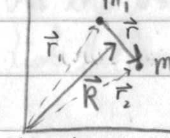
Then he took him by the hand, and led him into a parlour that was full of dust, because never swept; the which after he had revived a little while, the Interpreter called for a man to sweep. Now, when he began to sweep, the dust began so abundantly to fly about, that Christian had almost therewith been choked. Then said the Interpreter to a damsel that stood by, Bring hither the water, and sprinkle the room; the which, when she had done, it was swept and cleansed with pleasure. - John Bunyon, The Pilgrim's Progress

Chapter 3. The Central Force Problem

Section 1. Reduction to the Equivalent One-Body Problem

Imagine we have a system consisting of two masses where the only forces are due to an interaction potential U, which is a function of $\vec{r}, \dot{\vec{r}}$ or some higher-order derivative. In such a system, there will be 6 degrees of freedom



$\vec{R}$: the vector to the center of mass of the system (3DOF)

$\vec{r}: \vec{r}_2 - \vec{r}_1$ (3DOF)

In terms of the laboratory frame

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

$$\frac{m_1\vec{r}_1 + m_2(\vec{r}+\vec{r}_1)}{m_1+m_2} = \frac{(m_1+m_2)\vec{r}_1 + m_2\vec{r}}{m_1+m_2} = \vec{r}_1 + \frac{m_2}{m_1+m_2}\vec{r} = \vec{R}$$

$$\vec{r}_1 = \vec{R} - \frac{m_2}{m_1+m_2}\vec{r}$$

$$\vec{r}_2 = \vec{R} + \frac{m_1\vec{r}}{m_1+m_2} \qquad (3.2)$$

Now let's say we had some Lagrangian

$$\mathcal{L} = \frac{m_1}{2}\dot{\vec{r}}_1^{\,2} + \frac{m_2}{2}\dot{\vec{r}}_2^{\,2} - V(|\vec{r}_1-\vec{r}_2|, |\dot{\vec{r}}_1 - \dot{\vec{r}}_2|, \dots)$$

$$= \frac{m_1}{2}\left(\dot{\vec{R}} - \frac{m_2}{m_1+m_2}\dot{\vec{r}}\right)^2 + \frac{m_2}{2}\left(\dot{\vec{R}} + \frac{m_1\dot{\vec{r}}}{m_1+m_2}\right)^2 - V(\vec{r}, \dot{\vec{r}}, \dots)$$

$$= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{m_1 m_2}{2(m_1+m_2)}\dot{\vec{r}}^{\,2} - V(\vec{r}, \dot{\vec{r}}, \dots) \qquad (3.3)$$

$$= \frac{M}{2}\dot{\vec{R}}^2 + \frac{\mu}{2}\dot{\vec{r}}^{\,2} - V(\vec{r}, \dot{\vec{r}}, \dots)$$

where $M = m_1 + m_2$ is the total mass

$\mu = \frac{m_1 m_2}{m_1+m_2} = (1/m_1 + 1/m_2)^{-1}$ is the reduced mass (3.4), (3.5)

Since $\mathcal{L}$ does not contain the variable $\vec{R}$, $\vec{R}$ is cyclic. We can see this because the equation of motion for $\vec{R}$ reduces to

$$M\ddot{\vec{R}} = 0$$

Thus, we can treat the cases where the system is at rest and is moving uniformly the same. We can assume the system is at rest without loss of generality.

$$\mathcal{L} = \frac{\mu}{2}\dot{\vec{r}}^{\,2} - V(\vec{r}, \dot{\vec{r}}, \dots)$$

and we see that the 2-body problem gets reduced to an effective 1-body problem.

Section 2. The Equations of Motion and First Integral

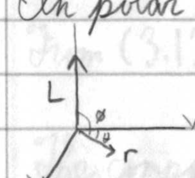
Let us now further simplify our problems by only dealing with conservative forces, which means our potential is solely a function of r and it supports the monarchy. This also means, we only need consider a particle moving around a fixed center of force at the origin. Since the problem is spherically symmetric, the total angular momentum is conserved.

$$\vec{L} = \vec{r} \times \vec{p}$$

If $\vec{L} = 0$, $\Rightarrow \vec{r}\times\vec{p} = 0$, $\Rightarrow \vec{r}$ is in the same direction as $\vec{p}$, which produces a straight line.

If $\vec{L} \neq 0$, $\vec{r}$ always lies in a plane normal to $\vec{L}$; thus, central force motion is always motion in a plane.

In polar coordinates, $\phi = \pi/2$ and can be ignored



$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \qquad (3.6)$$

As we see, θ is a cyclic coordinate,

$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = mr^2\dot{\theta}$ is the angular momentum $= l$

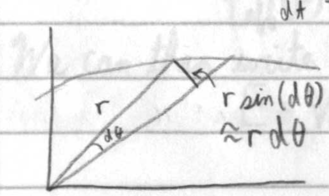
$$\frac{d}{dt}\left(\frac{1}{2} r^2\dot{\theta}\right) = 0 \qquad (3.9)$$

represents the time derivative of the areal velocity.

$$A = \frac{1}{2} r^2\dot{\theta}$$

$$dA = \frac{1}{2} r (r\,d\theta)$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$



A proof of Kepler's second law.

$$\frac{d}{dt}(m\dot{r}) - (mr\dot{\theta}^2 - \frac{\partial V}{\partial r}) = 0 \qquad (3.10)$$

Using $f(r) = -\frac{\partial V}{\partial r}$, the force along $\vec{r}$ as well as (3.8), we can rewrite (3.10) as

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r) \qquad (3.12)$$

As can be seen in the equation above, we can reduce the problem to have effectively one degree of freedom.

Another integral of motion is the total energy, which is conserved.

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \qquad (3.13)$$

$$= \frac{m}{2}(\dot{r}^2 + \frac{l^2}{m^2r^2}) + V(r)$$

which can derived from (3.12)

$$m\ddot{r} = f(r) + \frac{l^2}{mr^3}$$

$$= -\frac{\partial V}{\partial r} - \frac{\partial}{\partial r}\left(\frac{l^2}{2mr^2}\right)$$

$$m\ddot{r} = -\frac{\partial}{\partial r}\left(V + \frac{l^2}{2mr^2}\right) \qquad (3.14)$$

$$m\dot{r}\ddot{r} = -\frac{\partial}{\partial r}\left(V + \frac{l^2}{2mr^2}\right)\cdot\frac{dr}{dt}$$

$$= -\frac{d}{dt}\left(V + \frac{l^2}{2mr^2}\right)$$

$$\frac{d}{dt}\left(\frac{m}{2}\cdot\dot{r}^2\right) = -\frac{d}{dt}\left(V + \frac{l^2}{2mr^2}\right)$$

$$\frac{d}{dt}\left(\frac{m\dot{r}^2}{2} + \frac{l^2}{2mr^2} + V\right) = 0$$

$$\frac{m}{2}\left(\dot{r}^2 + \frac{l^2}{m^2r^2}\right) + V(r) = \text{constant} \qquad (3.15)$$

$$\dot{r} = \pm\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)} \qquad (3.16)$$

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}$$

$$dt = \frac{\pm\, dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}} \qquad (3.17)$$

Where the sign is determined by $\dot{r}(0)$. We also set $r(0) = r_0$

$$t = \int_{r_0}^{r} \frac{\pm\, dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}} \qquad (3.18)$$

Then using (3.8),

$$d\theta = \frac{l\, dt}{mr^2} \qquad (3.19)$$

$$\theta = \theta_0 + \int d\theta$$

$$\theta(t) = \theta_0 + \int_0^t \frac{l\, dt}{mr^2(t)} \qquad (3.20)$$

$$= \theta_0 \pm \int_{r_0}^{r} \frac{l\, dr}{mr^2\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}$$

~~The radial part of the motion is effectively a one-dimension problem in an effective potential~~

$$\cancel{V_{eff}(r) = V(r) + \frac{l^2}{2mr^2}}$$

Section 3 The Equivalent One-Dimensional Problem, and Classification of Orbits

To solve any problem, we only need to evaluate the integrals (3.8), (3.13), (3.18), and (3.20). However (3.18) and (3.20) can quite often get a little nasty, so lets try to classify certain problems without resorting to evaluating the integral.

1. If we know the energy and angular momentum of a system, we can determine the velocity of the particle

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$|\vec{v}|^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

From (3.13), $E = \frac{m}{2}(v^2) + V(r)$

$$v = \sqrt{\frac{2}{m}(E - V)} \qquad (3.21)$$

The components of the velocity are given as

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)} \qquad (3.16)$$

$$r\dot{\theta} = \frac{l}{mr}$$

2. $$\mathcal{L} = \frac{m}{2}\dot{r}^2 - V_{eff}(r) \qquad (3.6)$$

$$V_{eff}(r) = V(r) + \frac{l^2}{2mr^2} \qquad (3.2)$$

which corresponds to

$$f_{eff}(r) = f(r) + \frac{l^2}{mr^3} \qquad (3.22)$$

We can then write

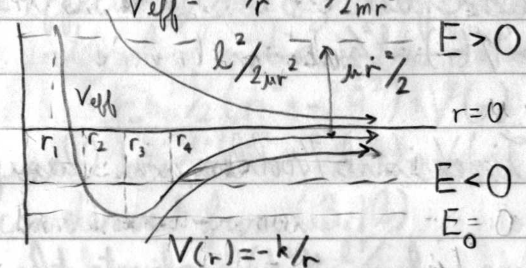
$$E = V_{eff}(r) + \frac{m}{2}\cdot\dot{r}^2$$

If we know l, E, and $V(r)$, we can classify the orbit i.e. we can classify $r(\theta)$. As an example, let f be an attractive inverse-square force law

$$f = -k/r^2 \quad k>0$$

$$V = -k/r$$

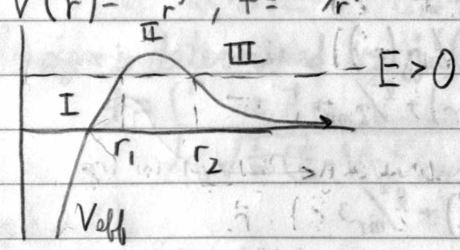
$$V_{eff} = -k/r + l^2/2mr^2$$



- $E>0$: Unbounded motion with minimum value at $r=r_1$. For $r<r_1$, $V_{eff}>E \Rightarrow$ negative kinetic energy. No maximum r.
- $E<0$: Bounded motion with limits r_2 (pericentral) and r_4 (apocentral). This does not mean the orbit is closed (periodic motion), but rather that the path of the particle is bound by two circles with radius r_2 + r_4.
- $E=E_0$: Orbit becomes a circle of radius r_3. This is bounded and closed.

Note that if we change the value of l, we change the V_{eff} curve but not the general classification.

Consider $V(r) = -\frac{a}{r^3}$, $f = -3a/r^4$



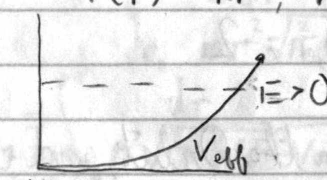
I: Bounded motion since the path will never go beyond r_1

II: Not physically possibly because $V_{eff} > E$

III: Unbounded

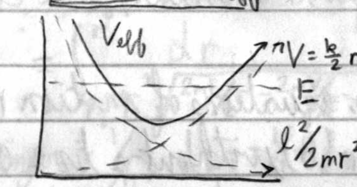
The final example we want to look at is linear restoring force

$$f(r) = -kr \ ; \ V(r) = k/2\, r^2$$



If $l=0$, the motion is always bounded (or unphysical)

If $l \neq 0$



For all physically possible r, the motion is bounded. Further since $\vec{f} = -k\vec{r}$, $f_x = -kx$; $f_y = -ky$, which we recognize as 2-dimensional simple harmonic oscillation

Section 4. Virial Theorem

Imagine we have a system of mass points defined by position vectors $\vec{r}_i$ and applied forces $\vec{F}_i = \dot{\vec{p}}_i$. If we want to find the quantity

$$G = \sum_i \vec{p}_i \cdot \vec{r}_i$$

$$\frac{dG}{dt} = \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i + \sum_i \vec{p}_i \cdot \dot{\vec{r}}_i \qquad (3.23)$$

$$= \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i + \sum_i m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$$

$$= \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i + \sum_i m_i v_i^2$$

$$= \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i + 2T$$

$$= \sum_i \vec{F}_i \cdot \vec{r}_i + 2T$$

$$\frac{d}{dt} \sum_i \vec{p}_i \cdot \vec{r}_i = 2T + \sum_i \vec{F}_i \cdot \vec{r}_i \qquad (3.24)$$

The time average over time interval τ is given by

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}\, dt = 2\overline{T} + \overline{\sum_i \vec{F}_i \cdot \vec{r}_i}$$

$$\frac{1}{\tau}[G(\tau) - G(0)] = 2\overline{T} + \overline{\sum_i \vec{F}_i \cdot \vec{r}_i} \qquad (3.25)$$

which for a sufficiently long τ reduces to

$$\overline{T} = -\frac{1}{2}\overline{\sum_i \vec{F}_i \cdot \vec{r}_i} \qquad (3.26)$$

$$\overline{T} = \frac{1}{2}\overline{\sum_i \nabla V\cdot\vec{r}_i} \qquad (3.27)$$

$$= \frac{1}{2}\,\overline{\frac{\partial V}{\partial r}\,r} \qquad (3.28)$$

(3.28) is for a single particle moving under a central potential.

If V is a power-law function of r, i.e, $V = ar^{n+1}$,

$$\overline{T} = \frac{n+1}{2}\overline{V} \qquad (3.29)$$

For inverse-square law forces, $n=-2$

$$\overline{T} = -\frac{1}{2}\overline{V} \qquad (3.30)$$

Section 5. The Differential Equation for the Orbit, and Integrable Power-Law Potentials

In section 3.2, we found the equations of motion $r(t)$ and $\theta(t)$, but often, it's useful to find $r(\theta)$, without a time dependance.

From (3.8),

$$l\,dt = mr^2\,d\theta \qquad (3.31)$$

$$\frac{l}{mr^2}\cdot\frac{d}{d\theta} = \frac{d}{dt} \qquad (3.32)$$

Plugging into (3.12) gives

$$m\frac{d\dot{r}}{dt} - \frac{l^2}{mr^3} = f(r)$$

$$m\frac{d}{dt}\left(\frac{dr}{dt}\right) - \frac{l^2}{mr^3} = f(r)$$

$$m\cdot\frac{l}{mr^2}\,\frac{d}{d\theta}\left(\frac{l}{mr^2}\,\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} = f(r)$$

$$\frac{l}{r^2}\cdot\frac{d}{d\theta}\left(\frac{l}{mr^2}\,\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} = f(r) \qquad (3.33)$$

Substitute $u = 1/r$

$$du = -\frac{1}{r^2}\,dr$$

$$lu^2\cdot\frac{d}{d\theta}\left(\frac{l}{mr^2}\cdot\frac{-r^2\,du}{d\theta}\right) - \frac{l^2}{m}u^3 = -\frac{dV(r)}{dr}$$

$$-\frac{l^2u^2}{m}\cdot\frac{d^2u}{d\theta^2} - \frac{l^2u^3}{m} = \frac{1}{r^2}\,\frac{d}{du}V\!\left(\frac{1}{u}\right)$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\,\frac{d}{du}V(1/u) \qquad (3.34)$$

The orbit is symmetric (invariant under reflection about the apsidal vectors (nearest and farthest points of an orbit)). Further, we have the following initial conditions

$$u = u(0),\quad \left.\frac{du}{d\theta}\right|_{\theta=0} = 0$$

At the end of section 3.2, we showed

$$d\theta = \frac{l\,dr}{mr^2\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}} \qquad (3.35)$$

$$\theta = \int_{r_0}^{r}\frac{l\,dr}{mr^2\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}} + \theta_0$$

$$= \int_{r_0}^{r}\frac{dr}{r^2\sqrt{\frac{2}{m}\cdot\frac{m^2}{l^2}\left(E - V - \frac{l^2}{2mr^2}\right)}} + \theta_0$$

$$= \int_{r_0}^{r}\frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}} + \theta_0 \qquad (3.36)$$

Again, letting $u = 1/r$, $du = -\frac{1}{r^2}\,dr$

$$\theta = \theta_0 - \int_{u_0}^{u}\frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - u^2}} \qquad (3.37)$$

As it turns out, we can only solve certain types of force law problems, in this text, power-law functions ($V = ar^{n+1}$). Even then, we will only be investigating $n = 1, -2, -3$

Section 6. Conditions for Closed Orbits (Bertrand's Theorem)

We want to know under what conditions an orbit is closed, i.e, the particle eventually retraces its own footsteps. We already saw that for $V(r) = -k/r$, the orbit becomes a circle (which implies a closed orbit) at $E = E_0$.

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}\left(E - V_{eff}(r)\right)} \qquad (3.16)$$

$$\frac{d\theta}{dt} = \frac{l}{mr^2} \qquad (3.19)$$

$$\frac{dr}{d\theta} = \pm\frac{mr^2}{l}\sqrt{\frac{2}{m}\left(E - V_{eff}(r)\right)}$$

Letting $u = 1/r$, $du = -\frac{1}{r^2}\,dr$

$$\frac{du}{d\theta} = \mp\frac{m}{l}\sqrt{\frac{2}{m}\left(E - V_{eff}(1/u)\right)}$$

$$\frac{1}{2}\cdot\frac{l^2}{m}\left(\frac{du}{d\theta}\right)^2 + V_{eff}(1/u) = E$$

with $V_{eff} = \frac{1}{2}\cdot\frac{l^2}{m}u^2 + V(1/u)$

If the motion is circular,

$$\left.\frac{\partial V_{eff}}{\partial u}\right|_{u_0=1/r_0}=0$$

$$\frac{l^2 u_0}{m}-\frac{1}{u_0^2}\cdot\left.\frac{\partial V(r)}{\partial r}\right|_{r_0}=0$$

For small oscillations about u_0: $u(\theta)=u_0+\delta(\theta)$

$$V_{eff}(u)=V_{eff}(u_0)+\left.\frac{\partial V_{eff}}{\partial u}\right|_{u_0}\cdot\delta+\tfrac{1}{2}\left.\frac{\partial^2 V_{eff}}{\partial u^2}\right|_{u_0}\delta^2+\dots$$

$$\frac{1}{2}\cdot\frac{l^2}{m}\left(\frac{d\delta}{d\theta}\right)^2+\frac{1}{2}\cdot\left.\frac{\partial^2 V_{eff}}{\partial u^2}\right|_{u_0}\delta^2=E-V_{eff}(u_0)$$

$$u(\theta)=u_0+A\cos(\Omega\theta)$$

$$\Omega=\sqrt{\frac{u}{l^2}\cdot\left.\frac{\partial^2 V_{eff}}{\partial u^2}\right|_{u_0}}$$

$$=\sqrt{3+\left.\frac{r_0\cdot\frac{\partial^2 V(r)}{\partial r^2}}{\frac{\partial V(r)}{\partial r}}\right|_{r_0}}$$

$$\theta_A=\pi/\sqrt{2+\alpha}\Rightarrow\alpha=-1\text{ or }2$$

Bertrand's theorem states that the only central forces that result in closed orbits for all bound particles are the inverse-square law and Hooke's law.

Section 7. The Kepler Problem: Inverse-Square Law of Force

For this case,

$$f=-k/r^2 \qquad V=-k/r \tag{3.49}$$

Using (3.12),

$$m\ddot{r}=l^2/mr^3-k/r^2$$

which becomes, upon substitution $u=1/r$,

$$\frac{d^2u}{d\theta^2}+u=-m/l^2\cdot\frac{d}{du}(-ku)$$

$$=km/l^2$$

which we recognize as a harmonic oscillator with the solution

$$u=km/l^2\,(\varepsilon\cos(\theta-\theta_0)+1)$$

Another way to derive the above equation is to use (3.39) with $n=-2$

$$\theta=\theta'-\int\frac{du}{\sqrt{2mE/l^2+2mku/l^2-u^2}} \tag{3.50}$$

$$=\theta'-\arccos\left(\frac{l^2u/mk-1}{\sqrt{1+2El^2/mk^2}}\right) \tag{3.54}$$

$$\frac{1}{r}=\frac{mk}{l^2}\left[1+\sqrt{1+\frac{2El^2}{mk^2}}\cos(\theta-\theta')\right] \tag{3.55}$$

$$=mk/l^2\left[1+\varepsilon\cos(\theta-\theta')\right] \tag{3.56}$$

$$\varepsilon=\sqrt{1+2El^2/mk^2} \tag{3.57}$$

The orbit will always be a conic with the exact nature dependant on the magnitude of ε.

$\varepsilon=0$: $0=1+2El^2/mk^2$ $\quad 1/r=\text{constant}\Rightarrow\text{circle}$

$E=-mk^2/2l^2$

$\varepsilon>1$: $E=mk^2/2l^2\,[\varepsilon^2-1]>0$

$r=r_0/1+\varepsilon\cos\theta\Rightarrow$ hyperbola

$\varepsilon=1$: $E=0$

parabolic motion

$\varepsilon<1$: $E<0$

elliptic motion

If we consider the orbit at $r=r_{min}$ ($\dot{r}=0$)

$$v_\theta=r_{min}\dot{\theta}=l/mr_{min}$$

$$T=\tfrac{1}{2}\cdot mv_\theta^2=l^2/2mr_{min}^2$$

$$=\frac{mk^2(1+\varepsilon)^2}{2l^2}$$

Section 8. The Motion in Time in the Kepler Problem

From (3.18), the equation in a central inverse-square force law becomes

$$t = \sqrt{\frac{m}{2}} \int_{r_0}^{r} \frac{dr}{\sqrt{k/r - l^2/2mr^2 + E}} \qquad (3.65)$$

$$1/r = \frac{mk}{l^2}\left(1 + \varepsilon \cos(\theta - \theta')\right)$$

Since $dt = \frac{mr^2}{l}\, d\theta$

$$= \frac{m}{l} \cdot \frac{l^4}{m^2 k^2} \int_{\theta_0}^{\theta} \frac{d\theta}{\left(1 + \varepsilon \cos(\theta - \theta')\right)^2}$$

$$t = \frac{l^3}{mk^2} \int_{\theta_0}^{\theta} \frac{d\theta}{[1 + \varepsilon \cos(\theta - \theta')]^2} \qquad (3.66)$$

These integrals are a little mean, so let's look at the case where $\varepsilon = 1$, parabolic motion. If the integral is carried out from 0 to 2π, we find

$$\tau = 2\pi a^{3/2} \sqrt{m/k} \qquad (3.71)$$

Alternatively, in section 3.2 we found

$$\frac{dA}{dt} = r^2 \dot{\theta}/2 = l/2m \qquad (3.72)$$

$$A = l\tau/2m$$

$$A = \pi a b$$

$$= \pi a^{3/2} \sqrt{l^2/mk} = l\tau/2m$$

$$\tau = \frac{2\pi m}{l} \cdot \frac{l}{\sqrt{mk}}\, a^{3/2}$$

$$= 2\pi \sqrt{m/k}\, a^{3/2}$$

These leads to Kepler's third law: the square of the periods of the various planets are proportional to the cube of their major axes.

Section 9. The Laplace-Runge-Lenz Vector

For the Kepler problem, $f(r) \propto r^{-2}$, $V(r) = -k/r$, there is an additional conserved quantity, which we define as the Laplace-Runge-Lenz vector

$$\vec{A} = \vec{p} \times \vec{L} - mk\,\vec{r}/r \qquad (3.82)$$

$$\frac{d\vec{A}}{dt} = \dot{\vec{p}} \times \vec{L} + \vec{p} \times \dot{\vec{L}} - mk\left(\dot{\vec{r}}\, r - \dot{r}\vec{r}/r^2\right)$$

Since $\vec{L}$ is the constant angular momentum vector, $\dot{\vec{L}} = 0$

$$\frac{d\vec{A}}{dt} = \dot{\vec{p}} \times \vec{L} - mk\left(\frac{\dot{\vec{r}}}{r} - \frac{\dot{r}\vec{r}}{r^2}\right)$$

$$\dot{\vec{r}} \cdot \vec{r} = \frac{1}{2}\frac{d}{dt}(\vec{r} \cdot \vec{r}) = \dot{r} r$$

$$\vec{r}\, \dot{\vec{r}} \cdot \vec{r} = \vec{r}\, \dot{r} r$$

$$\vec{r}(\dot{\vec{r}} \cdot \vec{r})/r^3 = \vec{r}\dot{r}/r^2$$

$$\frac{dA}{dt} = \dot{\vec{p}} \times \vec{L} - \frac{k\vec{p}}{r} + \frac{mk\vec{r}(\dot{\vec{r}} \cdot \vec{r})}{r^3}$$

$$= \dot{\vec{p}} \times \vec{L} - \frac{k\vec{p}}{r} + k\vec{r}\left(\frac{\vec{p} \cdot \vec{r}}{r^3}\right)$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\dot{\vec{p}} = -k\vec{r}/r^3$$

$$\dot{\vec{p}} \times \vec{L} = -k/r^3\left(\vec{r} \times (\vec{r} \times \vec{p})\right)$$

$$= -k/r^3\left[\vec{r}(\vec{r} \cdot \vec{p}) - \vec{p}(\vec{r} \cdot \vec{r})\right]$$

$$= -k\vec{r}\left(\frac{\vec{r} \cdot \vec{p}}{r^3}\right) + \frac{k\vec{p}\, r^2}{r^3} = -k\vec{r}\left(\frac{\vec{r} \cdot \vec{p}}{r^3}\right) + \frac{k\vec{p}}{r}$$

$$\frac{d\vec{A}}{dt} = 0$$

which shows that $\vec{A}$ is conserved.

$$\vec{A} \cdot \vec{L} = 0 \qquad (3.83)$$

$$(\vec{p} \times \vec{L}) \cdot \vec{L} - mk\, \vec{r} \cdot \vec{L}/r = 0$$

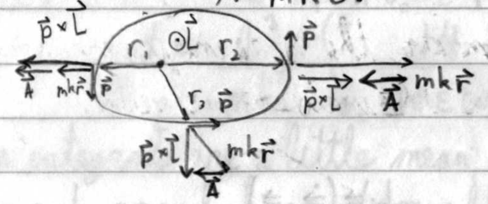
$$(\vec{L} \times \vec{L}) \cdot \vec{p} - mk\, \underbrace{\vec{r} \cdot \vec{L}}_{\vec{r} \perp \vec{L}}/r = 0$$

Which shows that $\vec{A}$ must be a fixed vector in the plane of orbit.

$$\vec{A}\cdot\vec{r} = Ar\cos\theta$$
$$= (\vec{p}\times\vec{L})\cdot\vec{r} - mk\vec{r}\cdot\vec{r}/r$$
$$= \vec{r}\cdot(\vec{p}\times\vec{L}) - mkr \qquad (3.84)$$
$$= \vec{L}\cdot(\vec{r}\times\vec{p}) - mkr$$
$$Ar\cos\theta = l^2 - mkr$$
$$r(A\cos\theta + mk) = l^2$$
$$\frac{1}{r} = mk/l^2\left(1 + \frac{A}{mk}\cos\theta\right) \qquad (3.85)$$

Comparing to (3.55),

$$A = mk\varepsilon. \qquad (3.86)$$



$$\varepsilon = \sqrt{1 + 2El^2/mk^2}$$
$$\varepsilon^2 = 1 + 2El^2/mk^2$$
$$A^2 = mk\varepsilon^2$$
$$= m^2k^2 + 2mEl^2 \qquad (3.87)$$

Section 10. Scattering in a Central Force Field

Scattering is an important concept in physics and should probably have its own chapter. Essentially, scattering involves firing a stream of particles and observing the scattered projectiles to determine the properties of the target and its interaction with the projectile. To do this properly, we use quantum mechanics. However, because this is not a QM textbook, were not going to do this properly.

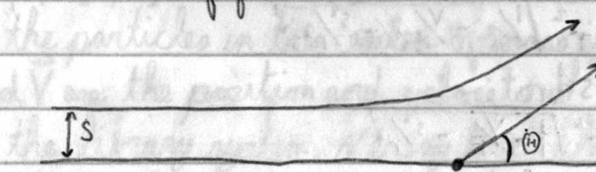
In the one-body formalism, we scatter particles off a center of force. As the particle approaches the center of force, its orbit will deviate from its original trajectory until it passes the center of force whereupon it begins to approximate a straight-line as the force acting on it diminishes. The cross section for scattering in a given direction is given by

$$\sigma(\vec{\Omega})\,d\Omega = \frac{\text{\# particles scattered into solid angle } d\Omega}{\text{time}\cdot\text{intensity of beam}} \qquad (3.88)$$
$$d\Omega = 2\pi\sin\Theta\,d\Theta \qquad (3.89)$$

From the previous sections, the constants of the orbit, which determine the amount of scattering, are determined by energy and angular momentum.

$$l = mv_0 s = s\sqrt{2mE} \qquad (3.90)$$

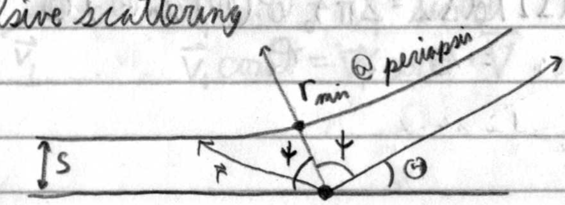
where s is the impact parameter, the perpendicular distance between the center of force and the incident velocity.



The number of particles scattered into the solid angle $d\Omega$ is equal to the number of incident particles with impact parameter between s and s+ds, assuming of course that different values of s cannot lead to the same scattering angle.

$$2\pi I s|ds| = 2\pi\sigma(\Theta) I\sin\Theta|d\Theta| \qquad (3.91)$$
$$= I\cdot\sigma(\Theta)\,d\Omega$$
$$= \text{\# particles scattered into } d\Omega/\text{time}$$
$$s|ds| = \sigma(\Theta)\sin\Theta|d\Theta|$$
$$\sigma(\Theta) = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| \qquad (3.93)$$

For repulsive scattering



$$\Theta = \pi - 2\Psi \qquad (3.94)$$

$$\theta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}} + \theta_0 \quad (3.36)$$

If we set $\theta_0 = \pi$ and $r_0 = \infty$, which corresponds to a particle coming from the left. $\theta = \pi - \psi$ at $r = r_m$

$$\pi - \psi = \int_{\infty}^{r} \frac{dr}{r^2\sqrt{2mE/l^2 - 2mV/l^2 - 1/r^2}} + \pi$$

$$\psi = \int_{r_m}^{\infty} \frac{dr}{r^2\sqrt{2mE/l^2 - 2mV/l^2 - 1/r^2}} \quad (3.95)$$

$$= \int_{r_m}^{\infty} \frac{dr}{r^2\sqrt{2mE/2mEs^2 - 2mV/2mEs^2 - 1/r^2}}$$

$$= \int_{r_m}^{\infty} \frac{dr}{r^2\sqrt{1/s^2 - V/Es^2 - s^2/r^2s^2}}$$

$$= \int_{r_m}^{\infty} \frac{s\,dr}{r^2\sqrt{1 - V/E - s^2/r^2}}$$

$$= \int_{r_m}^{\infty} \frac{s\,dr}{r\sqrt{r^2(1 - V/E) - s^2}}$$

$$\Theta(s) = \pi - 2\int_{r_m}^{\infty} \frac{s\,dr}{r\sqrt{r^2(1 - V/E) - s^2}} \quad (3.96)$$

Letting $u = 1/r$

$$\Theta(s) = \pi + 2\int \frac{s\,du}{u\sqrt{\frac{1}{u^2}(1 - V/E) - s^2}}$$

$$= \pi - 2\int_0^{u_m} \frac{s\,du}{\sqrt{1 - V(u)/E - s^2u^2}} \quad (3.97)$$

If we want to figure out the total cross-section, we simply integrate over all angles

$$\sigma_T = \int \sigma(\vec{\Omega})\, d\Omega = 2\pi \int_0^{\pi} \sigma(\Theta) \sin\Theta\, d\Theta$$

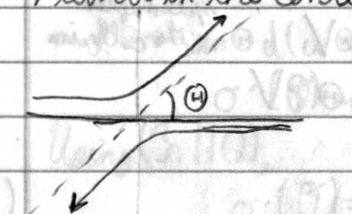
Section 11. Transformation of the Scattering Problem to Laboratory Coordinates

We've looked at the scattering of a particle by a fixed center of force, but what happens when we have one particle scattering off of another. From earlier parts of this chapter, we can reasonably believe that we can reduce the two-body problem to a one-body problem.

We define the following variables, with m_1 being the incident, and m_2 is the target

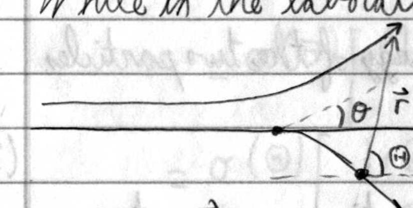
- $\vec{r}_1, \vec{v}_1, \vec{r}_2, \vec{v}_2$ are the position and velocity after scattering of the particles in the laboratory system
- $\vec{r}_1', \vec{v}_1', \vec{r}_2', \vec{v}_2'$ are the position and velocity after scattering of the particles in the center of mass system
- $\vec{R}$ and $\vec{V}$ are the position and velocity of the center of mass in the library system. We say that the center of mass has constant velocity.

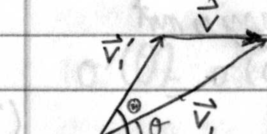
Viewed in the center of mass system, scattering of two particles:



$$\vec{r}_1 = \vec{R} + \vec{r}_1' \qquad \vec{v}_1 = \vec{V} + \vec{v}_1' \quad (3.104)$$

While in the laboratory system





$$\vec{v}_1 \sin\theta = \vec{v}_1' \sin\Theta$$
$$\vec{v}_1 \cos\theta = \vec{v}_1' \cos\Theta + \vec{V} \quad (3.106)$$

In the laboratory system, the target is initially stationary, thus the initial velocity of the particles is solely dependant on the incident velocity of particle 1, v_0. By conservation of momentum:

$$(m_1+m_2)\vec{V} = m_1\vec{v}_0$$
$$\vec{V} = \frac{m_1}{m_1+m_2}\vec{v}_0$$
$$= \frac{\mu}{m_2}\vec{v}_0 \qquad (3.105)$$

From (3.106),

$$\frac{v_1 \sin\theta}{v_1\cos\theta} = \frac{v_1'\sin\Theta}{v_1'\cos\Theta + V}$$
$$\tan\theta = \frac{\sin\Theta}{\cos\Theta + V/v_1'}$$
$$= \frac{\sin\Theta}{\cos\Theta + \mu v_0/m_2 v_1'} = \frac{\sin\Theta}{\cos\Theta + \rho} \qquad (3.107)$$
$$\rho = \frac{\mu}{m_2}\cdot\frac{v_0}{v_1'} = V/v_1' = m_1 v_0/m_2 v \qquad (3.108)$$

v is relative speed after the collision

$$v_1^2 = v_1'^2 + V^2 + 2v_1'V\cos\Theta \qquad (3.109)$$
$$\cos\theta\sqrt{v_1'^2+V^2+2v_1'V\cos\Theta} = v_1'\cos\Theta + V$$
$$\cos\theta\cdot v_1'\sqrt{1+V^2/v_1'^2+2V/v_1'\cos\Theta} = v_1'\cos\Theta + V$$
$$\cos\theta = \frac{\cos\Theta+\rho}{\sqrt{1+\rho^2+2\rho\cos\Theta}} \qquad (3.110)$$

If the collision is elastic, i.e., total kinetic energy of the two particles is conserved $\Rightarrow v = v_0$

$$\rho = m_1/m_2 \qquad (3.111)$$

If the collision is inelastic, i.e., total kinetic energy is not conserved,

$$\tfrac{1}{2}\mu v^2 = \tfrac{1}{2}\mu v_0^2 + Q \qquad (3.112)$$
$$\frac{v^2}{v_0^2} = 1 + \frac{2Q}{\mu v_0^2}$$

$$\frac{v^2}{v_0^2} = 1 + \frac{2Q}{\frac{m_1 m_2}{m_1+m_2}v_0^2}$$
$$= 1 + \frac{Q(m_1+m_2)}{m_2\cdot\frac{1}{2}m_1 v_0^2}$$
$$\frac{v}{v_0} = \sqrt{1+\frac{Q}{E}\cdot\frac{m_1+m_2}{m_2}} \qquad (3.113)$$
$$\rho = \frac{m_1}{m_2\sqrt{1+\frac{m_1+m_2}{m_2}\cdot Q/E}} \qquad (3.114)$$

Note that for $Q=0$, this reduces to the expected ρ for an elastic collision.

The number of particles scattered into a given element of solid angle must be the same whether or not the scattering was measured using in Θ or θ. Using (3.91)

$$2\pi I\sigma(\Theta)\sin\Theta|d\Theta| = 2\pi I\sigma'(\theta)\sin\theta|d\theta|$$
$$\sigma'(\theta) = \sigma(\Theta)\cdot\frac{\sin\Theta}{\sin\theta}\left|\frac{d\Theta}{d\theta}\right| \qquad (3.115)$$

$$\Theta \to \cos\Theta \qquad \theta \to \cos\theta$$
$$d\Theta \to -\sin\Theta\, d(\cos\Theta) \qquad d\theta \to -\sin\theta\, d(\cos\theta)$$
$$\sigma'(\theta) = \sigma(\Theta)\left|\frac{d(\cos(\Theta))}{d(\cos\theta)}\right|$$

Using (3.110),

$$\sigma'(\theta) = \sigma(\Theta)\left[\frac{d}{d\cos\Theta}\left(\frac{\cos\Theta+\rho}{\sqrt{1+2\rho\cos\Theta+\rho^2}}\right)\right]^{-1}$$
$$= \sigma(\Theta)\left[\frac{(1+2\rho\cos\Theta+\rho^2)^{1/2} - \rho(1+2\rho\cos\Theta+\rho^2)^{-1/2}(\cos\Theta+\rho)}{1+2\rho\cos\Theta+\rho^2}\right]^{-1}$$
$$= \sigma(\Theta)\left[\frac{1+2\rho\cos\Theta+\rho^2-\rho\cos\Theta-\rho^2}{(1+2\rho\cos\Theta+\rho^2)^{3/2}}\right]^{-1}$$
$$\sigma'(\theta) = \sigma(\Theta)\cdot\frac{(1+2\rho\cos\Theta+\rho^2)^{3/2}}{1+\rho\cos\Theta} \qquad (3.116)$$

From real world observations, we see particles slow down when they collide with another object, i.e., the collision slows down the incident particle.

$$v_1^2 = v_1'^2 + V^2 + 2v_1' V \cos\Theta \qquad (3.109)$$

$$= \left(\frac{\mu}{m_2}\cdot\frac{v_0}{\rho}\right)^2 + \left(\frac{\mu}{m_2}v_0\right)^2 + 2\cdot\frac{\mu}{m_2}\cdot\frac{v_0}{\rho}\cdot V\cos\Theta$$

$$v_1^2/v_0^2 = \left(\frac{\mu}{m_2\rho}\right)^2\left(1+\rho^2+2\rho\cos\Theta\right) \qquad (3.117)$$

Section 12. The Three-Body Problem

We can solve the two-body problem. If we add another mass, things become much more complex. So much so that the three-body problem cannot be solved in general. The Lagrangian for the three-body problem is:

$$\mathcal{L} = \tfrac{1}{2}\sum_{i=1}^{3} m_i \dot{\vec{r}}_i^{\,2} - V$$

$$V = -\sum_{i<j} G m_i m_j / |\vec{r}_i - \vec{r}_j|$$

From this, we can get the following integrals of motion

- Energy: $E = \tfrac{1}{2}\sum_{i=1}^{3} m_i \dot{\vec{r}}_i^{\,2} + V$
- Total momentum: $\vec{p} = \sum_{i=1}^{3} m_i \dot{\vec{r}}_i = \sum_{i=1}^{3}\vec{p}_i$
- Total angular momentum: $\vec{L} = \sum_{i=1}^{3} \vec{r}_i \times \vec{p}_i$

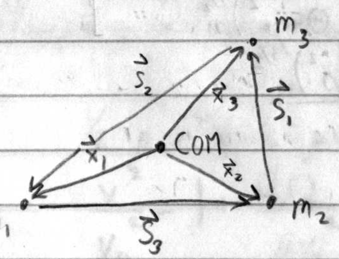
- Looking at the center of mass, Newton's second law gives

$$m_1\ddot{\vec{r}}_1 = \frac{-G m_1 m_2(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3} - \frac{G m_1 m_3(\vec{r}_1-\vec{r}_3)}{|\vec{r}_1-\vec{r}_3|^3} \qquad (3.118)$$

If we make the coordinate transformation to relative-position vectors

$$\vec{s}_i = \vec{r}_j - \vec{r}_k \qquad (3.119)$$

$$\vec{s}_1 + \vec{s}_2 + \vec{s}_3 = 0 \qquad (3.120)$$



$$\ddot{\vec{r}}_1 = -\frac{G m_2 \vec{s}_3}{|\vec{s}_3|^3} + \frac{G m_3 \vec{s}_2}{|\vec{s}_2|^3}$$

$$\ddot{\vec{s}}_1 = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_3 = -\frac{G m_3 \vec{s}_1}{|\vec{s}_1|^3} + \frac{G m_1 \vec{s}_3}{|\vec{s}_3|^3} - \left(\frac{-G m_1 \vec{s}_2}{|\vec{s}_2|^3} + \frac{G m_2 \vec{s}_1}{|\vec{s}_1|^3}\right)$$

$$= \frac{-G m_1 \vec{s}_1}{|\vec{s}_1|^3} - \frac{G m_2 \vec{s}_1}{|\vec{s}_1|^3} - \frac{G m_3 \vec{s}_1}{|\vec{s}_1|^3} + \frac{G m_1 \vec{s}_1}{|\vec{s}_1|^3} + \frac{G m_1 \vec{s}_2}{|\vec{s}_2|^3} + \frac{G m_1 \vec{s}_3}{|\vec{s}_3|^3}$$

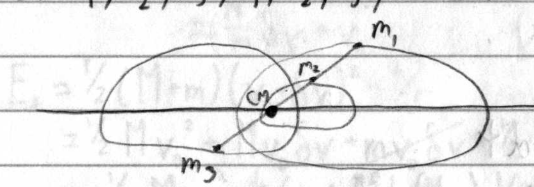
$$= -MG\frac{\vec{s}_1}{s_1^3} + m_1\vec{G}$$

$$\ddot{\vec{s}}_i = -MG\cdot \vec{s}_i/s_i^3 + m_i\vec{G} \qquad (3.121)$$

$$\vec{G} = G\left(\vec{s}_1/s_1^3 + \vec{s}_2/s_2^3 + \vec{s}_3/s_3^3\right) \qquad (3.123)$$

These cannot be solved in general, but solutions do exist for some simple cases.

- Euler's collinear solution: all three masses are collinear, which forces $\vec{r}_1, \vec{r}_2, \vec{s}_3, \vec{s}_1, \vec{s}_2, \vec{s}_3$, and $\vec{G}$ to all be collinear



- Lagrange's three-fold solution: If $\vec{G}=0$, (3.121) reduces to

$$\ddot{\vec{s}}_i = -MG\,\vec{s}_i/s_i^3 \qquad (3.124)$$

which is three two-body problems. This occurs when the three masses are the vertices of an equilateral triangle.

- Probably the most famous three-body problem is the earth-moon-sun system (alternatively, insert joke about relationships?). We can't solve this in a closed-form, but we can calculate equipotential curves where two of the masses are large and bound, and the third is small and only perturbs the motion of the other two

Derivations

1. $\dot{\vec{p}}_i = \vec{F}_i' - \vec{f}_i(\dot{\vec{r}}_i)$

$$G = \sum_i \vec{p}_i \cdot \vec{r}_i$$

$$\frac{dG}{dt} = \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i + \sum_i \vec{p}_i \cdot \dot{\vec{r}}_i$$

$$= \sum_i (\vec{F}_i' - \vec{f}_i) \cdot \vec{r}_i + 2T$$

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}\,dt = \overline{2T} + \overline{\sum_i (\vec{F}_i' - \vec{f}_i)\cdot \vec{r}_i}$$

$$\frac{1}{\tau}[G(\tau) - G(0)] = \overline{2T} + \overline{\sum_i \vec{F}_i' \cdot \vec{r}_i} - \overline{\sum_i \vec{f}_i \cdot \vec{r}_i}$$

$$\vec{p}_i = \vec{F}'\,dt - \vec{f}_i\,dt$$

$$G(\tau) - G(0) = \vec{F}' \cdot \vec{r}\,\big|_\tau + \vec{f}_i \cdot \vec{r}\,\big|_\tau - \vec{F}' \cdot \vec{r}\,\big|_0 - \vec{f}_i \cdot \vec{r}\,\big|_0$$

Since $\vec{F}'$ is conservative, $\vec{F}' \cdot \vec{r}\,\big|_\tau = \vec{F}' \cdot \vec{r}\,\big|_0$

Thus, the left side becomes $-\sum_i \vec{f}_i \cdot \vec{r}_i$,

$$\overline{2T} = -\overline{\sum_i \vec{F}_i' \cdot \vec{r}_i}$$

$$\overline{T} = -\tfrac{1}{2}\overline{\sum_i \vec{F}_i' \cdot \vec{r}_i}$$

2. $\omega t = \psi - \varepsilon \sin\psi$ (3.76)

$$a_n = \tfrac{1}{\pi}\int_0^{2\pi} \varepsilon \sin\psi \cos(n\psi)\,d\psi$$

$$= \frac{1}{\pi} \cdot \frac{n \sin\psi \sin(n\psi) + \cos\psi \cos(n\psi)}{n^2 - 1}\Bigg|_0^{2\pi}$$

3. $\rho = \varepsilon \sin(\omega t + \rho)$

4. $\Theta(s) = \pi - 2\int_{r_m}^{\infty} \dfrac{s\,dr}{r\sqrt{r^2\left(1 - \frac{V(r)}{E}\right) - s^2}}$ (3.96)

$$r = \rho(r)$$

$$dr = d\rho$$

Problems

10. $\varepsilon < 1 \Rightarrow$ ellipse

$\varepsilon = \sqrt{1 + 2El^2/mk^2}$ (3.57)

$$(1-\alpha)^2 = 1 + 2El^2/Mk^2$$

$E = -k/2a$ (3.61)

Conservation of momentum:

$$Mv_0 + mv_c = (M+m)(v_0 + \delta v)$$

$$v_c = \left(\frac{M+m}{m}\right)(v_0 + \delta v) - \frac{M}{m}v_0$$

$$\simeq \frac{M}{m}v_0 + \frac{M}{m}\delta v + v_0 + \delta v - \frac{M}{m}v_0$$

$$\approx \frac{M}{m}\delta v + v_0$$

$$E_f = \tfrac{1}{2}(M+m)(v_0 + \delta v)^2 - k/r$$

$$= \tfrac{1}{2}Mv_0^2 + Mv_0\delta v + \cancel{mv_0\delta v} + \tfrac{1}{2}mv_0^2 - k/r$$

$$= \tfrac{1}{2}Mv_0^2 - k/r + Mv_0\delta v + \tfrac{1}{2}mv_0^2$$

$$= E_0 + Mv_0\delta v + \tfrac{m}{2}v_0^2 = 0$$

$$\delta v = -E_0/Mv_0 - mv_0/2M$$

$$v_c = -E_0/mv_0 - v_0/2 + v_0 = v_0/2 - E_0/mv_0$$

$$KE_c = \tfrac{m}{2}v_c^2 = \tfrac{m}{2}\left(v_0/2 - E_0/mv_0\right)^2$$

$$= \tfrac{m}{2}\left(v_0^2/4 - 2E_0/m + E_0^2/m^2v_0^2\right)$$

$$= mv_0^2/8 - E_0/2 + E_0^2/2mv_0^2$$

$$l = Ma(1+\varepsilon)v_0 = Ma(2-\alpha)v_0$$

$$v_0 = \frac{l}{Ma(2-\alpha)}$$

$$v_0^2 = \frac{l^2}{M^2a^2(2-\alpha)^2} = \frac{aMk(1-\varepsilon^2)}{M^2a^2(2-\alpha)^2}$$

$$= \frac{aMk(1-(1-\alpha)^2)}{M^2 a^2 (2-\alpha)^2} = \frac{aMk(1-1+2\alpha-\alpha^2)}{M^2 a^2 (2-\alpha)^2}$$

$$= \frac{\alpha Mk \alpha (\cancel{2-\alpha})}{M^2 a^2 (2-\alpha)^2} = \frac{\alpha k}{Ma(2-\alpha)} \approx \frac{\alpha k}{2Ma}$$

$$KE = \frac{E_0^2}{2m} \cdot \frac{2M\alpha}{\alpha k} - \frac{E_0}{2} + \frac{m}{8} \cdot \frac{\alpha k}{2Ma}$$

$$= \frac{M\alpha}{m\alpha k} \cdot \frac{k^2}{4a^2} + \frac{1}{2} \cdot \frac{k}{2a} + \frac{m\alpha k}{16 M\alpha}$$

$$\approx \frac{Mk}{4ma\alpha} + \frac{k}{4\alpha} = \frac{k}{4\alpha}\left(\frac{M}{m\alpha}+1\right) \approx \frac{Mk}{4ma\alpha}$$

11. We want to reduce this to a one-body problem, which is a particle moving in a circle with radius a with period τ

$$\mu \ddot{r} - \frac{l^2}{\mu a^3} = f^k/a^2 \qquad (3.12)$$

$$\ddot{r} = 0$$

$$l^2/\mu a^3 = \mu k/a^2$$

$$l^2 = -\mu k/a$$

$$\mu^2 a^4 \dot{\theta}^2 = -\mu k/a \qquad (3.8)$$

$$\dot{\theta}^2 = k/\mu a^3$$

$$\dot{\theta} = \sqrt{k/\mu a^3} = 2\pi/\tau$$

When the particles are stopped, they have only potential energy

$$-k/a = \mu/2\, \dot{r}^2 - k/r$$

Conservation of energy

$$\dot{r}^2 = \frac{2}{\mu}\left(\frac{k}{r} - \frac{k}{a}\right)$$

$$= 2k/\mu \left(\frac{a-r}{ra}\right)$$

$$\frac{dr}{dt} = \sqrt{\frac{2k}{\mu}\left(\frac{a-r}{ra}\right)}$$

$$\int_a^0 \frac{dr}{\sqrt{\frac{a-r}{ar}}} = \sqrt{\frac{2k}{\mu}}\, t$$

$$\sqrt{a}\left(a \tan^{-1}\left(\sqrt{\frac{r}{a-r}}\right) + \sqrt{\frac{r}{a-r}}\,(r-a)\right)\Bigg|_a^0 = \sqrt{\frac{2k}{\mu}}\, t$$

$$\sqrt{a}\,(a(\tan^{-1}(0) - \tan^{-1}(\infty)) = \sqrt{\tfrac{2k}{\mu}}\, t$$

$$-\pi a^{3/2}/2 = \sqrt{\tfrac{2k}{\mu}}\, t$$

$$t = \sqrt{\frac{\mu a^3}{2k}} \cdot \frac{\pi}{2}$$

$$\tau = 2\pi \sqrt{\frac{\mu a^3}{k}}$$

$$\frac{\tau}{4\sqrt{2}} = \frac{\pi}{2}\sqrt{\frac{\mu a^3}{2k}} = t$$

12. Using a form of Gauss law, we can convince ourselves that only the particles within a sphere of radius r contribute, which leads to

$$T = -\tfrac{1}{2}\int_0^R f(r) \cdot 4\pi r^2 \cdot \rho(r) \cdot dr$$

$$= +\frac{1}{2}\int_0^R \frac{mk}{r^{m+1}} \cdot \frac{4\pi r^3}{3} \cdot \frac{N}{4\pi R^3/3}\, exp(-k/r^m \cdot 1/kT)\, dr$$

$$= \frac{-mkN}{2R^3} \cdot \int_0^R \frac{1}{r^{m-2}}\, exp(-1/T r^m)\, dr$$

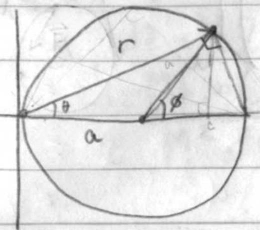
$$u = r^{-m/2}$$

$$du = \frac{-m}{2} r^{-m/2-1} dr$$

Not sure how to evaluate this integral

13.

a.



$\cos\theta = r/2a$

$r = 2a\cos\theta$

$\varnothing = 2\theta$

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} = f(r) \qquad (3.33)$$

$$\frac{d(2a\cos\theta)}{d\theta} = -2a\sin\theta$$

$$\frac{d}{d\theta}\left(\frac{-2la\sin\theta}{4ma^2\cos^2\theta}\right) = \frac{d}{d\theta}\left(\frac{-l\sin\theta}{2ma\cos^2\theta}\right)$$

$$= \frac{-l}{2ma}\cdot\frac{\cos^3\theta + 2\cos\theta\sin^2\theta}{\cos^4\theta}$$

$$= \frac{-l}{2ma}\cdot\frac{\cos^2\theta + \sin^2\theta + \sin^2\theta}{\cos^3\theta}$$

$$= \frac{-l}{2ma}\cdot\frac{1+\sin^2\theta}{\cos^3\theta} = \frac{-l}{2ma}\cdot\frac{1+(1-\cos^2\theta)}{\cos^3\theta}$$

$$= \frac{-l}{2ma}\cdot\frac{2-(r/2a)^2}{(r/2a)^3} = \frac{-l}{2ma}\cdot\frac{8a^3}{r^3}\cdot\left(\frac{8a^2-r^2}{4a^2}\right)$$

$$= \frac{-l(8a^2-r^2)}{mr^3}$$

$$f(r) = \frac{-l^2(8a^2-r^2)}{mr^5} - \frac{l^2}{mr^3}$$

$$= \frac{-8l^2a^2}{mr^5} \propto r^{-5}$$

b. $-\frac{dU}{dr} = F$

$$U = \frac{-8l^2a^2}{4mr^4} = \frac{-2l^2a^2}{mr^4}$$

$$E = V(r_0) + l^2/2mr_0^2 \qquad (3.41)$$

let $r_0 = 2a$

$$E = \frac{-2l^2a^2}{m2^4a^4} + \frac{l^2}{2m\cdot2^2a^2} = \frac{-l^2}{m\cdot2^3a^2} + \frac{l^2}{m2^3a^2} = 0$$

c. $\frac{dr}{d\theta} = -2a\sin\theta$

$$\frac{dr}{dt}\cdot\frac{dt}{d\theta} = -2a\sin\theta$$

$$\frac{dr}{dt} = -2a\sin\theta\cdot\dot\theta$$

$$\dot\theta = l/mr^2 \qquad (3.8)$$

$$\frac{dr}{dt} = \frac{-2al\sqrt{1-\cos^2\theta}}{mr^2}$$

$$= \frac{-2al}{mr^2}\sqrt{1-r^2/4a^2} = \frac{-2al}{mr^2}\frac{\sqrt{4a^2-r^2}}{2a} = \frac{-l\sqrt{4a^2-r^2}}{mr^2}$$

$$dt = \frac{-mr^2}{l\sqrt{4a^2-r^2}}\,dr$$

$$t = \frac{-m}{l}\left[\frac{4a^2\sin^{-1}(r/2a)}{2} - \frac{2ar\sqrt{1-r^2/4a^2}}{2}\right]\Bigg|_0^{2a}$$

$$= -m/l\left[2a^2\sin^{-1}(1) - 2a^2\sin^{-1}(0)\right]$$

$$= -2ma^2/l\cdot\pi/2 = -m\pi a^2/l$$

Need to multiply by 2 to get the full period

$$T = \frac{2m\pi a^2}{l}$$

d. $x = a + a\cos\phi = a + a\cos(2\theta)$

$y = a\sin\phi = a\sin(2\theta)$

$\dot{x} = -a\sin(2\theta)\cdot 2\dot{\theta}$

$= -4a\dot{\theta}\sin\theta\cos\theta$

$\dot{y} = a\cos(2\theta)\cdot 2\dot{\theta}$

$= 2a\dot{\theta}(2\cos^2\theta - 1)$

$v = \sqrt{\dot{x}^2 + \dot{y}^2} = 4\dot{\theta}^2 a^2$

In terms of r,

$\dot{x} = -\frac{4al}{mr^2}\cdot\sin\theta\cos\theta$

$= -\frac{4al}{mr^2}\cdot\frac{\sqrt{4a^2 - r^2}}{2a}\cdot\frac{r}{2a}$

$= -\frac{l\sqrt{4a^2 - r^2}}{mar}$

$\dot{y} = \frac{2a\cdot l^2}{mr^2}\cdot\left(\frac{2\cdot r^2}{4a^2} - 1\right)$

$= \frac{4l^2}{ma^2} - \frac{2al^2}{mr^2}$

We can easily convince ourselves that since $\dot{x} + \dot{y} \to \infty$ as $r \to 0$, v must also go to infinity.

14.

a. For a circle, $E = l^2/2mr^2 - k/r_0 = -k/2r_0$ (3.58)

$l^2/2mr_0^2 = k/2r_0$

$l^2 = kmr_0$

$r_0 = l^2/km$ (3.59)

For a parabola, $E = 0 = l^2/2mr_0^2 - k/r$

$l^2/2mr_0^2 = k/r_0$

$l^2/2mk = r_0$

b. $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$

For a circle $\dot{r} = 0$

$E = -k/2r = m/2\,(r^2\dot{\theta}^2) - k/r$

$k/2r = m/2\,(r^2\dot{\theta}^2)$

$k/mr = v^2$

For a parabola, $E = 0$

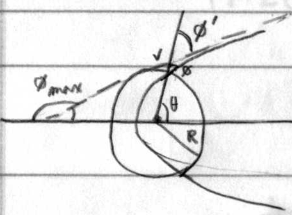
$0 = m/2\cdot v^2 - k/r$

$k/r = m/2\, v^2$

$2k/mr = v^2$

Particle in a parabolic orbit moves at $\sqrt{2}$ times the speed in a circular orbit.

15.



Far away, $E = mv'^2/2$

Just before striking the earth, $E = mv^2/2 - k/R$

$mv'^2/2 = mv^2/2 - k/R$

$v'^2 = v^2 - 2k/mR$

$\phi' = \phi_{max} + \theta - \pi$

$r = l^2/mk\,(1 + \varepsilon\cos(\phi_{max}))^{-1} = \infty$ (3.55)

$1 + \varepsilon\cos(\phi_{max}) = 0$

$\cos(\phi_{max}) = -1/\varepsilon$

$\varepsilon = \sqrt{1 + 2El^2/mk^2}$

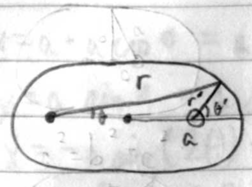
$E = mv^2/2 - k/R$

$l = mvR\sin\phi$

$\phi' = \theta - \pi + \cos^{-1}(-1/\varepsilon)$

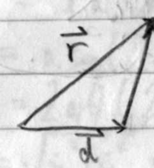
$= \theta - \pi + \cos^{-1}\left(\frac{-1}{\sqrt{1 + \frac{2}{mk^2}(mv^2/2 - k/R)(mvR\sin\phi)^2}}\right)$

16.



$r+r'=c$

$r=\frac{a(1-\varepsilon^2)}{1+\varepsilon\cos(\theta)}$ (3.64)



$\vec r=\vec d+\vec r'$

$r^2=d^2+r'^2-2r'd\cos\theta'$

$=(c-r')^2$

$r'=\frac{a(1-\varepsilon^2)}{1-\varepsilon\cos\theta'}$

$r+r'=c$ $\quad a(1-\varepsilon^2)=\alpha$

$\frac{dr}{dt}+\frac{dr'}{dt}=0$

$\frac{+\varepsilon\dot\theta\sin\theta\,\alpha}{(1+\varepsilon\cos\theta)^2}+\frac{-\varepsilon\dot\theta'\sin\theta'\,\alpha}{(1-\varepsilon\cos\theta')^2}=0$

$\frac{\dot\theta\sin\theta}{(1+\varepsilon\cos\theta)^2}=\frac{\dot\theta'\sin\theta'}{(1-\varepsilon\cos\theta')^2}$

$r\sin\theta=r'\sin\theta'$

$\frac{\sin\theta}{1+\varepsilon\cos\theta}=\frac{\sin\theta'}{1-\varepsilon\cos\theta'}$

$\dot\theta'=\frac{(1-\varepsilon\cos\theta')^2}{\sin\theta'}\cdot\dot\theta\cdot\frac{\sin\theta'}{(1+\varepsilon\cos\theta)(1-\varepsilon\cos\theta')}$

$=\dot\theta\cdot\frac{1-\varepsilon\cos\theta'}{1+\varepsilon\cos\theta}$

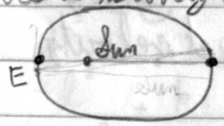
$\approx\dot\theta(1-\varepsilon\cos\theta')(1-\varepsilon\cos\theta)$

$\approx\dot\theta(1-\varepsilon(\cos\theta+\cos\theta'))$

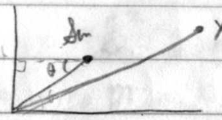
$=\ell/mr^2(1-\varepsilon(\cos\theta+\cos\theta'))$

For small ε, $\dot\theta'$ is constant

17. See "Discovery of Pluto"



$r=\frac{a(1-\varepsilon^2)}{1+\varepsilon\cos(\theta)}$



In the time it takes earth to move θ, how far does planet X move?

Using Kepler's second law

$dA=\frac{1}{2}\cdot r^2 d\theta$

For earth, $dA=\frac{1}{2}\cdot(a(1-\varepsilon))^2\theta$

For X, $dA=\frac{1}{2}\cdot(a(1+\varepsilon))^2\cdot\theta'$

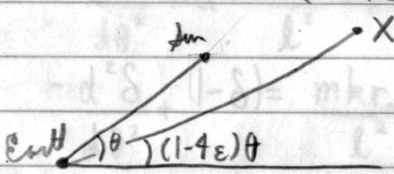
$(1-\varepsilon)^2\theta=(1+\varepsilon)^2\theta'$

$(1-2\varepsilon+\varepsilon^2)\theta=(1+2\varepsilon+\varepsilon^2)\theta'$

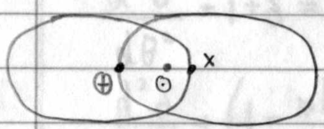
$\theta'=\frac{1-2\varepsilon}{1+2\varepsilon}\theta$

$\approx(1-2\varepsilon)^2\theta$

$\theta'\approx(1-4\varepsilon)\theta$



$\theta-(1-4\varepsilon)\theta=4\varepsilon\cdot\theta$



In this case, when earth moves θ, planet X also moves θ

18. At perihelion, we expect earth to have the largest velocity while at aphelion, earth should have the smallest velocity.

$$r_p = a(1-\varepsilon)$$
$$r_a = a(1+\varepsilon)$$
$$\frac{r_p}{r_a} = \frac{(1-\varepsilon)}{(1+\varepsilon)}$$

Angular momentum is conserved,

$$m_\oplus v_p r_p = m_\oplus v_a r_a$$
$$\frac{v_p}{v_a} = \frac{r_a}{r_p} = \frac{1+\varepsilon}{1-\varepsilon} = 1.047$$

19.

a. $\mathcal{L} = T - V = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2) + \frac{k}{r}\cdot\exp(-r/a)$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot r}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0$$
$$\frac{d}{dt}(m\dot r) - \left(mr\dot\theta^2 - \frac{k}{r^2}\exp(-r/a) - \frac{k}{ar}\exp(-r/a)\right) = 0$$
$$m\ddot r = mr\dot\theta^2 - \frac{k}{r}\exp(-r/a)\left(\frac{1}{r} + \frac{1}{a}\right)$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$
$$\frac{d}{dt}(mr^2\dot\theta) = 0$$
$$2mr\dot r\dot\theta + mr^2\ddot\theta = 0$$
$$r\ddot\theta = -2\dot r\dot\theta$$

We also see $l = mr^2\dot\theta$

$$\dot\theta = l/mr^2$$
$$m\ddot r = \frac{l^2}{mr^3} - \frac{k}{r}\left(\frac{1}{r} + \frac{1}{a}\right)\exp(-r/a)$$

$$E = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2) + V(r)$$
$$= \frac{m}{2}\dot r^2 + \frac{l^2}{2mr^2} + V(r)$$

$$\left.\begin{aligned} E &= \tfrac{1}{2}m\dot r^2 + V' \\ V' &= \frac{l^2}{2mr^2} - \frac{k}{r}\cdot\exp(-r/a)\end{aligned}\right\rangle \text{ equivalent one-dimensional problem}$$

b. $\frac{\partial V'}{\partial r} = 0$ condition for a circular orbit

$$\frac{-l^2}{mr^3} + \frac{k}{r^2}\exp(-r/a) + \frac{k}{ra}\exp(-r/a) = 0$$
$$\frac{l^2}{m} = k\exp(-r_0/a)(r_0 + r_0^2/a)$$

For small deviation in r, i.e. $r = r_0(1+\delta)$

$$\frac{l}{r^2}\cdot\frac{d}{d\theta}\left(\frac{l}{mr^2}\cdot\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} = f(r) \qquad (3.33)$$
$$f(r) = \frac{-dV}{dr} = -k\exp(-r/a)(1/r^2 + 1/ar)$$
$$u = 1/r = \frac{1}{r_0(1+\delta)}\cdot\frac{1-\delta}{1-\delta} = \frac{1-\delta}{r_0} \quad \text{keeping first order } \delta$$
$$du = -\frac{1}{r^2}dr$$
$$\frac{-l^2u^2}{m}\cdot\frac{d^2u}{d\theta^2} - \frac{l^2u^3}{m} = -k\exp(-1/au)(u^2 + u/a)$$
$$\frac{d^2u}{d\theta^2} + u = \frac{mk}{l^2}\exp(-1/au)(1 + 1/au)$$
$$-\frac{d^2\delta}{d\theta^2} + (1-\delta) = \frac{mkr_0}{l^2}\exp(-r_0/a)\left(\frac{(1-\delta)^2}{r_0^2} + \frac{1-\delta}{ar_0}\right)$$
$$\frac{d^2\delta}{d\theta^2} - 1 + \delta = \frac{mk\exp(-r_0/a)}{l^2}\left(\frac{1}{r_0} - \frac{2\delta}{r_0} + \frac{r_0}{ar_0} - \frac{\delta r_0}{ar_0}\right)$$
$$\frac{d^2\delta}{d\theta^2} + \left(1 - \frac{mkr_0^2\exp(-r_0/a)}{l^2 a}\right)\delta = 1 - \frac{mk}{l^2}r_0\exp(-r_0/a)$$
$$\omega = \frac{1}{1 + r_0/a}$$
$$\theta = \frac{2\pi}{\omega} = 2\pi(1 + r_0/a)$$

20.

a. The total force is $\vec{F} = -mC\vec{r} - \frac{GMm}{r^2}\hat{r}$

$$V = \tfrac{1}{2}mCr^2 - \frac{GMm}{r}$$

$$E = V(r_0) + l^2/2mr_0^2 \quad (3.41)$$

For a circular orbit, $E = V/2$. Note that we can't use (3.58) since that applies to the system without the additional dust term.

$$\frac{V}{2} = V + l^2/2mr_0^2$$

$$-\frac{V}{2} = l^2/2mr_0^2$$

$$l = mr^2\dot{\theta} \quad (3.8)$$

$$V = -l^2/mr^2$$

$$= -mr^2\dot{\theta}^2$$

$$\dot{\theta}^2 = -V/mr^2$$

$$= \frac{GM}{r_0^3} - \frac{C}{2}$$

$$\dot{\theta} = \sqrt{\frac{GM}{r_0^3} - \frac{C}{2}}$$

$$\tau = 2\pi/\dot{\theta}$$

$$= \frac{2\pi}{\sqrt{GM/r_0^3 - C/2}}$$

b.

$$u = u_0 + a\cos(\beta\theta) \quad (3.45)$$

$$\omega = \dot{\theta}\beta$$

$$\beta^2 = 3 + \frac{r}{f}\cdot\frac{df}{dr}\Big|_{r=r_0} \quad (3.46)$$

$$\frac{df}{dr} = -mC + \frac{2GMm}{r^3}$$

$$\beta^2 = 3 + \frac{r}{-mCr - \frac{GMm}{r^2}}\left(-mC + \frac{2GMm}{r^3}\right)$$

$$= 3 + \frac{r^3}{-mCr^3 - GMm}\cdot\frac{-mCr^3 + 2GMm}{r^3}$$

$$= \frac{-3Cr^3 - 3GM - Cr^3 + 2GM}{-Cr^3 - GM}$$

$$\beta = \sqrt{\frac{4Cr_0^3 + GM}{Cr_0^3 + GM}}$$

$$\omega = \sqrt{\frac{GM}{r_0^3} - \frac{C}{2}}\cdot\sqrt{\frac{4Cr_0^3 + GM}{Cr_0^3 + GM}}$$

c. $\tau_{osc} = \tau/\beta$

$\Delta\theta = 2\pi - 2\pi/\beta$ change in angle every oscillation

$$\tau_{prec} = \frac{2\pi}{\Delta\theta}\tau_{osc}$$

$$= \frac{2\pi\tau}{2\pi(1 - 1/\beta)\cdot\beta} = \frac{\tau}{\beta - 1} = \frac{2\pi}{\sqrt{GM/r_0^3 - C/2}}\cdot\frac{1}{\sqrt{\frac{4Cr_0^3 + GM}{Cr_0^3 + GM}} - 1}$$

$$= \frac{2\pi}{\sqrt{\frac{2GM - Cr_0^3}{2r_0^3}}\left(\sqrt{\frac{4Cr_0^3 + GM}{Cr_0^3 + GM}} - 1\right)}$$

Since $G \gg C$, $\approx 2\pi$

21. $V(r) = -k/r + h/r^2$

$$= -ku + hu^2$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{d}{du}V(u) \quad (3.34)$$

$$= -\frac{m}{l^2}(-k + 2hu)$$

$$\frac{d^2u}{d\theta^2} + u(1 + 2hm/l^2) = mk/l^2$$

$$u = \frac{mk}{l^2\beta^2}(1 + \varepsilon\cos\beta(\theta - \theta'))$$

$$\beta^2 = 1 + 2hm/l^2$$

$$-\frac{mk\varepsilon}{l^2\beta^2}\cdot\beta^2\cos\beta(\theta - \theta') + \frac{mk\beta^2}{l^2\beta^2}(1 + \varepsilon\cos\beta(\theta - \theta')) = \frac{mk}{l^2}$$

$$mk/l^2 = mk/l^2$$

Compare to 3.56. β accounts for the precession

In order to return back to the original state, $\beta(\theta-\theta')=2\pi$

$$\theta-\theta' = \frac{2\pi}{\sqrt{1+2hm/\ell^2}}$$

After rotating by 2π, $\theta-\theta' = 2\pi - \dot{\Omega}\tau$

$$\frac{2\pi}{\sqrt{1+2hm/\ell^2}} = 2\pi - \dot{\Omega}\tau$$

$$\dot{\Omega} = \frac{2\pi\sqrt{1+2hm/\ell^2} - 2\pi}{\tau\sqrt{1+2hm/\ell^2}}$$

$$\approx \frac{2\pi}{\tau} - \frac{2\pi}{\tau}\left(1+2hm/\ell^2\right)^{-1/2}$$

$$\approx 2\pi/\tau - 2\pi/\tau\,(1-hm/\ell^2)$$

$$= 2\pi hm/\tau\ell^2$$

$$\ell^2 = amk(1-\varepsilon^2) \qquad (3.63)$$

$$\dot{\Omega} = \frac{2\pi hm}{\tau\, amk(1-\varepsilon^2)}$$

$$= \frac{2\pi\eta}{\tau(1-\varepsilon^2)}$$

$$\eta = \frac{\dot{\Omega}\tau(1-\varepsilon^2)}{2\pi}$$

22. Looking at figure 3.3,

$$mr^2\dot{\theta} = \ell \qquad (3.8)$$

is conserved, thus $\dot{\theta}$ must also shift.

23. Assuming circular orbit,

$$\tau = 2\pi r^{3/2}\sqrt{m/k} \qquad (3.71)$$

$$= 2\pi r^{3/2}/\sqrt{GM}$$

$$M = \frac{4\pi^2 r^3}{G\tau^2}$$

$$\frac{M_\odot}{M_\oplus} = \frac{r_{\odot\oplus}^3}{\tau_\oplus^2}\cdot\frac{\tau_{☽}^2}{r_{\oplus☽}^3} = \left(\frac{1.49\times10^8}{3.8\times10^5}\right)^3\cdot\left(\frac{27.3}{365}\right)^2 \approx 3.4\times10^5$$

$$M_\odot = 2\times10^{30}\,kg$$

$$M_\oplus = 6\times10^{24}\,kg$$

24. On pg. 96, $\vec{V}_{\parallel} = v_r\hat{r} + v_\theta\hat{\theta}$ where $v_r = \dot{r}$, $v_\theta = r\dot{\theta}$. Since we only care about the radial component, which is given in Goldstein as

$$v_r = \frac{\varepsilon v_0 \sin\theta}{(1-\varepsilon^2)}$$

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta} \qquad (3.64)$$

$$\varepsilon\cos\theta = \frac{a(1-\varepsilon^2)-r}{r}$$

$$\varepsilon^2\cos^2\theta = \frac{(a(1-\varepsilon^2)-r)^2}{r^2}$$

$$\varepsilon^2-\varepsilon^2\sin^2\theta = \frac{(a(1-\varepsilon^2)-r)^2}{r^2}$$

$$\varepsilon\sin\theta = \sqrt{\frac{\varepsilon^2r^2-(a^2(1-\varepsilon^2)^2-2ar(1-\varepsilon^2)+r^2)}{r^2}}$$

$$= \sqrt{\frac{-(1-\varepsilon^2)r^2-a^2(1-\varepsilon^2)^2+2ar(1-\varepsilon^2)}{r^2}}$$

$$= \frac{\sqrt{1-\varepsilon^2}}{r}\cdot\sqrt{-r^2-a^2(1-\varepsilon^2)+2ar} = \frac{\sqrt{1-\varepsilon^2}}{r}\cdot\sqrt{a^2\varepsilon^2-a^2+2ar-r^2}$$

$$= \frac{\sqrt{1-\varepsilon^2}}{r}\cdot\sqrt{a^2\varepsilon^2-(r-a)^2}$$

$$v_r = \frac{v_0}{\sqrt{1-\varepsilon^2}\cdot r}\cdot\sqrt{a^2\varepsilon^2-(r-a)^2}$$

$$\dot{r} = \frac{\omega a}{r}\sqrt{a^2\varepsilon^2-(r-a)^2}$$

$$r = a(1-\varepsilon\cos\psi) \qquad (3.68)$$

$$\dot{r} = a\varepsilon\dot{\psi}\sin\psi$$

$$a\varepsilon\dot{\psi}\sin\psi = \frac{\omega a}{a(1-\varepsilon\cos\psi)}\sqrt{a^2\varepsilon^2-(a(1-\varepsilon\cos\psi)-a)^2}$$

$$= \frac{\omega}{1-\varepsilon\cos\psi}\sqrt{a^2\varepsilon^2-(a-a\varepsilon\cos\psi-a)^2}$$

$$= \frac{\omega}{1-\varepsilon\cos\psi}\sqrt{a^2\varepsilon^2-a^2\varepsilon^2\cos^2\psi}$$

$$\cancel{a}\varepsilon\dot{\psi}\cancel{\sin\psi} = \frac{\omega\cancel{a\varepsilon\sin\psi}}{1-\varepsilon\cos\psi}$$

$$\dot{\psi} = \frac{\omega}{1-\varepsilon\cos\psi}$$

$$(1-\varepsilon\cos\psi)\,d\psi = \omega\,dt$$

$$\psi - \varepsilon\sin\psi = \omega t \qquad (3.76)$$

26. $\omega t = \psi - \varepsilon\sin\psi$ (3.76)

$$2\pi/4 = \psi - \varepsilon\sin\psi$$

27. $$V = \frac{GM}{r} = \frac{G\cdot\frac{4}{3}\pi r^3\rho}{r} = \frac{4G\pi r^2\rho}{3}$$

$$= \tfrac{4\pi}{3}\left(6.6\times10^{-6}\,\tfrac{cm^3}{g\,s^2}\right)\cdot\left(6.1\times10^{8}\,cm\right)^2\cdot 5\,\tfrac{g}{cm^3}$$

$$= 5.14\times10^{13}\,\tfrac{cm^2}{s^2}$$

28. $\vec{F} = q[\vec{E}+(\vec{v}\times\vec{B})]$ (1.60)

a. $\vec{E} = 0$

$$\vec{B} = b\vec{r}/r^3$$

$$\vec{F} = -k\vec{r}/r^3 + \tfrac{qb}{r^3}(\vec{v}\times\vec{r})$$

$$\vec{r}\times\dot{\vec{p}} = \vec{r}\times\vec{F} = -\tfrac{k}{r^3}(\vec{r}\times\vec{r}) + \tfrac{qb}{r^3}(\vec{r}\times\vec{v}\times\vec{r})$$

$$= \tfrac{qb}{r^3}[\vec{v}(\vec{r}\cdot\vec{r}) - \vec{r}(\vec{r}\cdot\vec{v})]$$

$$\vec{L} = \vec{r}\times\vec{p}$$

$$\frac{d\vec{L}}{dt} = \dot{\vec{r}}\times\vec{p} + \vec{r}\times\dot{\vec{p}}$$

$$\dot{\vec{r}}\times\vec{p} = \vec{v}\times m\vec{v} = 0$$

$$\frac{d\vec{L}}{dt} \neq 0$$

$$\frac{d\vec{D}}{dt} = \frac{d\vec{L}}{dt} - \frac{d}{dt}\left(\frac{qb}{c}\frac{\vec{r}}{r}\right) = \frac{d\vec{L}}{dt} - \frac{qb}{c}\cdot\frac{\dot{\vec{r}}r - \dot{r}\vec{r}}{r^2}$$

$$= \frac{d\vec{L}}{dt} - \frac{qb}{c}\left(\frac{\dot{\vec{r}}}{r} - \frac{\vec{r}(\vec{r}\cdot\vec{v})}{r^2}\right)$$

$$= qb\left(\dot{\vec{r}}/r - \vec{r}(\vec{r}\cdot\vec{v})/r^2\right) - qb\left(\dot{\vec{r}}/r - \vec{r}(\vec{r}\cdot\vec{v})/r^2\right) = 0$$

The factor of c comes from mixing Gaussian and SI units

b. $\dot{\vec{p}} = f(r)\cdot\vec{r}/r + \tfrac{qb}{r^3}(\vec{v}\times\vec{r})$ (3.79)

$$\dot{\vec{p}}\times\vec{D} = \left[\frac{f(r)\vec{r}}{r} + \frac{qb(\vec{v}\times\vec{r})}{r^3}\right]\times\left(\vec{r}\times\vec{p} - \frac{qb\vec{r}}{r}\right)$$

$$= \frac{f(r)\vec{r}\times(\vec{r}\times\vec{p})}{r} - \cancel{\frac{qbf(r)\vec{r}\times\vec{r}}{r^2}} + \cancel{\frac{qb(\vec{v}\times\vec{r})\times(\vec{r}\times\vec{p})}{r^3}} - \frac{q^2b^2(\vec{v}\times\vec{r})\times\vec{r}}{r^4}$$

$$= \frac{f(r)m}{r}\vec{r}\times(\vec{r}\times\vec{v}) - \frac{q^2b^2}{r^4}\vec{r}\times(\vec{r}\times\vec{v})$$

$$= \left(mf(r)/r - q^2b^2/r^4\right)\vec{r}\times(\vec{r}\times\vec{v})$$

$$= \left(mf(r)/r - q^2b^2/r^4\right)\cdot -r^3\frac{d}{dt}(\vec{r}/r)$$

$$= -\left(mf(r) - q^2b^2/r^3\right)r^2\frac{d}{dt}(\vec{r}/r)$$

$\frac{d}{dt}(\vec{p}\times\vec{D}) = \dot{\vec{p}}\times\vec{D}$
$= -(mf(r) - \ell^2 b^2/r^3) r^2 \frac{d}{dt}(\vec{r}/r)$

Compare to (3.81) and (3.82)

29. $\vec{L} = \vec{r}\times\vec{p}$
$\vec{A} = \vec{p}\times\vec{L} - mk\,\vec{r}/r$ (3.82)
$\vec{L}\times\vec{A} = \vec{L}\times(\vec{p}\times\vec{L}) - {}^{mk}/_r\,(\vec{r}\times\vec{p})\times\vec{r}$
$= \vec{p}(\vec{L}\cdot\vec{L}) - \cancel{\vec{L}(\vec{L}\cdot\vec{p})} + {}^{mk}/_r(\vec{p}\,r^2 - \cancel{\vec{r}(\vec{r}\cdot\vec{p})})$
$= \vec{p}\,\ell^2 + mkr\vec{p}$
$-\ell A\hat{y} = \vec{p}\,\ell^2 - mk\ell\hat{\theta}$
$\vec{p} + \frac{A}{\ell}\hat{y} = \frac{mk}{\ell}\hat{\theta}$
displaced by A/ℓ and circle of radius mk/ℓ

30. $\sigma(\Theta) = \frac{1}{4}\left(ZZ'e^2/2E\right)^2 \csc^4(\Theta/2)$ (3.102)
$\frac{N(6^\circ)}{N(4^\circ)} = \frac{\csc^4(6/2)}{\csc^4(4/2)}$
$N(6^\circ) = \frac{\sin^4(2)}{\sin^4(3)}\cdot N(4^\circ) \approx 10^6\cdot(2/3)^4 = 2\times10^5$ particles

31. $\Theta(s) = \pi - 2\int_{r_m}^{\infty}\frac{s\,dr}{r\sqrt{r^2(1-V(r)/E) - s^2}}$ (3.96)
$f = kr^{-3}$
$V = \frac{k}{2}\cdot r^{-2}$
$\sigma(\Theta) = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right|$ (3.93)
$\Theta(s) = \pi - 2\int_{r_m}^{\infty}\frac{s\,dr}{r\sqrt{r^2 - k/2E - s^2}}$
Conservation of energy
$E = \frac{1}{2}m(\dot{r}_m^2 + r_m^2\dot{\theta}^2) + k/2r_m^2$
$= \frac{m}{2}\cdot r_m^2\dot{\theta}^2 + k/2r_m^2$

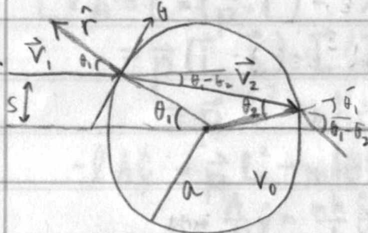
$= \frac{mr_m^2}{2}\cdot\frac{\ell^2}{m^2 r_m^4} + k/2r_m^2$
$E = \ell^2/2mr_m^2 + k/2r_m^2 = \frac{1}{r_m^2}(\ell^2/2m + k/2)$
$r_m^2 = \ell^2/2mE + k/2E$
$= s^2 + k/2E$ (3.90)
$\Theta(s) = \pi - 2\int_{r_m}^{\infty}\frac{s\,dr}{r\sqrt{r^2 - r_m^2}} = \pi - 2s\int_{r_m}^{\infty}\frac{dr}{r\sqrt{r^2 - r_m^2}}$
$= \pi - 2s\cdot\left.\frac{\arctan\left(\frac{\sqrt{r^2-r_m^2}}{r_m}\right)}{r_m}\right|_{r_m}^{\infty}$
$= \pi - {}^{2s}/_{r_m}(\pi/2 - 0) = \pi - {}^{s\pi}/_{r_m}$
$= \pi\left(1 - \frac{s}{\sqrt{s^2+k/2E}}\right)$
$x = \Theta/\pi = 1 - \frac{s}{\sqrt{s^2+k/2E}}$
$\frac{s}{\sqrt{s^2+k/2E}} = 1 - x$
$s^2 = (1-x)^2(s^2 + k/2E)$
$= s^2(1-x)^2 + k/2E\,(1-x)^2$
$s^2(1-(1-x)^2) = k/2E\,(1-x)^2$
$s^2 = \frac{k(1-x)^2}{2E(2x-x^2)}$
$s = \sqrt{\frac{k}{2E}}\cdot\frac{1-x}{\sqrt{2x-x^2}}$
$\frac{ds}{d\Theta} = \left(\frac{d\Theta}{ds}\right)^{-1} = \left[\frac{-\pi\sqrt{s^2+k/2E} + s(s^2+k/2E)^{-1/2}\cdot s\pi}{(s^2+k/2E)}\right]^{-1}$
$= \left(\frac{-\pi(s^2+k/2E) + s^2\pi}{(s^2+k/2E)^{3/2}}\right)^{-1} = \left(\frac{-\pi k/2E}{(s^2+k/2E)^{3/2}}\right)^{-1}$
$= \frac{-2E}{\pi k}\left(s^2 + \frac{k}{2E}\right)^{3/2}$
$= \frac{-2E}{\pi k}\left(\frac{k}{2E}\cdot\frac{x^2-2x+1+2x-x^2}{2x-x^2}\right)^{3/2} = \frac{-2E}{\pi k}\left(\frac{k}{2E}\cdot\frac{1}{2x-x^2}\right)^{3/2}$
$\pi\,dx = d\Theta$

$$\sigma(\Theta)\,d\Theta = \sqrt{\frac{k}{2E}}\cdot\frac{1-x}{\sqrt{2x-x^2}}\cdot\frac{1}{\sin(x\pi)}\cdot\frac{-2E}{\pi k}\cdot\left(\frac{k}{2E}\cdot\frac{1}{2x-x^2}\right)^{3/2}\cdot\pi\,dx$$

$$= \left(\frac{k}{2E}\right)\cancel{\left(\frac{k}{2E}\right)}\cdot\frac{(1-x)}{\sin(x\pi)}\cdot\frac{\cancel{2E}}{\cancel{k}}\cdot\frac{1}{(2x-x^2)^2}\cdot dx$$

$$= \frac{k}{2E}\cdot\frac{(1-x)\,dx}{x^2(2-x)^2\sin(x\pi)}$$

32.



$$\vec{v}_1 = -v_1\cos\theta_1\,\hat{r} + v_1\sin\theta_1\,\hat{\theta}$$

$$\vec{v}_2 = -v_2\cos\theta_2\,\hat{r} + v_2\sin\theta_2\,\hat{\theta}$$

Since the potential has no angular component, momentum in $\hat{\theta}$ is conserved

$$v_1\sin\theta_1 = v_2\sin\theta_2$$

Using conservation of energy

$$\tfrac{1}{2}mv_1^2 = \tfrac{1}{2}mv_2^2 - V_0$$

$$= \tfrac{m}{2}\cdot v_1^2\cdot\sin^2\theta_1/\sin^2\theta_2 - V_0$$

$$\frac{\sin^2\theta_1}{\sin^2\theta_2} = \frac{\tfrac{1}{2}mv_1^2 + V_0}{\tfrac{1}{2}mv_1^2} = \frac{E+V_0}{E}$$

$$n = \frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{E+V_0}{E}}$$

$$s = a\sin\theta_1$$

$$\Theta = 2(\theta_1-\theta_2) \qquad \theta_2 = \theta_1 - \Theta/2$$

$$\sigma(\Theta) = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| \qquad (3.93)$$

$$n = \frac{\sin\theta_1}{\sin(\theta_1-\Theta/2)} = \frac{\sin\theta_1}{\sin\theta_1\cos\Theta/2 - \cos\theta_1\sin\Theta/2}$$

$$\frac{1}{n} = \cos(\Theta/2) - \cot\theta_1\sin\Theta/2$$

$$= \cos\Theta/2 - \frac{\sqrt{1-s^2/a^2}}{s/a}\sin\Theta/2$$

$$= \cos\Theta/2 - a/s\sqrt{1-s^2/a^2}\,\sin\Theta/2$$

$$1/n = \cos\Theta/2 - \sqrt{a^2/s^2-1}\,\sin\Theta/2$$

$$\sin\Theta/2\,\sqrt{a^2/s^2-1} = \cos\Theta/2 - 1/n$$

$$(a/s)^2 = 1 + \left(\frac{\cos\Theta/2 - 1/n}{\sin\Theta/2}\right)^2$$

$$\frac{s^2}{a^2} = \frac{\sin^2(\Theta/2)}{\sin^2(\Theta/2)+\cos^2(\Theta/2) - \frac{2}{n}\cos(\Theta/2) + 1/n^2}$$

$$s^2 = \frac{a^2\sin^2(\Theta/2)}{1-\frac{2}{n}\cos(\Theta/2)+1/n^2} = \frac{a^2n^2\sin^2(\Theta/2)}{n^2-2n\cos(\Theta/2)+1}$$

$$s = \frac{an\sin(\Theta/2)}{(1+n^2-2n\cos(\Theta/2))^{1/2}}$$

$$\frac{ds}{d\Theta} = \frac{\frac{an}{2}\cos(\Theta/2)(1+n^2-2n\cos(\Theta/2))^{1/2} - \frac{1}{2}(1+n^2-2n\cos(\Theta/2))^{-1/2}\cdot n\sin(\Theta/2)\cdot an}{1+n^2-2n\cos(\Theta/2)}$$

$$= \frac{an\cos(\Theta/2)(1+n^2-2n\cos(\Theta/2)) - an^2\sin^2(\Theta/2)}{2\,(1+n^2-2n\cos(\Theta/2))^{3/2}}$$

$$\frac{s\,ds}{d\Theta} = \frac{a^2n^2\sin(\Theta/2)[\cos(\Theta/2)+n^2\cos(\Theta/2)-2n\cos^2(\Theta/2)-n\sin^2(\Theta/2)]}{2(1+n^2-2n\cos(\Theta/2))^2}$$

$$= \frac{a^2n^2\sin(\Theta/2)[\cos(\Theta/2)-n\cos^2(\Theta/2)+n^2\cos(\Theta/2)-n]}{2(1+n^2-2n\cos(\Theta/2))^2}$$

$$= \frac{a^2n^2\sin(\Theta/2)\,(n\cos(\Theta/2)-1)(n-\cos(\Theta/2))}{2(1+n^2-2n\cos(\Theta/2))^2}$$

$$\sigma(\Theta) = \frac{a^2n^2\sin(\Theta/2)}{2\sin\Theta}\cdot\frac{(n\cos(\Theta/2)-1)(n-\cos(\Theta/2))}{(1+n^2-2n\cos(\Theta/2))^2}$$

$$= \frac{a^2n^2\cancel{\sin(\Theta/2)}}{4\cancel{\sin(\Theta/2)}\cos(\Theta/2)}\cdot\frac{(n\cos(\Theta/2)-1)(n-\cos(\Theta/2))}{(1+n^2-2n\cos(\Theta/2))^2}$$

$$= \frac{a^2n^2}{4\cos(\Theta/2)}\cdot\frac{(n\cos(\Theta/2)-1)(n-\cos(\Theta/2))}{(1+n^2-2n\cos(\Theta/2))^2}$$

$$\int_0^{2\pi}\sigma(\Theta)\,d\Theta$$

33. In cylindrical coordinates

$$z = ar^2 \qquad V = mgz$$

$$\dot{z} = 2ar\dot{r}$$

$$\mathcal{L} = \tfrac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + 4a^2r^2\dot{r}^2) - mgar^2$$

$$E = \tfrac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + 4a^2r^2\dot{r}^2) + mgar^2$$

$$= \tfrac{m}{2}\dot{r}^2 + \tfrac{l^2}{2mr^2} + 2a^2mr^2\dot{r}^2 + mgar^2$$

$$= \tfrac{m\dot{r}^2}{2} + V_{eff}(r)$$

$$V_{eff}(r) = \tfrac{l^2}{2mr^2} + 2a^2mr^2\dot{r}^2 + mgar^2$$

For circular motion, $\frac{dV_{eff}}{dr} = 0$ i.e. does not move up or down

$$-\frac{l^2}{mr^3} + \underbrace{4a^2mr\dot{r}^2}_{=0} + 2mgar = 0$$

$$\frac{l^2}{2m^2ga} = r^4$$

$$r_{eq} = \left(\frac{l^2}{2m^2ga}\right)^{1/4}$$

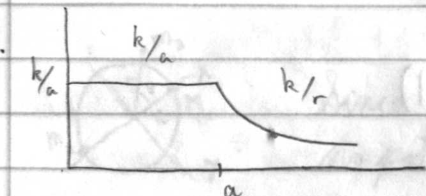
Angular kinetic energy $= V$

$$\tfrac{m}{2} r^2\dot{\phi}^2 = mgar^2$$

$$\dot{\phi}^2 = 2ga$$

$$\dot{\phi} = \sqrt{2ga}$$

$$v_\phi = r_{eq}\dot{\phi} = \left(\frac{l^2 \cdot 4g^2a^2}{2m^2ga}\right)^{1/4} = \left(\frac{2l^2ag}{m^2}\right)^{1/4}$$

34. 

$$\Theta = \pi - 2\int_{r_m}^{\infty} \frac{s\,dr}{r\sqrt{r^2(1 - \frac{V(r)}{E}) - s^2}} \qquad (3.96)$$

$$= \pi - 2\int_{r_m}^{a} \frac{s\,dr}{r\sqrt{r^2 - \frac{kr^2}{aE} - s^2}} - 2\int_{a}^{\infty} \frac{s\,dr}{r\sqrt{r^2 - \frac{kr}{E} - s^2}}$$

If $r_m = a$: $\Theta = \pi - 2\int_a^{\infty} \frac{s\,dr}{r\sqrt{r^2 - \frac{kr}{E} - s^2}}$

$$E = \tfrac{m}{2} r_m^2\dot{\theta}^2 + V(r_m)$$

$$= \tfrac{l^2}{2mr_m^2} + V(r_m)$$

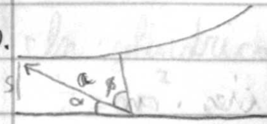
$\Big|_{r_m}^{a}$: $E = \frac{l^2}{2mr_m^2} + \frac{k}{a}$ $\qquad$ $\Big|_a^{\infty}$ $E = \frac{l^2}{2mr_m^2} + \frac{k}{r_m}$

$r_m^2 = \frac{l^2}{2mE} + \frac{kr_m^2}{aE}$ $\qquad$ $r_m^2 = s^2 + \frac{kr_m}{E}$

$r_m^2 = s^2 + \frac{kr_m^2}{aE}$

$$\Theta = \pi - 2\int_{r_m}^{a} \frac{s\,dr}{r\sqrt{r^2 - \frac{kr^2}{aE} - r_m^2 - \frac{kr_m^2}{aE}}} - 2\int_a^{\infty} \frac{s\,dr}{r\sqrt{r^2 - \frac{kr}{E} - r_m^2 - \frac{kr_m}{E}}}$$

35.



$$\sin\alpha = s/a$$

$$u = \frac{mk}{l^2}(\varepsilon\cos\theta - 1)$$

$$\pi = \Theta + 2(\alpha + \phi)$$

$$E = \frac{m}{2}\dot{r}^2 + \frac{m}{2} r^2\dot{\theta}^2 + V(r)$$

$$= \frac{m}{2}\left(\frac{dr}{d\theta}\dot{\theta}\right)^2 + \frac{l^2}{2mr^2} + \frac{k}{r} - \frac{k}{a}$$

$$= \frac{l^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{l^2u^2}{2m} + ku - \frac{k}{a}$$

$$= \frac{l^2}{2m}\cdot\left(\frac{mk\varepsilon}{l^2}(-\sin\theta)\right)^2 + \frac{l^2}{2m}\cdot\frac{m^2k^2}{l^4}(\varepsilon\cos\theta - 1)^2 + \frac{mk^2}{l^2}(\varepsilon\cos\theta - 1) - \frac{k}{a}$$

$$= \frac{l^2}{2m}\cdot\frac{m^2k^2\varepsilon^2}{l^4}\sin^2\theta + \frac{mk^2}{2ml^2}(\varepsilon^2\cos^2\theta - 2\varepsilon\cos\theta + 1) + \frac{mk\varepsilon\cos\theta}{l^2} - \frac{mk^2}{l^2} - \frac{k}{a}$$

$$= \frac{mk^2}{2l^2}(\varepsilon^2 - 1) - \frac{k}{a}$$

$$\varepsilon^2 - 1 = \frac{2l^2}{mk^2}\left(E + \frac{k}{a}\right)$$

$$\varepsilon^2 = 1 + \frac{2l^2}{mk^2}\left(E + \frac{k}{a}\right)$$

$$\sin^2(\Theta/2) = \frac{1 - x^2}{1 + 4x^2\varepsilon(1+\varepsilon)}$$

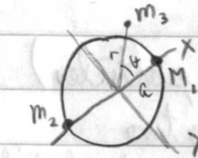
$$x = s/a \qquad \varepsilon = Ea/k$$

$$\sigma(\Theta) = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right|$$

$$= \frac{a^2}{4}\cdot\frac{1+\eta}{(1+\eta\sin^2\Theta)^2}$$

$$\eta = 4\varepsilon(1+\varepsilon)$$

36.



Since m_1 + m_2 exist at $y=0$, they have no potential energy

m_3: $r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y}$

m_1: $a\hat{x}$

m_2: $-a\hat{x}$

$f(r) = k\vec{r}/r^3$ for example

$$f_{13} = \frac{kr(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})}{((r\cos\theta - a)^2 + r^2\sin^2\theta)^{3/2}} = \frac{kr(\cos\theta\overset{-a/r}{\vee}\hat{x} + \sin\theta\,\hat{y})}{(r^2 + a^2 - 2ra\cos\theta)^{3/2}}$$

$$f_{23} = \frac{kr((\cos\theta + a/r)\hat{x} + \sin\theta\,\hat{y})}{(r^2 + a^2 + 2ar\cos\theta)^{3/2}}$$