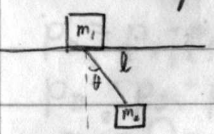
I wanted you to see what real real courage is, instead of getting the idea that courage is a man with a gun in his hand. It's when you know you're licked before you begin but you begin anyway and you see it through no matter what. You rarely win, but sometimes you do.

2014-2015 — Atticus Finch (Harper Lee, To Kill a Mockingbird)

1* A pendulum of mass m_2 and length l is constrained to move in a single plane. The point of support is attached to a mass m_1 which can freely move with no friction on a horizontal line in the same plane, as shown in the figure below. In what follows assume the masses shown as blocks in the figure may be treated as point masses.



a. Find the Lagrangian of the system in terms of suitable generalized coordinates.

x_1 and $y_1 = 0$ for m_1 are not dependant on anything else, so let's look at the motion of m_2

$$x_2 = x_1 + l\sin\theta \qquad \dot{x}_2 = \dot{x}_1 + l\dot{\theta}\cos\theta$$
$$y_2 = -l\cos\theta \qquad \dot{y}_2 = l\dot{\theta}\sin\theta$$

So our generalized coordinates are x and θ.

$$\mathcal{L} = \tfrac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) - m g y_2$$
$$= \tfrac{m_1}{2}(\dot{x}^2) + \tfrac{m_2}{2}(\dot{x}^2 + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2) + m_2 g l\cos\theta$$

b. $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\tfrac{d}{dt}(m_1\dot{x} + m_2\dot{x} + m_2 l\dot{\theta}\cos\theta) = 0$$
$$(m_1+m_2)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\tfrac{d}{dt}(m_2\dot{x}l\cos\theta + m_2 l^2\dot{\theta}) - (-m_2\dot{x}l\dot{\theta}\sin\theta - mgl\sin\theta) = 0$$
$$m_2\ddot{x}l\cos\theta - m_2\dot{x}l\dot{\theta}\sin\theta + m_2 l^2\ddot{\theta} + m_2\dot{x}l\dot{\theta}\sin\theta + m_2 g l\sin\theta = 0$$
$$\ddot{x}\cos\theta + l\ddot{\theta} + g\sin\theta = 0$$

c. Going back to Lagrangian, let $\vec{x} = \begin{pmatrix} x \\ \theta \end{pmatrix}$

$$T = \begin{pmatrix} m_1+m_2 & m_2 l \\ m_2 l & m_2 l^2 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & 0 \\ 0 & +m_2 g l \end{pmatrix}$$

$$\mathcal{L} = \tfrac{1}{2}\left[(m_1+m_2)\dot{x}^2 + 2m_2 l\dot{x}\dot{\theta} + m_2 l^2\dot{\theta}^2\right] - \tfrac{1}{2} m_2 g l\theta^2$$
$$= \tfrac{1}{2}\dot{\vec{x}}\,T\,\dot{\vec{x}} - \tfrac{1}{2}\vec{x}\,V\,\vec{x}$$

$$\det(V - \lambda T) = \begin{pmatrix} -(m_1+m_2)\lambda & -m_2 l\lambda \\ -m_2 l\lambda & +m_2 g l - m_2 l^2\lambda \end{pmatrix} = 0$$
$$= (m_1+m_2)\lambda(-m_2 g l + m_2 l^2\lambda) - m_2^2 l^2\lambda^2 = 0$$
$$= m_2 l\lambda[(m_1+m_2)(-g + l\lambda) - m_2 l\lambda] = 0$$
$$= m_2 l\lambda[-(m_1+m_2)g + m_1 l\lambda] = 0$$
$$\lambda = 0,\ {}^{(m_1+m_2)g}/_{m_1 l}$$
$$\omega^2 = \lambda$$

$$|0\rangle: \begin{pmatrix} 0 & 0 \\ 0 & m_2 g l \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left|\sqrt{{}^{(m_1+m_2)g}/_{m_1 l}}\right\rangle: \begin{pmatrix} -(m_1+m_2)\cdot{}^{(m_1+m_2)g}/_{m_1 l} & m_2 l\cdot{}^{(m_1+m_2)g}/_{m_1 l} \\ m_2 l\cdot{}^{(m_1+m_2)g}/_{m_1 l} & -m_2 g l + m_2 l^2\cdot{}^{(m_1+m_2)g}/_{m_1 l} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\frac{m_1+m_2}{l}a + m_2 b = 0$$

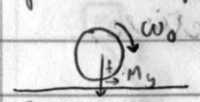
$$\left|\sqrt{{}^{(m_1+m_2)g}/_{m_1 l}}\right\rangle = \begin{pmatrix} -m_2 \\ {}^{m_1+m_2}/_{l} \end{pmatrix}$$

d. Find the frequency of small oscillations of the system.

$$\therefore\ \ddot{x} = -l\ddot{\theta} - g\sin\theta$$
$$(m_1+m_2)(-l\ddot{\theta} - g\sin\theta) + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta = 0$$
$$-m_1 l\ddot{\varepsilon} - m_1 g\varepsilon - m_2 g\varepsilon - m_2 l\dot{\theta}^2\varepsilon = 0$$
$$\ddot{\varepsilon} = -\frac{(m_1 g + m_2 g + m_2 l\dot{\theta}^2)}{m_1 l}\varepsilon$$

$$\omega = \sqrt{\frac{(m_1 g + m_2 g + m_2 l\dot{\theta}^2)}{m_1 l}} = \sqrt{\frac{(m_1 g + m_2 g)}{m_1 l}}$$

2*. A rigid, axially symmetric wheel has mass M, radius a, and moment of inertia I about its axis. The wheel is spun about this axis with constant angular speed ω_0, and is then released in an upright position on a horizontal plane. The wheel slips for a time τ and then rolls without slipping. The coefficient of kinetic fiction between the wheel and the plane is μ



a. Write down the equations of motion for the system.

- We need to break the problem into slipping and not slipping portions. We'll start with slipping

$$m\ddot{x} = mg\,\mu = I\ddot{\theta}$$

$$\ddot{\theta} = \frac{mg\mu}{I}$$

When not slipping

$$\mathcal{L} = \tfrac{1}{2}\cdot m\dot{x}^2 + \tfrac{1}{2} I\dot{\theta}^2$$

$$m\ddot{x} = 0$$

$$I\ddot{\theta} = 0$$

b. Find the time τ.

- $$\dot{\theta} = \frac{Mg\mu\, t}{I} + \omega_0$$

$$\ddot{x} = \mu g$$

$$\dot{x} = \mu g t = a\dot{\theta}$$

$$= \frac{Mg\mu t}{I} + \omega_0$$

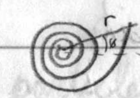
$$g\mu t(1 - M/I) = \omega_0$$

$$\tau = \frac{\omega_0}{g\mu(1 - M/I)}$$

c. Determine the center of mass speed of the rolling wheel for $t \ge \tau$.

- $$\dot{x} = \mu g t = \mu g \tau = \frac{\omega_0}{1 - M/I}$$

3. Find the force law for a central force that allows a particle to move in a logarithmic spiral orbit, given by $r = k\cdot \exp(\alpha\phi)$, where r and ϕ are the polar coordinates for the particle, and k and α are constants.



- $$m\ddot{r} - \ell^2/mr^3 = f(r)$$

$$m\ddot{r} - \cancel{m^2}\cancel{r^4}\dot{\theta}^2/\cancel{mr^3} = f(r)$$

$$m\ddot{r} - mr\dot{\theta}^2 = f(r)$$

$$\frac{d^2u}{d\phi^2} + u = \frac{-m}{\ell^2 u^2} f(u)$$

$$u = 1/r = 1/k \cdot \exp(-\alpha\phi)$$

$$\frac{du}{d\phi} = -\alpha/k \cdot \exp(-\alpha\phi)$$

$$\frac{d^2u}{d\phi^2} = \alpha^2/k \cdot \exp(-\alpha\phi)$$

$$\frac{\alpha^2 \exp(-\alpha\phi)}{k} + \frac{\exp(-\alpha\phi)}{k} = \frac{-mk^2 \cancel{\exp(-2\alpha\phi)}}{\ell^2 \exp(-2\alpha\phi)} f(u)$$

$$(1+\alpha^2)u = \frac{-m}{\ell^2 u^2}\cdot f(u)$$

$$f = -\frac{\ell^2(1+\alpha^2)u^3}{m}$$

- $$= \frac{-\ell^2(1+\alpha^2)\cancel{\exp(-3\alpha\phi)}}{mk^3\exp(-3\alpha\phi)} = \frac{-\ell^2(1+\alpha^2)}{mr^3}$$

4. A localized distribution of charge has a charge density

$$\rho(\vec{r}) = \left(\frac{e}{32\pi a_0^3}\right)\left(\frac{r}{a_0}\right)^2 \exp(-r/a_0)\cos^2\theta$$

with e the electron charge and a_0 the Bohr radius,

$a_0 = \hbar^2/me^2 = 0.527\times10^{-10}$ m

a. Make a multipole expansion of the potential due to this charge density and determine all the non-vanishing multipole moments.

We want m = 0, and we can see that our charge density will depend on $P_0(\cos\theta)$ and $P_2(\cos\theta)$.

$P_0(\cos\theta) = \cos\theta$

$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$

$\cos^2\theta = \frac{2}{3}\cdot P_2(\cos\theta) + \frac{1}{3}\cdot P_0(\cos\theta)$

$$\rho(\vec{r}) = \left(\frac{e}{32\pi a_0^3}\right)\left(\frac{r}{a_0}\right)^2 \exp(-r/a_0)\left(\tfrac{2}{3}\cdot P_2(\cos\theta) + \tfrac{1}{3}\cdot P_0(\cos\theta)\right)$$

$$q_{\ell m} = \int \sqrt{\tfrac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta')\, r'^\ell \rho(\vec{r}')\, d^3x'$$

$$q_{00} = \sqrt{\frac{1}{4\pi}} \int_0^{\infty}\int_{-1}^{1}\int_0^{2\pi} P_0(\cos\theta') \left(\frac{e}{32\pi a_0^3}\right)\left(\frac{r'}{a_0}\right)^2 \exp(-r'/a_0)\left(\tfrac{1}{3}P_0(\cos\theta')\right) r'^2\, dr'\, d(\cos\theta')\, d\phi'$$

$$= \sqrt{\frac{1}{4\pi}}\cdot\frac{2}{3}\cdot\frac{e\cdot 2\pi}{32\pi a_0^3\, a_0^2}\int_0^{\infty} r^4 \exp(-r/a_0)\, dr$$

$$= \frac{e}{\sqrt{4\pi}\cdot 24 a_0^5}\left(4!\,a_0^5\right) = \frac{e}{\sqrt{4\pi}}$$

$$q_{20} = \sqrt{\frac{5}{4\pi}}\cdot\frac{2\pi e}{32\pi a_0^5}\cdot\frac{2}{3}\cdot\frac{2}{5}\int_0^{\infty} r^6 \exp(-r/a_0)\, dr$$

$$= \sqrt{\frac{5}{4\pi}}\ \frac{e}{60 a_0^5}\cdot 6!\,a_0^7 = 12 e\, a_0^2 \sqrt{\frac{5}{4\pi}}$$

b. Write down an expression for the potential at large distances

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_\ell \frac{1}{2\ell+1}\cdot\frac{q_{\ell m}}{r^{\ell+1}}\, Y_{\ell m}(\theta,\phi)$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_{00}\, P_0(\cos\theta)}{\sqrt{4\pi}\, r} + \sqrt{\frac{5}{4\pi}}\ \frac{q_{20}\, P_2(\cos\theta)}{5\cdot r^3}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{e\cdot \cos\theta}{4\pi r} + \frac{6 e a_0^2 (3\cos^2\theta - 1)}{4\pi r^3}\right]$$

5*. A particle of charge q and mass m moves in static, uniform electric and magnetic fields $\vec{E}$ and $\vec{B}$, with $\vec{E}$ in the $\hat{x}$ direction and $\vec{B}$ in the $\hat{y}$ direction. The fields satisfy $|\vec{E}| < |\vec{B}|$ in cgs units (in MKSA units, $|\vec{E}| < c|\vec{B}|$).

a. Find the equations of motion that determine the particle trajectory in these fields.

For a non-relativistic particle, $\vec{F} = q\vec{E} + \vec{v}\times\vec{B}$.

For a relativistic particle: $\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{u}}{c}\vec{B}\right)$

$$\frac{dE}{dt} = q\vec{u}\cdot\vec{E}$$

b. Find the most general solution to these equations, and explain what minimal set of parameters beyond those above would need to be provided to fully specify the trajectory.

$m\ddot{\vec{x}} = qE\hat{x} + \dot{\vec{x}}\times B\hat{y}$

$$\vec{v}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & B & 0 \end{vmatrix} = (-B\dot{z}, -(0), \dot{x}B)$$

$$m\ddot{x} = qE - qB\dot{z}$$
$$m\ddot{y} = 0 \qquad y = v_y t + y_0$$
$$m\ddot{z} = qB\dot{x}$$
$$m\dot{z} = qBx + z_0$$
$$m\ddot{x} = qE - qB\left(\tfrac{qB}{m}x + z_0/m\right)$$
$$\ddot{x} = q/m\, E - qBz_0/m - q^2B^2/m^2\, x$$
$$x(t) = A\cos\left(qB/m\, t - (q/m\, E - qBz_0/m)\right)$$
$$\frac{m\,dz}{dt} = qBx(t) + z_0$$
$$mz(t) = qB\cdot \sin\left(qBt/m - (qE/m - qBz_0/m)\right)\cdot m/qB + z_0 t + z_0'$$

$$x(t) = \cos\left(qBt/m - (qE/m - qBv_z/m)\right) + x_0$$
$$y(t) = v_y t + y_0$$
$$z(t) = \sin\left(qBt/m - (qE/m - qBv_z/m)\right) + v_z t + z_0$$

We'll also need the initial position and velocity of the particle. Although, we could probably set the initial position to the origin.

6*. A monatomic ideal gas at an initial temperature T_0 expands from V_0 to $2V_0$. Calculate the work done per mole by the gas and the heat absorbed by it, assuming.

a. the gas expands at constant temperature T, or

$$PV = nk_BT = RT$$
$$W = \int_{V_0}^{2V_0} P\,dV = RT\int_{V_0}^{2V_0}\frac{dV}{V} = RT_0 \ln 2$$
$$C_V = \frac{\partial U}{\partial T}$$
$$\partial U = C_V\,dT = 0 \Rightarrow Q = \Delta U + W = RT_0 \ln 2$$

b. The gas expands at constant pressure P

$$W = \int_{V_0}^{2V_0} P\,dV = PV_0 = RT_0$$
$$C_V = 3/2 \cdot RT$$
$$\partial U = 3/2\, RT\,dT = 3/2\, P\,dV = 3/2\, PV_0 = 3/2\, RT_0$$
$$Q = 5/2\, RT_0$$

7. A three-dimensional harmonic oscillator has energy levels

$$E(n_1, n_2, n_3) = (n_1 + n_2 + n_3 + 3/2)\hbar\omega$$

with $n_1, n_2, n_3 = 0, 1, 2, \ldots$

a. What is the average of the three dimensional harmonic oscillator when it is at temperature T.

$$Z = \sum_{n=0} \exp(-\beta n\hbar\omega) = \frac{1}{1-\exp(-\beta\hbar\omega)}$$

For a single oscillator, $Z = \dfrac{\exp(-\beta\hbar\omega/2)}{1-\exp(-\beta\hbar\omega)}$

$$\sinh(x) = \frac{\exp(2x)-1}{2\exp(x)}$$
$$Z_1 = \frac{1}{2\sinh(\beta\hbar\omega/2)} \quad \cancel{\exp\left(\tfrac{-3\beta\hbar\omega}{2}\right)}$$
$$\Rightarrow Z = \frac{1}{8\sinh^3(\beta\hbar\omega/2)} = 1/8\,\sinh^{-3}(\beta\hbar\omega/2)\ \cancel{\exp\left(\tfrac{-3\beta\hbar\omega}{2}\right)}$$

$$\langle E\rangle = -\frac{1}{Z}\cdot\frac{\partial Z}{\partial \beta}$$

$$\frac{\partial Z}{\partial \beta} = \frac{-3}{8}\cdot\frac{\cosh(\beta\hbar\omega/2)}{\sinh^4(\beta\hbar\omega/2)}\cdot\frac{\hbar\omega}{2}$$

c. $\langle E\rangle = \frac{3}{2}\hbar\omega\coth(\beta\hbar\omega/2)$

b. What is the specific heat of the three dimensional harmonic oscillator at temperature T.

· $C_V = \frac{\partial U}{\partial T}$

$$= \frac{\partial}{\partial T}\left(\frac{3\hbar\omega}{2}\cdot\coth\left(\hbar\omega/2k_BT\right)\right) = \frac{3\hbar\omega}{2}\cdot -\mathrm{csch}^2\left(\hbar\omega/2k_BT\right)\cdot\frac{-\hbar\omega}{2k_BT^2}$$

$$= \frac{3}{4}(\beta\hbar\omega)^2\left(\frac{2}{\exp(\beta\hbar\omega/2)-\exp(-\beta\hbar\omega/2)}\right)^2$$

$$= \frac{3(\beta\hbar\omega)^2}{(\exp(\beta\hbar\omega/2)-\exp(-\beta\hbar\omega/2))^2}$$

c. Find expressions for the specific heat that are appropriate for for very high temperature ($k_BT >> \hbar\omega$) and for very low temperature ($k_BT << \hbar\omega$).

· At high temperature, $\beta << \hbar\omega$

$$\frac{3(\beta\hbar\omega)^2\exp(\beta\hbar\omega/2)}{(\exp(\beta\hbar\omega)-1)^2} = \frac{3(\beta\hbar\omega)^2(1+\beta\hbar\omega)}{(1-1-(\beta\hbar\omega))^2} \approx \frac{3\hbar}{2}\cdot\frac{k_B}{T}$$

· At low temperature, $\beta >> \hbar\omega$

$$\frac{3(\beta\hbar\omega)^2}{\exp(\beta\hbar\omega)}$$

8. A classical gas of N non-interacting particles of mass m in equilibrium is confined to a container of cross-sectional area A and height h. Gravity acts on the particles and presents a potential $V(z)=mgz$

a. Evaluate the partition function Z and the Helmholtz free energy of this system.

· The total energy of a particle, $E = \vec{p}^2/2m + mgz$

$$Z_C = \frac{1}{N!}\left[\frac{A}{h^3}\int_0^h \exp(-\beta p^2/2m)\exp(-\beta mgz)\,dz\,d^3p\right]^N$$

$$= \frac{1}{N!}\left[\frac{A}{h^3}\cdot\frac{\exp(-\beta mgz)}{-\beta mg}\Big|_0^h\left(\int_{-\infty}^{\infty}\exp\left(-\frac{\beta p^2}{2m}\right)dp\right)^3\right]^N$$

$$= \frac{1}{N!}\left[\frac{A(1-\exp(-\beta mgh))}{\beta mgh^3}\cdot\left(\sqrt{\frac{2\pi m}{\beta}}\right)^3\right]^N \qquad \lambda = \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{1/2}$$

$$= \frac{1}{N!}\left[\frac{A(1-\exp(-\beta mgh))}{\beta mg\lambda^3}\right]^N$$

$$A = -kT\ln Z$$

$$= -kT\left[-\ln N! + N\ln(A(1-\exp(-\beta mgh))) - N\ln(\beta mg\lambda^3)\right]$$

b. $\langle \bar{z}\rangle = \langle E\rangle = -\frac{1}{Z}\cdot\frac{\partial Z}{\partial\beta}$

$$= -\cancel{N!}\left[\frac{\cancel{\beta mg\lambda^3}}{A(1-\exp(-\beta mgh))}\right]^N\cdot\frac{N}{\cancel{N!}}\left[\frac{A(1-\exp(-\beta mgh))}{\cancel{\beta mg\lambda^3}}\right]^{N-1}$$

$$\cdot\frac{Amgh\exp(-\beta mgh)\cdot\beta mg\lambda^3 - mg\lambda^3A(1-\exp(-\beta mgh))}{(\beta mg\lambda^3)^2}$$

$$= \frac{N(1-\exp(-\beta mgh)-\beta mgh\exp(-\beta mgh))}{\beta(1-\exp(-\beta mgh))}$$

c. If $h = 10\,m$ and the particles are uranium atoms, at what temperature does $mgh/k_B T = 1$? Express your answer in Kelvins.

$T = mgh/k_B$

$m \approx 238 \cdot m_p = 2\times10^2 \cdot 2\times10^{-27}\,kg$

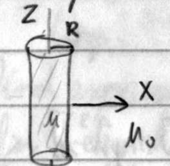
$g = 10\,m/s^2$

$h = 10\,m$

$k_B = 1\times10^{-23}\,J/K$

$$T = \frac{4\times10^{-25}\,kg \cdot 10\,m/s^2 \cdot 10\,m}{10^{-23}\,J/K} \approx 4\,K$$

9*. Consider a space in which there is an external magnetic field of the form $\vec{B}_0 = B_0\hat{x}$. An infinitely long cylinder with a circular cross-section of radius R and magnetic permeability μ is placed in the space, with axis along the $\hat{z}$ direction.



a. Find the total magnetic field $\vec{B}$ everywhere in space (both inside and outside the cylinder.

$\vec{B} = \mu_0\vec{H}$ $\qquad \vec{H} = -\nabla\Phi = -\left(\frac{\partial\Phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\cdot\frac{\partial\Phi}{\partial\phi}\hat{\phi}\right)$

$\vec{H} = B_0/\mu_0\,\hat{x}$

Inside: $\Phi_{in}(\rho,\phi) = A\rho\cos\phi$

Outside: $\Phi_{out}(\rho,\phi) = \frac{\beta}{\rho}\cos\phi - \frac{B_0}{\mu_0}\rho\cos\phi$

$$\frac{\partial\Phi_{in}}{\partial\phi} = \frac{\partial\Phi_{out}}{\partial\phi}\bigg|_{\rho=R}$$

$$\mu\frac{\partial\Phi_{in}}{\partial\rho} = \mu_0\frac{\partial\Phi_{out}}{\partial\rho}\bigg|_{\rho=R}$$

$-AR\sin\phi = -\frac{\beta}{R}\sin\phi + \frac{B_0 R}{\mu_0}\cdot\rho\sin\phi$

$\mu(A\cos\phi) = \mu_0\left(-\beta/R^2\cdot\cos\phi - \frac{B_0}{\mu_0}\cos\phi\right)$

$-AR^2 = -\beta + H_0R^2 \qquad AR^2 = \beta - H_0R^2$

$\mu AR^2 = -\mu_0\beta - \mu_0 H_0 R^2$

$\mu(\beta - H_0R^2) = -\mu_0\beta - \mu_0H_0R^2$

$\beta(\mu+\mu_0) = (\mu-\mu_0)H_0R^2$

$$\beta = \frac{(\mu-\mu_0)}{\mu+\mu_0}H_0R^2$$

$$AR^2 = \frac{(\mu-\mu_0)}{\mu+\mu_0}H_0R^2 - H_0R^2 = H_0R^2\left[\frac{\mu-\mu_0-\mu-\mu_0}{\mu+\mu_0}\right]$$

$$A = \frac{-2H_0\mu_0}{\mu+\mu_0}$$

$$\Phi_{in} = \frac{-2H_0\mu_0}{\mu+\mu_0}\rho\cos\phi$$

$$\vec{H}_{in} = \frac{-2B_0}{\mu+\mu_0}\left(\cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}\right)$$

$$\Phi_{out} = \frac{(\mu-\mu_0)B_0R^2}{\mu_0(\mu+\mu_0)\rho}\cos\phi - H_0\rho\cos\phi$$

$$\vec{H}_{out} = \left[\frac{(\mu-\mu_0)B_0R^2\cos\phi}{\mu_0(\mu+\mu_0)\rho^2} - H_0\cos\phi\right]\hat{\rho} + \left[\frac{(\mu-\mu_0)B_0R^2\sin\phi}{\mu_0(\mu+\mu_0)\rho^2} - H_0\sin\phi\right]\hat{\phi}$$

b. Suppose a uniform current density $\vec{J} = J\hat{z}$ flows through the cylinder. Find the net force per unit length $\vec{f}$ on the cylinder due to this cylinder.

· $\vec{F} = q\vec{v} \times \vec{B}$

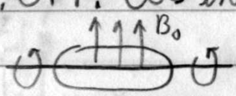
$= I\vec{l} \times \vec{B}$

$= \vec{J}A l \times \vec{B}$

$= J\pi R^2 l \hat{z} \times \vec{B}$

$$\vec{f} = \frac{J\pi R^2 \cdot 2B_0}{\mu + \mu_0}\left[\sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}\right]$$

10. A thin copper wire of cross-sectional area A is bent into ring of radius a, and rotates around an axis perpendicular to a static, uniform magnetic field B_0 of magnitude 0.01T. Its initial frequency of rotation is ω_0



a. Find an expression for the voltage in the ring as a function of time

· $\varepsilon = \frac{\partial \Phi}{\partial t}$

$\Phi = \pi a^2 B_0 \cos(\omega_0 t)$

$\varepsilon = \pi a^2 \omega_0 B_0 \sin(\omega_0 t)$

b. Calculate the average resistive energy loss per rotation due to the induced current in the ring, assuming that the conductivity of Cu in SI units is $\sigma = 6 \times 10^7$ S/m

· $\vec{\nabla} \times \vec{B} = \mu_0 I$

$P = E/t = IV$ $\qquad V = IR$

$= V^2/R$ $\qquad R = \sigma L/A$

$$P = \frac{\pi^2 a^4 \omega_0^2 B_0^2 \sin^2(\omega_0 t)}{\sigma^2 \cdot 4\pi^2 a^2} \cdot A$$

$$= \frac{a^2 \omega_0^2 B_0^2 A \sin^2(\omega_0 t)}{4\sigma^2}$$

c. $E_{lost} = \frac{a^2 \omega_0 B_0^2 A}{8\sigma^2}$

c. Calculate the time τ it takes for the rings rotational frequency to decrease to 1/e of its original value, assuming that all the energy goes into Joule heating and that the density of Cu is $\rho = 8920\ \mathrm{kg/m^3}$.

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11. Suppose a neutron scatters from a heavy spin-½ nucleus with the interaction potential

$$V(\vec{r}) = A\delta(\vec{r}) + B\delta(\vec{r})\vec{S}_1\cdot\vec{S}_2$$

where $\vec{S}_1$ is the neutron spin and $\vec{S}_2$ is the nuclear spin. $\vec{r}$ is the vector pointing from the heavy nucleus to the neutron. In this problem we will ignore the recoil of the nucleus.

a. Suppose the two particles are in a spin-singlet state. What is the differential cross section for scattering in the Born approximation.

· $$f = \frac{-2m}{\hbar^2}\cdot\frac{1}{4\pi}\int V(\vec{r})\exp\left(i(\vec{k}-\vec{k}')\vec{r}\right)dr$$

$$= \frac{-m}{2\pi\hbar^2}\left[A + B\vec{S}_1\cdot\vec{S}_2\right]$$

$$\langle f\rangle = \frac{-m}{2\pi\hbar^2}\left[A + B\langle j|\vec{S}_1\cdot\vec{S}_2|j\rangle\right]$$

$$\vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1\cdot\vec{S}_2 \qquad \vec{S}_1\cdot\vec{S}_2 = \frac{S^2 - S_1^2 - S_2^2}{2}$$

$$\vec{S}_1\cdot\vec{S}_2 = \vec{S}^2 - \vec{S}_1^2 - \vec{S}_2$$

$\vec{S}^2 = 0$ since singlet

$$\vec{S}_1^2|j\rangle = \vec{S}_2^2|j\rangle = \hbar^2 j(j+1)|j\rangle$$

$$\langle f\rangle = \frac{-m}{2\pi\hbar^2}\left[A + B\cdot\left(\frac{0 - \hbar^2\cdot 3/4 - \hbar^2\cdot 3/4}{2}\right)\right]$$

$$= \frac{-m}{2\pi\hbar^2}\left[A - \frac{3B\hbar^2}{4}\right]$$

$$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{m^2}{4\pi^2\hbar^4}\left(A - \frac{3B\hbar^2}{4}\right)^2$$

b. Suppose the two particles are in a spin triplet state. What is the differential cross section for scattering in the Born approximation?

· The only thing that changes is $S^2|j\rangle = \hbar^2$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left[A + \frac{B\hbar^2}{4}\right]^2$$

c. Suppose the particles are initially spin-unpolarized, each with equal probability to be up or down. What is the average differential cross-section for scattering in this situation?

Take the average of part a and b.

$$\cancel{\frac{d\sigma}{d\Omega} = \frac{1}{4}\cdot\frac{m^2}{4\pi^2\hbar^4}\left[\left(A - \frac{3B\hbar^2}{4}\right)^2 + \left(A + \frac{B\hbar^2}{4}\right)^2\right]}$$

$$\cancel{= \frac{m^2}{16\pi^2\hbar^4}\left(2A^2 - \frac{4AB\hbar^2}{4} + \frac{10B^2\hbar^4}{16}\right)}$$

$$\cancel{= \frac{m^2}{8\pi^2\hbar^4}\left(A^2 - \frac{AB\hbar^2}{2} + \frac{5B^2\hbar^4}{16}\right)}$$

Count triplet state three times

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\frac{d\sigma}{d\Omega}\bigg|_{s=0} + \frac{3}{4}\cdot\frac{d\sigma}{d\Omega}\bigg|_{s=1}$$

$$= \frac{m^2}{16\pi^2\hbar^4}\left(A - \frac{3B\hbar^2}{4}\right)^2 + \frac{3m^2}{16\pi^2\hbar^4}\left(A + \frac{B\hbar^2}{4}\right)^2$$

$$= \frac{m^2}{16\pi^2\hbar^4}\left(A^2 - \frac{6AB\hbar^2}{4} + \frac{9B^2\hbar^4}{16} + 3A^2 + \frac{6AB\hbar^2}{4} + \frac{3B^2\hbar^4}{16}\right)$$

$$= \frac{m^2}{4\pi^2\hbar^4}\left(A^2 + \frac{3B^2\hbar^4}{16}\right)$$

12*. Consider a spin-$\frac{1}{2}$ free particle with quantum states denoted by $|\vec{p}, \vec{s}\rangle$, with $\vec{p}$ the three-dimensional momentum and $\vec{s}$ the spin. The energy of the particle with momentum $\vec{p}$ in this situation is given by E_p (with $p = |\vec{p}|$), i.e. $\hat{H}_0 |\vec{p}, \vec{s}\rangle = E_p |\vec{p}, \vec{s}\rangle$

a. Suppose a potential of the form $\hat{H}_1 = \gamma \vec{B}\cdot\vec{s}$ is turned on, where γ is a constant and $\vec{B} = B\hat{z}$ is an external magnetic field along the $\hat{z}$ direction. Find the exact eigenvalues of the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$ in terms of E_p, and the corresponding eigenstates in terms of the state vector $|\vec{p}, \vec{s}\rangle$

- $\vec{B}\cdot\vec{s} = B s_z$

$\gamma B s_z |\vec{p}, \vec{s}\rangle = \gamma B \cdot \hbar/2 \, |\vec{p}, \vec{s}\rangle$

$\hat{H} |\vec{p}, \vec{s}\rangle = (E_p + \hbar\gamma B/2) |\vec{p}, \vec{s}\rangle$

$\hat{H} |\vec{p}, \vec{s}\rangle = (E_p - \hbar\gamma B/2) |\vec{p}, -\vec{s}\rangle$ assuming spin along z-axis

b. Suppose a different potential is turned on, with the form $\hat{H}_2 = \lambda \vec{p}\cdot\vec{s}$ where λ is a given constant. Find all the eigenvalues and the corresponding eigenstates of the new Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_2$. (Note $\hat{H}_1$ is not included in the Hamiltonian for this part).

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13*. Consider a two quantum bit system, whose state space is spanned by $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

A controlled not (cNOT) gate flips the value of the second bit ($0 \leftrightarrow 1$) if the first bit has the value 1, and leaves the second bit unchanged if the value of the first bit is 0.

a. Write down the 4x4 unitary matrix corresponding to the cNOT operation. (Make sure to specify the ordering of states you are using to represent the matrix).

- $$\begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{pmatrix} \rightarrow \begin{pmatrix} |00\rangle \\ |01\rangle \\ |11\rangle \\ |10\rangle \end{pmatrix}$$

b. Suppose the bits are initially in the state $|00\rangle$. The first (control bit) is acted upon by a Hadamard gate $[U_H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}]$, and then a cNOT gate acts upon both bits. What is the state of the system after this protocol?

Say $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$U_H |0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}[|1\rangle - |0\rangle]$$

$\text{cNOT}[\frac{1}{\sqrt{2}}(|10\rangle - |00\rangle)]$

$= \frac{1}{\sqrt{2}}(|11\rangle - |00\rangle)$

c. $|\langle 00 | \frac{1}{\sqrt{2}}(|11\rangle - |00\rangle)|^2 = \frac{1}{2}$

$|\langle 11 | \frac{1}{\sqrt{2}}(|11\rangle - |00\rangle)|^2 = \frac{1}{2}$

You will always find the first bit in either 0 or 1.

You have a 50% chance of finding the second state in 0 and a 50% chance of finding it in 1.

14. Consider a hydrogen atom with electron in a state of principal quantum number $n=2$. For this problem, we will not consider the electron spin, so there are four states at the same energy:

$$|n,l,m\rangle = |2,1,1\rangle,\ |2,1,0\rangle,\ |2,1,-1\rangle,\ |2,0,0\rangle$$

An electric field applied in the $\hat{z}$ direction introduces a perturbation $\Delta H = -e\mathcal{E}z$.

a. Use the symmetries of the wavefunctions to show that only two of the matrix elements $\langle 2,l_1,m_1|\Delta H|2,l_2,m_2\rangle$ are different than zero. Find the values of these matrix elements.

- This is the linear Stark effect. $m=0$ and $l=1$ and $l=0$, thus the only non-vanishing elements are

$$\langle 2,1,0|-e\mathcal{E}z|2,0,0\rangle$$
$$\langle 2,0,0|-e\mathcal{E}z|2,1,0\rangle$$

We only need to evaluate one element since they are degenerate.

$\langle 2,1,0|-e\mathcal{E}z|2,0,0\rangle$

$$= -e\mathcal{E}\int_0^{2\pi}\int_{-1}^{1}\int_0^{\infty}\left(\frac{1}{6a_0^3}\right)^{1/2}\cdot\frac{r\exp(-r/2a_0)}{2a_0}\cdot\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta\cdot r\cos\theta$$
$$\cdot\left(\frac{1}{2a_0^3}\right)^{1/2}\left(1-\frac{r}{2a_0}\right)\exp(-r/2a_0)\cdot\frac{1}{\sqrt{4\pi}}r^2\,dr\,d(\cos\theta)\,d\phi$$

$$= \frac{-e\mathcal{E}\cdot 2\pi}{2a_0^3\cdot 4\pi}\int_{-1}^{1}\int_0^{\infty} r^4\cos^2\theta\exp(-r/a_0)\left(1-\frac{r}{2a_0}\right)dr\,d(\cos\theta)$$

$$= \frac{-e\mathcal{E}}{4a_0^3}\int_0^{\infty} r^4\exp(-r/a_0)\left(1-\frac{r}{2a_0}\right)\cdot\left.\frac{\cos^3\theta}{3}\right|_{-1}^{1}dr$$

$$= \frac{-e\mathcal{E}}{6a_0^3}\left[\int_0^{\infty} r^4\exp(-r/a_0)\,dr - \int_0^{\infty}\frac{r^5}{2a_0}\exp(-r/a_0)\,dr\right]$$

$$= \frac{-e\mathcal{E}}{6a_0^3}\left[\frac{\exp(-r/a_0)}{-1/a_0}\left[\frac{(-1)^4 4!}{(-1/a_0)^4}\right] - \frac{\exp(-r/a_0)}{2a_0\cdot(-1/a_0)}\left(\frac{(-1)^5 5!}{(-1/a_0)^5}\right)\right]\Bigg|_0^{\infty}$$

$$= \frac{-e\mathcal{E}}{6a_0^3}\left[a_0\cdot 4!\,a_0^4 + \frac{5!(-1)\,a_0^5}{-2\cdot(-1)}\right]$$

$$= \frac{e\mathcal{E}}{6}a_0^2\left(\frac{5!}{2}-4!\right) = 16e\mathcal{E}a_0^2$$

b. Use degenerate perturbation theory to find the energies of the electron states for the Hamiltonian $H_0+\Delta H$ with H_0 the Hamiltonian for an electron in a hydrogen atom, up to order $\mathcal{E}$.

- The ground state of Hydrogen atom for $n=1$ is $-13.6\,eV$

$$E_0 = -\frac{13.6}{n^2} = -\frac{13.6}{4} = -3.4\,eV$$

$$\begin{pmatrix} -\lambda & 16e\varepsilon a_0^2 & & \\ 16e\varepsilon a_0^2 & -\lambda & & \\ & & -\lambda & \\ & & & -\lambda \end{pmatrix}$$

$$\det = -\lambda\begin{bmatrix} -\lambda & & \\ & -\lambda & \\ & & -\lambda \end{bmatrix} - 16e\varepsilon a_0^2\begin{bmatrix} 16e\varepsilon a_0^2 & & \\ & -\lambda & \\ & & -\lambda \end{bmatrix}$$

$$= -\lambda(-\lambda)(\lambda^2) - 16e\varepsilon a_0^2(16e\varepsilon a_0^2)\lambda^2 = 0$$

$$= \lambda^2(\lambda^2 - (16e\varepsilon a_0^2)^2) = 0$$

$$\lambda = \pm 16e\varepsilon a_0$$

$$E = -3.4\,eV \pm 16e\varepsilon a_0$$

Note that radial wavefunctions and spherical harmonics are given

$$R_{20}(r) = (1/2a_0^3)^{1/2}(1-r/2a_0)\exp(-r/2a_0)$$
$$R_{21}(r) = (1/6a_0^3)^{1/2}(r/2a_0)\exp(-r/2a_0)$$
$$a_0 = \hbar^2/me^2$$
$$Y_0^0 = 1/\sqrt{4\pi}$$
$$Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\exp(\pm i\phi) = \left(\frac{3}{8\pi}\right)^{1/2}\cdot\frac{x\pm iy}{r}$$
$$Y_1^0 = (3/4\pi)^{1/2}\cos\theta = (3/4\pi)^{1/2}\cdot\frac{z}{r}$$

15*. One way to produce a gamma-ray photon is via backscattering of a laser on a relativistic particle beam. Find the wavelength λ' of a photon scattered at 180° via the head-on collision of a laser beam of wavelength λ with a beam of electrons of mass m and energy E.

$\xrightarrow{E_{B\gamma}} \leftarrow E_{Be}$ $\qquad E = hf \qquad f\lambda = c$

$\xleftarrow{E_{A\gamma}} \leftarrow E_{Ae}$ $\qquad E_\gamma = hc/\lambda$

$$P_{B\gamma} = (E_{B\gamma}, \vec{p}_{B\gamma}) \qquad P_{B\gamma} + P_{Be} = P_{A\gamma} + P_{Ae}$$
$$P_{A\gamma} = (E_{A\gamma}, \vec{p}_{A\gamma}) \qquad P_{B\gamma}^2 + P_{Be}^2 + 2P_{B\gamma}P_{Be} = P_{A\gamma}^2 + P_{Ae}^2 + 2P_{A\gamma}P_{Ae}$$
$$P_{Be} = (E_{Be}, \vec{p}_{Be}) \qquad \cancel{m_\gamma^2} + m_e^2 + 2(E_{B\gamma}E_{Be} + |\vec{p}_{B\gamma}||\vec{p}_{Be}|)$$
$$P_{Ae} = (E_{Ae}, \vec{p}_{Ae}) \qquad = \cancel{m_\gamma^2} + m_e^2 + 2(E_{A\gamma}E_{Ae} - |\vec{p}_{A\gamma}||\vec{p}_{Ae}|)$$

$$E_{B\gamma}E_{Be} + E_{B\gamma}p_{Be} = E_{A\gamma}E_{Ae} - E_{A\gamma}p_{Ae}$$
$$E_\gamma E + E_\gamma p = E_\gamma' E' - E_\gamma' p'$$
$$E_{B\gamma} + E_{Be} = E_{A\gamma} + E_{Ae}$$
$$E_\gamma + E = E_\gamma' + E' \qquad E_\gamma' = E_\gamma + E - E'$$
$$E_\gamma - p = -E_\gamma' - p' \qquad E^2 = p^2 + m_e^2$$
$$E'^2 = p'^2 + m_e^2$$
$$E_\gamma - \sqrt{E^2 - m_e^2} = -E_\gamma' - \sqrt{E'^2 - m_e^2}$$
$$E'^2 - m_e^2 = (E_\gamma' + E_\gamma + \sqrt{E^2 - m_e^2})^2$$
$$E'^2 = m_e^2 + (E_\gamma' + E_\gamma + \sqrt{E^2 - m_e^2})^2$$

$$E_\gamma' = E_\gamma + E - [m_e^2 + (E_\gamma' + E_\gamma + \sqrt{E^2 - m_e^2})^2]^{1/2}$$

Solve for E_γ'

16.

a. Compare the momentum and kinetic energy of a bullet in flight to those of the continental United States due to continental drift.

- Say the US is 3000 miles across (covers three time zones), and it's about 3 times wide as it is long.

$$3\times10^6 \text{ miles}^2 \cdot \left(\frac{5\times10^3 \text{ m}}{3 \text{ miles}}\right)^2 = \frac{25}{3}\times10^{12} \text{ m}^2 \approx \frac{25}{3}\times10^{12} \text{ m}^3$$

Say density ~ 10 kg/m³ and drift speed ~ 10 cm/year

$$\frac{25}{3}\times10^{12} \text{ m}^3 \cdot \frac{10 \text{ kg}}{\text{m}^3} \cdot \frac{10^{-1} \text{ m}}{3.6\times10^7 \text{ s}} \approx 2\times10^5 \text{ kg m/s} \quad \text{momentum}$$

$$\frac{1}{2}\frac{25\times10^{12}}{3}\text{m}^3 \cdot \frac{10 \text{ kg}}{\text{m}^3} \cdot \left(\frac{10^{-1} \text{ m}}{3.6\times10^7 \text{ s}}\right)^2 \approx \frac{10}{3.16}\times10^{-3} \text{ kg m}^2/\text{s}^2$$
$$\approx 3\times10^{-4} \text{ kg} \cdot \text{m}^2/\text{s}^2$$

Say a brown bess has a muzzle velocity of 300 m/s and the bullet is 10 g

$$p = 10^{-2} \text{ kg} \cdot 3\times10^2 \text{ m/s} = 3 \text{ kg m/s}$$
$$E = \tfrac{1}{2}10^{-2} \text{ kg} \cdot (3\times10^2 \text{ m/s})^2 = 5\times10^2 \text{ kg m}^2/\text{s}^2$$

b. Estimate the temperature change of a steel ball bearing (1 cm radius) striking the ground after being dropped from your hand.

- $E = mgh = V\rho gh = \frac{4}{3}\pi(10^{-2} \text{ m})^3 \cdot 10 \text{ kg/m}^3 \cdot 10 \text{ m/s}^2$
$= 4\times10^{-4}$ J

$E/q = mC\Delta T$

$$\Delta T = q/cm = \frac{gh}{100} = 1 \text{ K}$$