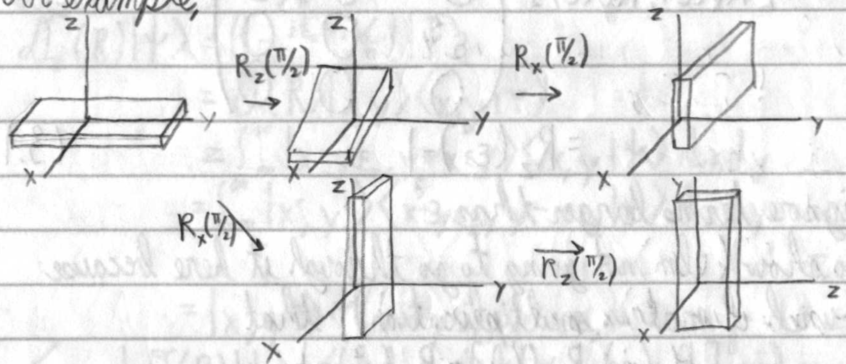
Section 1. Rotations and Angular Momentum Commutation Relations

Subsection Finite versus Infinitesimal Rotations

In classical mechanics, we can show that rotations around the same axis commute while rotations around different axes do not i.e., If you rotate by θ around the z-axis then rotate by θ' around the z-axis again, this is equivalent to rotating by θ' then θ. However, if you rotate by θ around the z-axis followed by a θ' rotation around the x-axis, this is different than the reverse.

For example,



We'll follow Sakurai's notation and say that ϕ (the angle of rotation) is positive when rotation is counterclockwise when viewed from the positive side of the concerned axis.

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & \\ \sin\phi & \cos\phi & \\ & & 1 \end{pmatrix} \approx \begin{pmatrix} 1-\varepsilon^2/2 & -\varepsilon & \\ \varepsilon & 1-\varepsilon^2/2 & \\ & & 1 \end{pmatrix} \quad (3.1.3),$$

$$R_x(\phi) = \begin{pmatrix} 1 & & \\ & \cos\phi & -\sin\phi \\ & \sin\phi & \cos\phi \end{pmatrix} \approx \begin{pmatrix} 1 & & \\ & 1-\varepsilon^2/2 & -\varepsilon \\ & \varepsilon & 1-\varepsilon^2/2 \end{pmatrix} \quad (3.1.4),$$

$$R_y(\phi) = \underbrace{\begin{pmatrix} \cos\phi & & \sin\phi \\ & 1 & \\ -\sin\phi & & \cos\phi \end{pmatrix}}_{\text{finite rotations}} \approx \underbrace{\begin{pmatrix} 1-\varepsilon^2/2 & & \varepsilon \\ & 1 & \\ -\varepsilon & & 1-\varepsilon^2/2 \end{pmatrix}}_{\text{infinitesimal rotations}} \quad (3.1.5)$$

We can also show

$$R_x(\varepsilon)R_y(\varepsilon) = \begin{pmatrix} 1-\varepsilon^2/2 & 0 & \varepsilon \\ \varepsilon^2 & 1-\varepsilon^2/2 & -\varepsilon \\ -\varepsilon & \varepsilon & 1-\varepsilon^2 \end{pmatrix} \quad (3.1.6a)$$

$$R_y(\varepsilon)R_x(\varepsilon) = \begin{pmatrix} 1-\varepsilon^2/2 & \varepsilon^2 & \varepsilon \\ 0 & 1-\varepsilon^2/2 & -\varepsilon \\ -\varepsilon & \varepsilon & 1-\varepsilon^2 \end{pmatrix} \quad (3.1.6b)$$

which brings us back to "rotations about different axes do not commute."

$$[R_x(\varepsilon), R_y(\varepsilon)] = \begin{pmatrix} 0 & -\varepsilon^2 & 0 \\ \varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= R_z(\varepsilon^2) - 1 \quad (3.1.7)$$

where we ignore terms larger than ε^2

We can also show (I'm not going to go through it here because it's just a bunch of matrix multiplication), that

$$[R_z(\varepsilon), R_x(\varepsilon)] = R_y(\varepsilon^2) - 1$$
$$[R_y(\varepsilon), R_z(\varepsilon)] = R_x(\varepsilon^2) - 1$$

We'll actually use this result later to derive the angular momentum commutation relations.

Subsection Infinitesimal Rotations in Quantum Mechanics

$$|\alpha\rangle_R = \mathcal{D}(R)|\alpha\rangle \quad (3.1.10)$$

If we write the previous section in bra-ket notation, we get the above equation. Written this way, we see parallels with (1.6.12) and (2.1.5). This leads us to the conclusion that we can write $\mathcal{D}(R)$ as

$$\mathcal{D}(R) = \mathbb{1}_\varepsilon = 1 - iG\varepsilon \quad (3.1.11)$$

where G is a hermitian operator.

$$G \to J_k/\hbar, \quad \varepsilon \to d\phi \quad (3.1.14)$$

$$\mathcal{D}(\hat{n}, d\phi) = 1 - i\left(\frac{\mathbf{J}\cdot\hat{n}}{\hbar}\right)d\phi \quad (3.1.15)$$

where we have defined J_k as the generator of infinitesimal rotation around the k-axis. J_k is also known as the angular momentum operator. With this in hand, what happens when we let $\mathcal{D}_z(R)$ act on $|x\rangle$?

$$\mathcal{D}_z(R)|x,y,z\rangle = |x\cos\phi_0 - y\sin\phi_0, x\sin\phi_0 + y\cos\phi_0\rangle \quad \text{(Shankar 12.2.6)}$$
$$\approx |x - y\varepsilon_z, y + x\varepsilon_z\rangle \quad \text{(Shankar 12.2.8)}$$

(Note, I decided to bring in Shankar here because I think its a great resource. Seriously, why wasn't I using it before?)

- Shankar Problem 12.2.1

$$\mathcal{D}_z(R) = 1 - i\varepsilon_z J_z/\hbar$$
$$\mathcal{D}_z(R)|\psi\rangle = (1 - i\varepsilon_z J_z/\hbar)|\psi\rangle$$
$$= \mathcal{D}(R)|x,y\rangle\langle x,y|\psi\rangle$$
$$= \int_{-\infty}^{\infty} |x - y\varepsilon_z, y + x\varepsilon_z\rangle\langle x,y|\psi\rangle\, dx\, dy$$
$$= \int_{-\infty}^{\infty} |x', y'\rangle\langle x' + y\varepsilon_z, y - x\varepsilon_z|\psi\rangle\, dx\, dy \qquad x' = x - y\varepsilon_z, \quad y' = y + x\varepsilon_z$$
$$= |x', y'\rangle\psi(x' + y\varepsilon_z, y' - x\varepsilon_z)$$
$$\langle x,y|\mathcal{D}(R)|\psi\rangle = \langle x,y|x',y'\rangle\psi(x' + y\varepsilon_z, y' - x\varepsilon_z)$$
$$= \delta(x - x')\delta(y - y')\psi(x' + y\varepsilon_z, y' - x\varepsilon_z)$$
$$= \psi(x + y\varepsilon_z, y - x\varepsilon_z) \quad \text{(Shankar 12.2.9)}$$

So how does this relate to the rotation matrix in the previous subsection? To obtain a finite rotation around the z-axis

$$\mathcal{D}_z(\phi) = \lim_{N\to\infty}\left[1 - i\left(\frac{J_z}{\hbar}\right)\left(\frac{\phi}{N}\right)\right]^N$$
$$= \exp(-iJ_z\phi/\hbar)$$
$$\approx 1 - iJ_z\phi/\hbar - J_z^2\phi^2/2\hbar^2 \quad (3.1.16)$$

$$[R_x, R_y] = (1 - iJ_x\phi/\hbar - J_x^2\phi^2/2\hbar^2)(1 - iJ_y\phi/\hbar - J_y^2\phi^2/2\hbar^2)$$
$$- (1 - iJ_y\phi/\hbar - J_y^2\phi^2/2\hbar^2)(1 - iJ_x\phi/\hbar - J_x^2\phi^2/2\hbar^2)$$
$$= 1 - iJ_z\phi^2/\hbar - 1 = -iJ_z\phi^2/\hbar$$

We can easily convince ourselves that only ϕ^2 terms survive, and what we get is

$$-\frac{J_y^2\phi^2}{2\hbar^2} - \frac{J_xJ_y\phi^2}{\hbar^2} - \frac{J_x^2\phi^2}{2\hbar^2} + \frac{J_x^2\phi^2}{2\hbar^2} + \frac{J_yJ_x\phi^2}{\hbar^2} + \frac{J_y^2\phi^2}{2\hbar^2} = -\frac{iJ_z\phi^2}{\hbar}$$

$$-[J_x, J_y]\frac{\phi^2}{\hbar^2} = -\frac{iJ_z\phi^2}{\hbar}$$

$$[J_x, J_y] = i\hbar J_z \qquad (3.1.19)$$

This can be expanded for all combinations of x, y, z

$$[J_i, J_j] = i\hbar\, \varepsilon_{ijk} J_k \qquad (3.1.20)$$

This is the fundamental commutation relation of angular momentum.

Section 2. Spin ½ Systems and Finite Rotations

Subsection Rotation Operator for Spin ½

Another relation that might be useful to know is

$$\{J_i, J_j\} = (\hbar^2/2)\delta_{ij}$$

Back in Chapter 1, we found S_x, S_y, S_z, the spin operators, which we'll rewrite here.

$$S_x = (\hbar/2)[(|+\rangle\langle-|) + (|-\rangle\langle+|)]$$
$$S_y = (i\hbar/2)[-(|+\rangle\langle-|) + (|-\rangle\langle+|)]$$
$$S_z = (\hbar/2)[(|+\rangle\langle+|) - (|-\rangle\langle-|)] \qquad (3.2.1)$$

Also as was shown in problem 1.8, it satisfies the commutation relation

$$[S_i, S_j] = i\hbar\, \varepsilon_{ijk} S_k$$

I would like to stress here that $\vec{J}$ and $\vec{S}$ represent different things. Even though they follow the same commutation relations, they are distinct entities. In fact, as we will find out later on, $\vec{J} = \vec{L} + \vec{S}$ where $\vec{L}$ is the orbital angular momentum. For those following along in Shankar will notice he uses $\vec{L}$ instead of $\vec{J}$. This is fine since $\vec{J} = \vec{L}$ for a single particle system. I should also probably say that Sakurai claims there is no theoretical justification that $\vec{S}$ and $\vec{J}$ have the same commutation relations, and I'm inclined to believe him since I didn't see any justification in Shankar either.

Just as we used J to define finite rotation around some axis, we can similarly define finite rotation of a spin-½ system.

$$|\alpha\rangle_R = \mathcal{D}_z(\phi)|\alpha\rangle \qquad (3.2.2)$$
$$\mathcal{D}_z(\phi) = \exp(-iS_z\phi/\hbar) \qquad (3.2.3)$$

To prove this is a rotation, let's look at $\langle S_x\rangle$

$$\langle S_x\rangle = {}_R\langle\alpha|S_x|\alpha_R\rangle = \langle\alpha|\mathcal{D}_z^\dagger(\phi)S_x\mathcal{D}_z(\phi)|\alpha\rangle$$
$$= \langle\alpha|\exp(iS_z\phi/\hbar)S_x\exp(iS_z\phi/\hbar)|\alpha\rangle \qquad (3.2.4)$$

- $$\exp(iS_z\phi/\hbar) = 1 + \frac{i\phi}{\hbar}\cdot\frac{\hbar}{2}[(|+\rangle\langle+|) - (|-\rangle\langle-|)]$$
$$= 1 + \frac{i\phi}{2}[(|+\rangle\langle+|) - (|-\rangle\langle-|)]$$
$$\exp(-iS_z\phi/\hbar) = 1 - \frac{i\phi}{2}[(|+\rangle\langle+|) - (|-\rangle\langle-|)]$$

$$\exp(iS_z\phi/\hbar)S_x\exp(-iS_z\phi/\hbar)$$
$$= \frac{\hbar}{2}[\exp(iS_z\phi/\hbar)|+\rangle\langle-|\exp(-iS_z\phi/\hbar) + \exp(iS_z\phi/\hbar)|-\rangle\langle+|\exp(-iS_z\phi/\hbar)]$$

$$S_z|+\rangle = \frac{\hbar}{2}|+\rangle$$
$$S_z|-\rangle = -\frac{\hbar}{2}|-\rangle$$

$$\exp(iS_z\phi/2)S_x\exp(-iS_z\phi/2) = \frac{\hbar}{2}[\exp(i\phi/2)|+\rangle\langle-|\exp(\tfrac{i\phi}{2}) + \exp(-\tfrac{i\phi}{2})|-\rangle\langle+|\exp(-i\phi/2)]$$
$$= \frac{\hbar}{2}[(|+\rangle\langle-|)(\cos\phi + i\sin\phi) + (|-\rangle\langle+|)(\cos\phi - i\sin\phi)]$$
$$= \frac{\hbar}{2}[(|+\rangle\langle-|) + (|-\rangle\langle+|)]\cos\phi - \frac{i\hbar}{2}[(|-\rangle\langle+|) - (|+\rangle\langle-|)]\sin\phi$$
$$= S_x\cos\phi - S_y\sin\phi$$

- $$\exp(iS_z\phi/\hbar)S_x\exp(-iS_z\phi/\hbar)$$
$$\approx \left(1 + \frac{iS_z\phi}{\hbar} - \frac{S_z^2\phi^2}{2\hbar^2}\right)S_x\left(1 - \frac{iS_z\phi}{\hbar} - \frac{S_z^2\phi^2}{2\hbar^2}\right)$$
$$= S_x + \frac{i\phi}{\hbar}(S_zS_x - S_xS_z) - \frac{\phi^2}{2\hbar^2}(S_z^2S_x - 2S_zS_xS_z + S_xS_z^2)$$
$$= S_x + \frac{i\phi}{\hbar}[S_z, S_x] - \frac{\phi^2}{2\hbar^2}[S_z, [S_z, S_x]]$$
$$= S_x + \frac{i\phi}{\hbar}[i\hbar S_y] - \frac{\phi^2}{2\hbar^2}[S_z, i\hbar S_y]$$
$$= S_x - \phi S_y - \frac{\phi^2}{2\hbar^2}[i\hbar\cdot -i\hbar S_x]$$
$$= S_x - \phi S_y - \frac{\phi^2}{2}S_x = S_x(1 - \phi^2/2) - S_y(\phi)$$
$$\approx S_x\cos\phi - S_y\sin\phi$$

We showed that

$$\langle S_x \rangle = \langle S_y \rangle \cos\phi - \langle S_y \rangle \sin\phi \quad (3.2.8)$$

and we can follow the same steps to show

$$\langle S_y \rangle = \langle S_y \rangle \cos\phi + \langle S_x \rangle \sin\phi \quad (3.2.9)$$

$$\langle S_z \rangle = \langle S_z \rangle \quad (3.2.10)$$

In addition, if we look back at (3.1.3), (3.1.4), (3.1.5), we see that the expectation value of the spin operator behaves as vector under the classical rotation matrices. It should be a fairly short step to convince ourselves that the same holds true for the angular momentum operator.

Subsection Spin Precession Revisited

Back is section 2.1, we looked at a spin-½ system with magnetic moment $e\hbar/2m_ec$ subjected to an external magnetic field $\vec{B}$. Since were in the section on spin-½ systems, let's take another look at that problem again.

$$\mathcal{H} = -\left(\frac{e}{m_eC}\right)\vec{S}\cdot\vec{B} = \omega S_z \quad (3.2.16)$$

$$\omega = |e|B/m_eC \quad (3.2.17)$$

$$\mathcal{U}(t,0) = \exp\left(-i\mathcal{H}t/\hbar\right) = \exp\left(-iS_z\omega t/\hbar\right) \quad (3.2.18)$$

which we compare to (3.2.3), the operator of finite rotation, which implies that $\phi = \omega t$. If we then look at the time evolution of the expectation values.

$$\langle S_x \rangle_t = \mathcal{U}^+ S_x \mathcal{U}$$

$$= \langle S_x \rangle_{t=0} \cos(\omega t) - \langle S_y \rangle_{t=0} \sin(\omega t) \quad (3.2.19a)$$

$$\langle S_y \rangle_t = \langle S_y \rangle_{t=0} \cos(\omega t) + \langle S_x \rangle_{t=0} \sin(\omega t) \quad (3.2.19b)$$

$$\langle S_z \rangle_t = \langle S_z \rangle_{t=0} \quad (3.2.19c)$$

We can easily see that at $t = 2\pi/\omega$, the spin returns to its original direction.

I notice that there was a bit in the previous subsection that I forgot, so let's put that here.

What happens when we let the rotation operator act upon some ket

$$|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle \quad (3.2.13)$$

$$\exp\left(-iS_z\phi/\hbar\right)|\alpha\rangle = \exp\left(-i\phi/2\right)|+\rangle\langle+|\alpha\rangle + \exp\left(i\phi/2\right)|-\rangle\langle-|\alpha\rangle \quad (3.2.14)$$

Normally, if we rotate by 2π, we expect to get back to the original state.

$$\exp(0)|\alpha\rangle = \exp(0)|+\rangle\langle+|\alpha\rangle + \exp(0)|-\rangle\langle-|\alpha\rangle$$

$$|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle$$

$$\exp\left(-iS_z\cdot 2\pi/\hbar\right)|\alpha\rangle = \exp(-i\pi)|+\rangle\langle+|\alpha\rangle + \exp(i\pi)|-\rangle\langle-|\alpha\rangle$$

$$= -|+\rangle\langle+|\alpha\rangle - |-\rangle\langle-|\alpha\rangle = -|\alpha\rangle$$

$$\exp\left(-iS_z\cdot 4\pi/\hbar\right)|\alpha\rangle = \exp(-i2\pi)|+\rangle\langle+|\alpha\rangle + \exp(i2\pi)|-\rangle\langle-|\alpha\rangle$$

$$= |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle = |\alpha$$

We need to rotate by 4π to get back to the original state.

If we apply this logic to the spin precession problem, we get

$$|\alpha, t=0; t\rangle = \exp\left(-i\omega t/2\right)|+\rangle\langle+|\alpha\rangle + \exp\left(i\omega t/2\right)|-\rangle\langle-|\alpha\rangle \quad (3.2.20)$$

The time it takes to get back to the original state is $t = 4\pi/\omega$, or put another way, the period of the state ket is twice as long as the period for spin precession. Note that this is not entirely true as there are some small corrections due to quantum electrodynamics, a topic beyond the scope of my knowledge.

Subsection Neutron Interferometry Experiment to Study 2π Rotations

(3.2.15) says that a rotation by 2π will lead to a relative minus sign, but what is the physical meaning of this negative? If everything in the universe were multiplied by a minus sign, there would be no observable difference. In order to observe this difference, we need to look at an unrotated state and a rotated state, which we can do by way of neutron interferometry.

Subsection Pauli Two-Component Formalism

Here, we introduce a tool for manipulating spin-½ systems.
Historically, it was thought that scalars, vectors, and tensors were all that was needed to fully describe physics.
Scalars describe spin-0 particles, vectors describe spin-1 particles, and tensors describe spin-2 particles. But what about spin-½ particles? We need to use something called a spinor. Now, I should probably state that mathematicians came up with the spinor first, then physicists observed this natural phenomenon and claimed it. Just like differential geometry.
It turns out spinors are kind of complicated, so for all intents and purposes, we can treat spinors as two-component column matrices. However, the difference comes when we finitely rotate. As seen in the previous subsections, we need to rotate by 4π to get back to the original state. A fun example of this behavior is the Balinese cup dance (please don't ask me to do this, I'm not flexible enough).
For the spin-½ case, let's define the basis

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |+\rangle \qquad \chi_+^\dagger = (1 \;\; 0) = \langle +|$$

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |-\rangle \qquad \chi_-^\dagger = (0 \;\; 1) = \langle -| \qquad (3.2.26)$$

From this, we can create a general spinor

$$\chi = \begin{pmatrix} \langle +|\alpha\rangle \\ \langle -|\alpha\rangle \end{pmatrix} = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = c_+\chi_+ + c_-\chi_- \qquad (3.2.28)$$

where c_+ and c_- are complex numbers. Similarly,

$$\chi^\dagger = (c_+^* \;\; c_-^*) \qquad (3.2.29)$$

Now what happens if we look at the expectation value of $\langle S_x \rangle$ in terms of spinors?

$$\begin{aligned}\langle\alpha|S_x|\alpha\rangle &= \sum_{a'=+,-}\sum_{a''=+,-}\langle\alpha|a'\rangle\langle a'|S_x|a''\rangle\langle a''|\alpha\rangle \\ &= \langle\alpha|+\rangle\langle+|S_x|+\rangle\langle+|\alpha\rangle \\ &\quad +\langle\alpha|+\rangle\langle+|S_x|-\rangle\langle-|\alpha\rangle \\ &\quad +\langle\alpha|-\rangle\langle-|S_x|+\rangle\langle+|\alpha\rangle \\ &\quad +\langle\alpha|-\rangle\langle-|S_x|-\rangle\langle-|\alpha\rangle \\ &= \hbar/2\left[\langle\alpha|+\rangle\, 0\cdot\langle+|\alpha\rangle + \langle\alpha|+\rangle\langle-|\alpha\rangle + \langle\alpha|-\rangle\langle+|\alpha\rangle + 0\cdot\langle\alpha|-\rangle\langle-|\alpha\rangle\right] \\ &= \hbar/2\,\left(\langle\alpha|+\rangle \;\; \langle\alpha|-\rangle\right)\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\langle+|\alpha\rangle \\ \langle-|\alpha\rangle\end{pmatrix} \\ &= \hbar/2\;\chi^\dagger\sigma_x\chi \qquad (3.2.31)\end{aligned}$$

where σ_x is a Pauli matrix, defined as

$$\vec{S} = \hbar/2\;\vec{\sigma} \qquad \text{(Shankar 14.3.30)}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix};\; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix};\; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} \qquad (3.2.32)$$

It's probably a good idea to remember these matrices as well as some of their properties

1. Anticommute
$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \qquad (3.2.34)$$
2. Commutation relation
$$[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k \qquad (3.2.35)$$
3. For cyclic permutations of i, j, k
$$\sigma_i\sigma_j = i\sigma_k \qquad (3.2.26)$$
4. Some assorted properties
$$\sigma_i^\dagger = \sigma_i$$
$$\det\sigma_i = -1$$
$$\mathrm{Tr}\,\sigma_i = 0$$
$$\sigma_i^2 = I \qquad (3.2.37)$$
$$\vec{\sigma}\cdot\vec{a} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3\end{pmatrix} \qquad (3.2.38)$$
$$(\vec{\sigma}\cdot\vec{a})^2 = |\vec{a}|^2 \qquad (3.2.41)$$

· Shankar Problem 14.3.4

Prove $$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b}) \qquad (3.2.39)$$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \sum_j\sum_k\left(\tfrac{1}{2}\{\sigma_j,\sigma_k\} + \tfrac{1}{2}[\sigma_j,\sigma_k]\right)a_j b_k$$
$$= \sum_j\sum_k(\delta_{jk} + i\varepsilon_{jkl}\sigma_l)a_j b_k$$
$$= \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

Alternatively

$$M = \sum m_\alpha \sigma_\alpha \qquad \text{(Shankar 14.3.42)}$$
$$m_\beta = \tfrac{1}{2}\operatorname{Tr}(M\sigma_\beta) \qquad \text{(Shankar 14.3.43)}$$
$$M = \sum_{\alpha=0}^{3}\tfrac{1}{2}\operatorname{Tr}((M\sigma_\alpha)\sigma_\alpha)$$
$$= \tfrac{1}{2}\operatorname{Tr}(M\sigma_0) \qquad \text{since } \sigma_1, \sigma_2, \sigma_3 \text{ are all traceless}$$
$$= \tfrac{1}{2}\operatorname{Tr}(M\sigma_0) \qquad \sigma_0 = I$$
$$= \tfrac{1}{2}\sum_i\sum_j a_i b_j \operatorname{Tr}(\sigma_i\sigma_j)$$
$$= \vec{a}\cdot\vec{b}$$

$$\operatorname{Tr}(M\sigma_k) = \sum_{i,j} a_i b_j \operatorname{Tr}(\sigma_i\sigma_j\sigma_k)$$
$$= \tfrac{1}{2}\sum_{i,j} a_i b_j \operatorname{Tr}(\sigma_i\sigma_j\sigma_k + \sigma_k\sigma_i\sigma_j)$$
$$= \tfrac{1}{2}\sum_{i,j} a_i b_j \operatorname{Tr}([\sigma_k,\sigma_i]\sigma_j + \sigma_i\{\sigma_k,\sigma_j\}$$
$$= \tfrac{1}{2}\sum_{i,j,l} a_i b_j \operatorname{Tr}(2i\varepsilon_{kil}\sigma_l\sigma_j + 2\sigma_i\delta_{kj})$$
$$= i\sum_{ijl} a_i b_j \operatorname{Tr}(\varepsilon_{kil}\cdot 2\delta_{lj})$$
$$= 2i(\vec{a}\times\vec{b})_k$$

$$M = \tfrac{1}{2}\operatorname{Tr}(M\sigma_0)\sigma_0 + \tfrac{1}{2}\sum\operatorname{Tr}(M\sigma_k)\sigma_k$$
$$= \vec{a}\cdot\vec{b} + i\vec{\sigma}(\vec{a}\times\vec{b})$$

Subsection Rotations in the Two-Component Formalism

Using (3.2.3) and the Pauli matrices,

$$\mathcal{D}(\hat{n},\phi) = \exp\left(-i\vec{\sigma}\cdot\hat{n}\phi/2\right) \qquad (3.2.42)$$
$$= 1 - i(\vec{\sigma}\cdot\hat{n})\phi/2 - (\vec{\sigma}\cdot\hat{n})^2/2\cdot(\phi/2)^2 + \dots$$

Since $$(\vec{\sigma}\cdot\hat{n})^n = \begin{cases} 1 & n \text{ even} \\ (\vec{\sigma}\cdot\hat{n}) & n \text{ odd}\end{cases} \qquad (3.2.43)$$

which we can get by (3.2.41), (3.2.42) becomes

$$\exp\left(-i\vec{\sigma}\cdot\hat{n}\phi/2\right) = I\cos(\phi/2) - i(\vec{\sigma}\cdot\hat{n})\sin(\phi/2) \qquad (3.2.44)$$
$$= \begin{pmatrix}\cos(\phi/2) - in_z\sin(\phi/2) & (-in_x - n_y)\sin(\phi/2) \\ (-in_x + n_y)\sin(\phi/2) & \cos(\phi/2) + in_z\sin(\phi/2)\end{pmatrix} \qquad (3.2.45)$$

We can act $\exp(-i\vec{\sigma}\cdot\hat{n}\phi/2)$ on spinor χ just as we would act $\exp(-i\vec{S}\cdot\hat{n}\phi/\hbar)$ on the state $|\alpha\rangle$, which leads us to wonder what the expectation value of $\vec{\sigma}$ is. A proof of this can be seen around (3.2.5) or (3.2.6).

$$\langle\sigma_x\rangle = \exp(i\sigma_z\phi/2)\,\sigma_x\exp(-i\sigma_z\phi/2)$$
$$= \langle\sigma_x\rangle\cos\phi - \langle\sigma_y\rangle\sin\phi$$
$$\langle\sigma_y\rangle = \langle\sigma_y\rangle\cos\phi + \langle\sigma_x\rangle\sin\phi$$
$$\langle\sigma_z\rangle = \langle\sigma_z\rangle \qquad (3.2.48)$$

Just as we saw that a spin-½ system will pick up a relative minus after a 2π rotation, we see that

$$\exp\left(-i\vec{\sigma}\cdot\hat{n}\cdot 2\pi/2\right) = -1 \qquad (3.2.49)$$

One thing we can do now that we have spinors is solve problem 1.9 again. That is, let's solve

$$\vec{S}\cdot\hat{n}|\vec{S}\cdot\hat{n};+\rangle = \tfrac{\hbar}{2}|\vec{S}\cdot\hat{n};+\rangle \qquad (3.2.51)$$

where $\hat{n} = (\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta)$

(3.2.50) is equivalent to

$$\vec{\sigma}\cdot\hat{n}\chi = \chi \qquad (3.2.50)$$

First, we recognize that we can get our $\hat{n}$ by taking the spin-up state ($\chi = \binom{1}{0}$) and rotating first by an angle β around the y-axis followed by a rotation of α around the z-axis. From (3.2.44), this is equivalent to

$$\chi = [\cos(\alpha/2) - i\sigma_z\sin(\alpha/2)][\cos(\beta/2) - i\sigma_y\sin(\beta/2)]\begin{pmatrix}1\\0\end{pmatrix}$$
$$= \begin{pmatrix}\cos(\alpha/2) - i\sin(\alpha/2) & 0 \\ 0 & \cos(\alpha/2) + i\sin(\alpha/2)\end{pmatrix}\begin{pmatrix}\cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix} \exp(^{-i\alpha}/_2) & \\ & \exp(^{i\alpha}/_2) \end{pmatrix} \begin{pmatrix} \cos(^{\beta}/_2) \\ \sin(^{\beta}/_2) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(^{\beta}/_2)\exp(^{-i\alpha}/_2) \\ \sin(^{\beta}/_2)\exp(^{i\alpha}/_2) \end{pmatrix} \quad (3.2.52)$$

which is what we got if we shift the phase difference by $^{\alpha}/_2$.

Section 3. O(3), SU(2), and Euler Rotations

Subsection Orthogonal Group

In general, three real numbers are needed to characterize a rotation: the polar and azimuthal angles and the rotation angle. Alternatively, the three Cartesian components of the vector. What about the 3×3 rotation matrix? We would expect there to be 9 entries, but the orthogonality condition

$$RR^T = R^TR = 1 \quad (3.3.1)$$

provides 6 equations resulting in the rotation matrix actually having only 3 independent numbers.

It can also be shown that the rotation matrices form a group i.e.

1. Closure (the product of any two elements of the group returns another element of the group)

$$(R_1R_2)(R_1R_2)^T = R_1R_2R_2^TR_1^T$$
$$= R_1R_1^T = 1 \quad (3.3.2)$$

2. Multiplication is associative. This is true for all matrix multiplication
3. There is an Identity matrix, here, no rotation
4. There is an inverse, here, rotation in the opposite direction

This group is also known as O(3) where O stands for orthogonal and 3 stands for 3 dimensions. Occassionally, this group is further delineated as $O(3)_+$ where + stands for determinant of 1.

Subsection Unitary unimodular group

The rotation matrices (3.1.3) are O(3), but clearly, the spinors cannot form O(3). Instead, (3.2.45) forms SU(2) where S stands for special (determinant = 1) and U stands for unitary.

We can show (3.2.45) is SU(2) by comparing it to the most general unitary unimodular matrix

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad (3.3.7)$$

$$\det U = aa^* - b(-b^*) = aa^* + bb^*$$
$$= |a|^2 + |b|^2 = 1 \quad (3.3.8)$$

Unitary by $U^\dagger U = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$

$$= \begin{pmatrix} aa^* + bb^* & 0 \\ 0 & bb^* + aa^* \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad (3.3.9)$$

Comparing to (3.2.45) leads to

$$a = \cos(^{\phi}/_2) - in_z \sin(^{\phi}/_2)$$
$$b = (-in_x - n_y)\sin(^{\phi}/_2) \quad (3.3.10)$$

Checking group condition

1. $\begin{pmatrix} a_1 & b_1 \\ -b_1^* & a_1^* \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ -b_2^* & a_2^* \end{pmatrix} = \begin{pmatrix} a_1a_2 - b_1b_2^* & a_1b_2 + a_2^*b_1 \\ -a_2b_1^* - a_1^*b_2^* & -b_1^*b_2 + a_1^*a_2^* \end{pmatrix}$
2. Is a matrix
3. Identity $= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$
4. $U^{-1} = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}$

$$\begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} aa^* + bb^* & 0 \\ 0 & bb^* + aa^* \end{pmatrix}$$
$$= I$$

Subsection Euler Rotations

- A way to characterize the most general rotation in three dimension is to use the three Euler angles, or the three Euler rotations.

1. Rotate the rigid body counterclockwise about the z-axis by angle α
   Imagine we had a y'-axis embedded in the body such that it gets rotated when we rotate about the z-axis.
2. Rotate by an angle β about the the y'-axis,
   Again, imagine we had a z'-axis embedded in the body
3. Rotate by an angle γ about the z'-axis

- In terms of $O(3)$ matrices

$$R(\alpha,\beta,\gamma) = R_{z'}(\gamma) R_{y'}(\beta) R_z(\alpha) \qquad (3.3.15)$$

To visualize, take a card or something until you convince yourself it works. It's a little bit difficult to mix primed and unprimed axes, so let's convert

$$R_{y'}(\beta) = R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha) \qquad (3.3.16)$$

$$R_{z'}(\gamma) = R_{y'}(\beta) R_z(\gamma) R_{y'}^{-1}(\beta) \qquad (3.3.17)$$

I encourage you to convince yourself that these return the same motion by going through them yourself

$$\begin{aligned} R(\alpha,\beta,\gamma) &= R_{z'}(\gamma) R_{y'}(\beta) R_z(\alpha) \\ &= R_{y'}(\beta) R_z(\gamma) R_{y'}^{-1}(\beta) R_{y'}(\beta) R_z(\alpha) \\ &= R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha) R_z(\gamma) R_z(\alpha) \\ &= R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha) R_z(\alpha) R_z(\gamma) \\ &= R_z(\alpha) R_y(\beta) R_z(\gamma) \qquad (3.3.19) \end{aligned}$$

- For spin-½ systems, we simply convert this to the 2×2 matrix representation

$$\begin{aligned} \mathcal{D}(\alpha,\beta,\gamma) &= \mathcal{D}_z(\alpha)\mathcal{D}_y(\beta)\mathcal{D}_z(\gamma) \qquad (3.3.20) \\ &= \exp(-i\sigma_z\alpha/2)\exp(-i\sigma_y\beta/2)\exp(-i\sigma_z\gamma/2) \end{aligned}$$

$$= \begin{pmatrix} \exp(-i\alpha/2) & \\ & \exp(i\alpha/2) \end{pmatrix}\begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix}\begin{pmatrix} \exp(-i\gamma/2) & \\ & \exp(i\gamma/2) \end{pmatrix}$$

$$= \begin{pmatrix} \exp(-i(\alpha+\gamma)/2)\cos(\beta/2) & -\exp(-i(\alpha-\gamma)/2)\sin(\beta/2) \\ \exp(i(\alpha-\gamma)/2)\sin(\beta/2) & \exp(i(\alpha+\gamma)/2)\sin(\beta/2) \end{pmatrix} \qquad (3.3.21)$$

Section 4. Density Operators and Pure versus Mixed Ensembles

Subsection Polarized vs. Unpolarized Beams

- So far we've been dealing with ensembles, i.e., everything in a system can be described by the same state ket. But what happens when we can characterize part of the system using $|\alpha\rangle$ and the other part using $|\beta\rangle$. We introduce the idea of fractional population, or probability weight. For example, if we have an ensemble of atoms with completely random spin orientation. Half the members are characterized by $|+\rangle$ while the other half is characterized by $|-\rangle$.

$$w_+ = 0.5 \qquad w_- = 0.5 \qquad (3.4.3)$$

Because there is no information on the relative phase between spin-up and spin-down states, we refer to this as an incoherent mixture. Note that this is not

$$\tfrac{1}{\sqrt{2}}|+\rangle + \tfrac{1}{\sqrt{2}}|-\rangle \qquad (3.4.4)$$

since that contains information on the phase difference. Another way to think about this is to think of a mixed ensemble as different entities rather than a coherent linear superposition.

"The Census report, like most surveys, had cost an awful lot of money and didn't tell anybody anything they didn't already know - except that every single person in the Galaxy had 2.4 legs and owned a hyena." - Douglas Adams, Hitchhikers Guide to the Galaxy.

This illustrates the danger of taking superpositions when they're not needed.

The ensemble in the Stern-Gerlach experiment are in initially an unpolarized state because there is no preferred spin direction. After going through an SG apparatus, the beam is now in a polarized state.

Subsection Ensemble Averages and Density Operator

Suppose we want to make a measurement of observable A on a mixed ensemble. The ensemble average is given by

$$[A] = \sum_i w_i \langle \alpha^{(i)} | A | \alpha^{(i)} \rangle$$
$$= \sum_i \sum_{a'} w_i |\langle a' | \alpha^{(i)} \rangle|^2 a' \quad (3.4.6)$$
$$A|\alpha^{(i)}\rangle = a'|a'\rangle$$
$$[A] = \sum_i w_i \sum_{b'} \sum_{b''} \langle \alpha^{(i)} | b' \rangle \langle b' | A | b'' \rangle \langle b'' | \alpha^{(i)} \rangle$$
$$= \sum_{b'} \sum_{b''} \left(\sum_i w_i \langle b'' | \alpha^{(i)} \rangle \langle \alpha^{(i)} | b' \rangle \right) \langle b' | A | b'' \rangle \quad (3.4.7)$$

We can then define the density operator

$$\rho = \sum_i w_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}| \quad (3.4.8)$$

as well as the density matrix

$$\langle b'' | \rho | b' \rangle = \sum_i w_i \langle b'' | \alpha^{(i)} \rangle \langle \alpha^{(i)} | b' \rangle \quad (3.4.9)$$

From this,

$$[A] = \mathrm{Tr}(\rho A) \quad (3.4.10)$$

The density operator is Hermitian and

$$\mathrm{Tr}(\rho) = 1 \quad (3.4.11)$$

In a pure ensemble,

$$\mathrm{Tr}(\rho^2) = 1 \quad (3.4.15)$$

As an example, say we have a completely polarized beam with $S_x \pm$.

$$\rho = |S_x;\pm\rangle\langle S_x;\pm| = \tfrac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle) \cdot \tfrac{1}{\sqrt{2}}(\langle+| \pm \langle-|)$$
$$= \begin{pmatrix} 1/2 & \pm 1/2 \\ \pm 1/2 & 1/2 \end{pmatrix} \quad (3.4.19)$$

Subsection Time Evolution of Ensembles

If an ensemble is to be left undisturbed, the fractional population w_i cannot change. Thus change in ρ is solely dependant on the state ket $|\alpha^{(i)}\rangle$.

$$|\alpha^{(i)}, t_0 = 0; t\rangle = \mathcal{U}|\alpha, t_0 = 0\rangle$$
$$\mathcal{U}(t) = \exp(-iHt/\hbar)$$
$$\rho(t) = \sum_i w_i \mathcal{U}|\alpha^{(i)}\rangle\langle\alpha^{(i)}|\mathcal{U}^\dagger$$
$$i\hbar\frac{d\rho}{dt} = i\hbar\sum_i w_i \left[\frac{d\mathcal{U}}{dt}|\alpha^{(i)}\rangle\langle\alpha^{(i)}|\mathcal{U}^\dagger + \mathcal{U}|\alpha^{(i)}\rangle\langle\alpha^{(i)}|\frac{d\mathcal{U}^\dagger}{dt} \right]$$
$$= i\hbar\sum_i w_i \left[-\frac{i}{\hbar}H\mathcal{U}|\alpha^{(i)}\rangle\langle\alpha^{(i)}|\mathcal{U}^\dagger + \frac{i}{\hbar}\mathcal{U}|\alpha^{(i)}\rangle\langle\alpha^{(i)}|H\mathcal{U}^\dagger \right]$$
$$= \sum_i w_i \left(H|\alpha^{(i)}, t_0; t\rangle\langle\alpha^{(i)}, t_0; t| - |\alpha^{(i)}, t_0; t\rangle\langle\alpha^{(i)}, t_0; t|H \right)$$
$$= -[\rho, H] \quad (3.4.29)$$

Subsection Continuum Generalizations

If we have a base ket characterized by continuous eigenvalues

$$[A] = \sum_{b'}\sum_{b''} \langle b''|\rho|b'\rangle\langle b'|A|b''\rangle \quad (3.4.10)$$
$$\rightarrow [A] = \iint \langle \vec{x}''|\rho|\vec{x}'\rangle\langle\vec{x}'|A|\vec{x}''\rangle d^3x'\, d^3x'' \quad (3.4.32)$$

$$\langle b''|\rho|b'\rangle = \sum_i w_i \langle b''|a^{(i)}\rangle\langle a^{(i)}|b'\rangle \quad (3.4.9)$$
$$\rightarrow \langle\vec{x}''|\rho|\vec{x}'\rangle = \langle\vec{x}''|\left(\sum_i w_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}|\right)|\vec{x}'\rangle$$
$$= \sum_i w_i \psi_i(\vec{x}'')\psi_i^*(\vec{x}') \quad (3.4.33)$$

Subsection Quantum Statistical Mechanics

For a completely random ensemble,

$$\rho = \frac{1}{N}\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \quad (3.4.39)$$

which shows that all states are equally populated.

If we define σ, a quantitative measure of disorder

$$\sigma = -\mathrm{Tr}(\rho \ln \rho) \quad (3.4.35)$$
$$S = k\sigma \quad (3.4.41)$$

where S is entropy

We see that for a completely random ensemble (3.4.34)

$$\sigma = \ln N \tag{3.4.37}$$

For a pure ensemble, (3.4.16)

$$\sigma = 0 \tag{3.4.38}$$

Section 5. Eigenvalues and Eigenstates of Angular Momentum

Subsection Commutation Relations and the Ladder Operators

$$\vec{J}^2 = J_x J_x + J_y J_y + J_z J_z \tag{3.5.1}$$

$$[\vec{J}^2, J_k] = 0 \tag{3.5.2}$$

$$\begin{aligned}[J_x J_x + J_y J_y + J_z J_z, J_z] &= J_x[J_x, J_z] + [J_x, J_z]J_x \\ &+ J_y[J_y, J_z] + [J_y, J_z]J_y \\ &+ \cancel{J_z[J_z, J_z]} + \cancel{[J_z, J_z]J_z} \\ &= J_x(-i\hbar J_y) + (-i\hbar J_y)J_x + J_y(i\hbar J_x) + (i\hbar J_x)J_y = 0\end{aligned}$$

We can convince ourselves that this also works for J_x and J_y

Define the ladder operators

$$J_\pm = J_x \pm i J_y \tag{3.5.5}$$

which we can compare to (1.4.19).

$$\begin{aligned}[J_+, J_-] &= [J_x + iJ_y, J_x - iJ_y] \\ &= \cancel{[J_x, J_x]} + i[J_y, J_x] - i[J_x, J_y] + \cancel{[J_y, J_y]} \\ &= i \cdot -i\hbar J_z - i \cdot i\hbar J_z = \hbar J_z + \hbar J_z = 2\hbar J_z\end{aligned} \tag{3.5.6a}$$

$$\begin{aligned}[J_z, J_\pm] &= [J_z, J_x \pm iJ_y] \\ &= [J_z, J_x] \pm i[J_z, J_y] \\ &= i\hbar J_y \pm i \cdot i\hbar(-J_x) \\ &= \pm\hbar J_x \pm i\hbar J_y = \pm\hbar J_\pm\end{aligned} \tag{3.5.6b}$$

$$\begin{aligned}[\vec{J}^2, J_\pm] &= [\vec{J}^2, J_x \pm iJ_y] \\ &= [\vec{J}^2, J_x] \pm i[\vec{J}^2, J_y] = 0\end{aligned} \tag{3.5.7}$$

If we denote the eigenvalues of

$$\vec{J}^2|a,b\rangle = a|a,b\rangle \tag{3.5.4a}$$

$$J_z|a,b\rangle = b|a,b\rangle \tag{3.5.4b}$$

J_z acting on $J_\pm|a,b\rangle$

$$\begin{aligned}J_z(J_\pm|a,b\rangle) &= ([J_z, J_\pm] + J_\pm J_z)|a,b\rangle \\ &= (\pm\hbar J_\pm + J_\pm b)|a,b\rangle \\ &= (b \pm \hbar) J_\pm |a,b\rangle\end{aligned} \tag{3.5.8}$$

$J_\pm$ increments J_z eigenvalues by a factor of $\hbar$ at a time. But what about J^2

$$\begin{aligned}J^2(J_\pm|a,b\rangle) &= ([J^2, J_\pm] + J_\pm J^2)|a,b\rangle \\ &= J_\pm a|a,b\rangle \\ &= a(J_\pm|a,b\rangle)\end{aligned} \tag{3.5.11}$$

Nothing happens.

$$J_\pm|a,b\rangle = c_\pm|a, b\pm\hbar\rangle \tag{3.5.12}$$

Subsection Eigenvalues of J^2 and J_z

$$J^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle \tag{3.5.34a}$$

$$J_z|j,m\rangle = m\hbar|j,m\rangle \tag{3.5.34b}$$

where j is either an integer or half-integer and m spans $-j$ to j.

Subsection Matrix Elements of Angular-Momentum Operators

$$\begin{aligned}\langle j',m'|J^2|jm\rangle &= \langle j'm'|j(j+1)\hbar^2|jm\rangle \\ &= j(j+1)\hbar^2\delta_{jj'}\delta_{mm'}\end{aligned} \tag{3.5.35a}$$

$$\langle j'm'|J_z|jm\rangle = m\hbar\delta_{jj'}\delta_{mm'} \tag{3.5.35b}$$

$$\langle j'm'|J_\pm|jm\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,\hbar\,\delta_{jj'}\delta_{m',(m\pm1)} \tag{3.5.41}$$

For example, for $j=1$:

$$J^2 = \hbar^2\begin{bmatrix}2 & & \\ & 2 & \\ & & 2\end{bmatrix} \qquad J_z = \hbar\begin{bmatrix}-1 & & \\ & 0 & \\ & & 1\end{bmatrix}$$

$$J_+ = \hbar\begin{bmatrix}0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0\end{bmatrix} \qquad J_- = \hbar\begin{bmatrix}0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0\end{bmatrix}$$

Subsection Representations of the Rotation Operator

From (3.1.16), we have

$$\mathcal{D}^{(j)}_{m'm}(R) = \langle j m' | \exp\left(\frac{-i \vec{J}\cdot\hat{n}\phi}{\hbar}\right) | j m \rangle \qquad (3.5.42)$$

Note that there is no j' since

$$J^2 \mathcal{D}^{(j)}_{m'm}(R) | j m \rangle$$
$$= \mathcal{D}(R) J^2 | j, m \rangle \qquad \text{since } [J^2, J_k] = 0$$
$$= \mathcal{D}(R)\, j(j+1)\hbar^2 | j, m \rangle$$
$$= j(j+1)\hbar^2 [\mathcal{D}(R) | j, m \rangle] \qquad (3.5.43)$$

The matrix elements in (3.5.42) are called the Wigner functions.

Section 6 Orbital Angular Momentum

We start by defining the orbital angular momentum

$$\vec{L} = \vec{x} \times \vec{p} \qquad (3.6.1)$$

Some textbooks start by using the orbital angular momentum, which is the same as the angular momentum $\vec{J}$ for a single particle when spin-angular momentum is zero.

Subsection Orbital Angular Momentum as Rotation Generator

$$[L_i, L_j] = i \varepsilon_{ijk} \hbar L_k \qquad (3.6.2)$$

Compare to (3.1.20). We used $\vec{J}$ as the rotation generator, so can we do the same for $\vec{L}$?

$$\left[1 - i\left(\frac{\delta\phi}{\hbar}\right) L_z\right] | x' y' z' \rangle = \left[1 - i\left(\frac{p_y}{\hbar}\right)(\delta\phi\, x') + i\left(\frac{p_x}{\hbar}\right)(\delta\phi\, y')\right] | x' y' z' \rangle$$
$$= | x' - y'\delta\phi, y' + x'\delta\phi, z' \rangle \qquad (3.6.5)$$

$$\langle r, \theta, \phi | \left[1 - i\left(\frac{\delta\phi}{\hbar}\right) L_z\right] | \alpha \rangle = \langle r, \theta, \phi - \delta\phi | \alpha \rangle$$
$$= \langle r, \theta, \phi | \alpha \rangle - \delta\phi \cdot \frac{\partial}{\partial\phi} \langle r, \theta, \phi | \alpha \rangle \qquad (3.6.8)$$

$$\langle \vec{x}' | L_z | \alpha \rangle = -i\hbar \frac{\partial}{\partial\phi} \langle \vec{x}' | \alpha \rangle \qquad (3.6.9)$$

$$\vec{L}^2 = \vec{x}^2 \vec{p}^2 - (\vec{x}\cdot\vec{p})^2 + i\hbar\, \vec{x}\cdot\vec{p} \qquad (3.6.16)$$

Subsection Spherical Harmonics