab. What is the maximum energy of muon neutrinos (ν_μ) that can be produced through the decay $\pi^+\to\mu^+ + \nu_\mu$? Assume the neutrinos are massless, and use $m_\pi = 0.1396\ \mathrm{GeV}/c^2$ and $m_\mu = 0.1057\ \mathrm{GeV}/c^2$.

- $p_\pi = (E_\pi, \vec{p}_\pi)$ $\quad p_\pi^2 = E_\pi^2 - \vec{p}_\pi^{\,2} = m_\pi^2$
  $p_\mu = (E_\mu, \vec{p}_\mu)$ $\quad p_{\mu}^2 = m_{\mu}^2$
  $p_\nu = (E_\nu, \vec{p}_\nu)$ $\quad p_\nu^2 = m_\nu^2 = 0$

From conservation of momentum, $p_\pi = p_\mu + p_\nu$
$$p_\nu = p_\pi - p_\mu$$
$$p_\nu^2 = p_\pi^2 - 2p_\pi\cdot p_\mu + p_\mu^2$$
$$0 = m_\pi^2 + m_\mu^2 - 2(E_\pi E_\mu + \vec{p}_\pi\cdot\vec{p}_\mu)$$
$$= m_\pi^2 + m_\mu^2 - 2E_\pi E_\mu - 2\sqrt{E_\pi^2-m_\pi^2}\sqrt{E_\mu^2-m_\mu^2}\cdot\cos\theta$$

We can convince ourselves that the maximum E_μ occurs when $\theta = 0°$

$$m_\pi^2 + m_\mu^2 - 2E_\pi E_\mu = 2(E_\pi^2-m_\pi^2)^{1/2}(E_\mu^2-m_\mu^2)^{1/2}$$
$$(m_\pi^2+m_\mu^2-2E_\pi E_\mu)^2 = 4(\cancel{E_\pi^2E_\mu^2} - E_\pi^2m_\mu^2 - E_\mu^2m_\pi^2 + m_\pi^2m_\mu^2)$$
$$(m_\pi^2+m_\mu^2)^2 - 4E_\pi E_\mu(m_\pi^2+m_\mu^2) + \cancel{4E_\pi^2E_\mu^2} = \sim$$
$$4m_\pi^2E_\mu^2 - 4E_\pi(m_\pi^2+m_\mu^2)E_\mu + (m_\pi^2+m_\mu^2)^2 - 4m_\pi^2m_\mu^2 + 4E_\pi^2m_\mu^2 = 0$$
$$4m_\pi^2E_\mu^2 - 4E_\pi(m_\pi^2+m_\mu^2)E_\mu + (m_\pi^2-m_\mu^2)^2 + 4E_\pi^2m_\mu^2 = 0$$
$$E_\mu = \frac{4E_\pi(m_\pi^2+m_\mu^2) \pm \sqrt{16(m_\pi^2+m_\mu^2)^2E_\pi^2 - 4\cdot 4(m_\pi^2)((m_\pi^2-m_\mu^2)^2+4E_\pi^2m_\mu^2)}}{8m_\pi^2}$$
$$= \frac{4E_\pi(m_\pi^2+m_\mu^2) \pm \sqrt{16(m_\pi^2-m_\mu^2)^2E_\pi^2 - 16m_\pi^2(m_\pi^2-m_\mu^2)^2}}{8m_\pi^2}$$
$$= \frac{4E_\pi(m_\pi^2+m_\mu^2) \pm 4(m_\pi^2-m_\mu^2)\sqrt{E_\pi^2-m_\pi^2}}{8m_\pi^2}$$
$$= \frac{E_\pi(m_\pi^2+m_\mu^2) + (m_\pi^2-m_\mu^2)\sqrt{E_\pi^2-m_\pi^2}}{2m_\pi^2}$$

with unit $c=1$

(note, plugging into a calculator gives around 9.8 GeV).

16*. The Trinity test in New Mexico in 1945 was the first explosion of the atomic bomb. From a series of pictures of the test published in popular news a few years later, British physicist G.I. Taylor was able to use the time stamps and length scales to estimate the energy released by the explosion, a number which was classified at the time.
Above is one of these images. The energy of the blast (or shock) wave constitutes approximately 50% of the energy of the nuclear explosion (the remainder is in the form of thermal energy and nuclear radiation energy). The shock wave is characterized by an abrupt change in the density of air, which under normal circumstances is approximately $1.2\ ^{kg}/_{m^3}$. It is reasonable to assume that the radius of the shock wave depends on the energy of the blast and the density of air into which it is expanding. Using this image, with time since detonation and distance scale bar as shown, estimate the blast energy in TNT equivalent. Note that 1 kg of TNT = $4.\times10^6$ J.

The time scale is 0.006 s and the radius is between 50 and 100 m. Using dimensional analysis, $[E] = ^{kg\,m^2}/_{s^2}$

$$E \propto \rho r^5/t^2$$

$$= \frac{1.2\,kg}{m^3}\cdot\frac{(100)^5\,m^5}{(6\times10^{-3})^2\,s^2}\cdot\frac{1\,kg\,TNT}{4\times10^6\,J}\cdot\frac{1\ ton\ TNT}{10^3\,kg\,TNT}$$

$$= \frac{1.2\times10^{\cancel{10}\,7}}{36\times\cancel{10^{-6}}\cdot4\times\cancel{10^6}\cdot\cancel{10^3}}\ ton\ TNT$$

$$= \frac{10^6}{12} \simeq 10^5\ ton\ TNT$$

Since this only accounts for half the energy, the total blast energy is about 20 kTon of TNT.

Which matches with what is expected. Remember a speech by President Truman in which he stated an American airplane had dropped a bomb on Hiroshima. A bomb that had more than twenty-thousand tons of TNT.