Graduate Classical Mechanics

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Acknowledgements

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Contents

Chapter 1

Newtonian Mechanics

"If I have seen further it is by standing on ye shoulders of Giants." - Sir Isaac Newton (1676)

When Sir Isaac Newton published the Principia in 1686, he laid out the basis of all physics from particles colliding in an accelerator to planets orbiting around each other. Granted, there are more nuances to these systems (quantum mechanics and general relativity respectively), but to understand these phenomena, it is important to have a solid foundation on which to build.

Before the discover of quantum mechanics and general relativity, it was thought that physics was done, that we had discovered everything and the only thing remaining was more precise measurements. This statement was proven false, and it was found that classical mechanics does not provide a perfect interpretation of the world around us. That being said, why study classical mechanics if it's wrong? I can think of two reasons. First, classical mechanics provides a good approximation for most situations. Quantum mechanics really only kicks in around the Planck scale. For anything bigger, we can use classical mechanics, and it gives us pretty much the correct answer. Relativistic corrections are needed as we approach the speed of light, but the fastest man-made object (a nuclear powered steel plate) only reached upwards of 66 kilometers per second, on the order of ten-thousand times less than the speed of light. Most things we deal with are on classical scales. It would be rather tedious if we had to make relativistic corrections every time we tried to do a block on a ramp type of problem. Second, classical mechanics, as stated before, is the starting point of physics. Much of the math and language we use later is developed here.

1.1 Newton's Laws

We'll start with the very basics: Newton's three laws. If you've taken any physics class, you will almost certainly have had these drilled into your head. In this section, we'll restate them and set up some notation.

1.1.1 First Law

Imagine we have a particle, which in classical mechanics does not need to be subatomic (and indeed should not be subatomic). It could be anything from a billiard ball to a planet. In any case, we can specify its position at some time by a vector $\vec{x}(t)$. If we ask that particle where it came from and where it's going, we need to know its velocity, $\vec{v}(t)$ (1.1). When Newton wrote the Principia, he dealt with momentum, defined to be the product of a particle's mass and velocity (1.2). In your undergraduate classes, mass was generally kept constant, but now that we're at the graduate level, we can see what happens when the mass is not constant. Finally, we can ask what happens if the velocity changes. This gives us the acceleration of the system (1.3), which as we shall see shortly, is tied to the force. There are higher order terms (jerk, snap, crackle, pop, lock, and drop if I remember correctly), but we will largely be ignoring these terms.

$$\vec{v}(t) = \dot{\vec{x}} = \frac{d\vec{x}}{dt} \tag{1.1}$$

$$\vec{p}(t) = m(t)\ \vec{v}(t) \tag{1.2}$$

$$\vec{a}(t) = \ddot{\vec{x}} = \frac{d^2\vec{x}}{dt^2} \tag{1.3}$$

If we have a particle moving in not a straight line, it is often easier to use angular coordinates. For velocity and acceleration, they follow the same formulas laid out above. Position is often denoted by $\vec{r}(t)$. N.B., I will likely use $\vec{r}$ and $\vec{x}$ interchangeably. The difference is angular momentum (1.4).

$$\vec{L}(t) = \vec{r}(t) \times \vec{p}(t) \tag{1.4}$$

Newton's First Law may sound obvious to us, but when it was first stated, it flew in the face of thousands of years of science (referred to as Aristotelian thought). Imagine we have a particle with a constant mass. If we know its velocity and position at some time t_0, then by looking at equation (1.1), the position a time Δt later is given by,

$$\vec{x}(t_0 + \Delta t) = \vec{x}(t_0) + \vec{v}(t_0)\Delta t$$

However, this is not always true in certain frames of reference (the Earth is not an inertial frame, which we'll look at when we get to centrifugal motion). A frame that does satisfy this is an inertial frame. Newton's first law is the statement that such frames exist. Further, these inertial frames are invariant under translation, rotation, and boosting. That is, if we move our frame somewhere else, rotate it, or move it at a constant velocity, physics still acts the same. That physics is the same in all inertial frames of reference has important implications for the universe. This is referred to as the principle of relativity. There is no special point, direction, or velocity in the universe.

One way to think about the principle of relativity is by picturing an elevator. If you are isolated inside that elevator, it is impossible to tell if it is at rest or moving at a constant velocity. The only way to tell what is going on in the elevator is if it is speeding up or slowing down. An accelerating frame is not an inertial frame, so physics will behave differently (in this case, you will feel lighter or heavier).

1.1.2 Second Law

Newton's second law is probably his best known. It is a concise way to write the equations of motion for a particle. We define some quantity force $\vec{F}$ as the change in momentum (1.5). The angular analog is torque (1.6). If we know the initial position and velocity of a particle, we can determine its position at any time.

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{1.5}$$

$$\vec{N} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt} \tag{1.6}$$

One thing we should note is that when we start looking at the forces between particles, we have to introduce the concept of a field. In Newtonian thinking, forces are created and destroyed instantaneously. However, this doesn't quite sit well with Einstein's claim that the speed of light is the ultimate speed limit in the universe. Imagine if the sun were to disappear. According to Newton, Earth would then fly off in a tangential direction. The gravitational force from the sun would be travelling faster than the speed of light, but Einstein says that can't happen. So how do we reconcile these two opposing thoughts? We imagine that there is some underlying sheet (or field) that manifests these forces. Eventually you'll learn that things like particles are actually the product of quantizing these fields, but for now, we don't have to worry about that.

1.1.3 Third Law

Newton's third law states that if there are two particles, the force of one particle acting on the other is equal in magnitude and opposite in direction to the force from the other particle, $\vec{F}_{ij} = -\vec{F}_{ji}$. We can imagine that if this weren't true and we pushed our hands together, they would actually go through each other since the acceleration of one hand would be greater than the acceleration of the other.

The other consequence of Newton's third law is that if there are no external forces in a system, the net force is zero. Looking at equation (1.5), if $\vec{F} = 0$, the momentum of the system does not change. Momentum is a conserved quantity. We can also show that angular momentum is a conserved quantity. If there are no external torques, angular momentum does not change.

There are several conserved quantities in physics, but the main one we deal with in classical mechanics is the conservation of energy. To first order, we can break energy into two categories: kinetic and potential. If we move our particle from point 1 to point 2, we do some work on that particle (1.7). If we put work into a system, we change its energy (1.8).

If you've ever been on a roller-coaster, you know that the cart comes to a stop when its reached the top of that first hill only to pick up speed as it goes down until its moving quite fast. At the start, the cart has all potential energy (1.10) (it has the potential to move), and at the end, that potential energy has grown into kinetic energy (1.9). If we can write the total energy of a system as $T + V$, the energy of the system is conserved, and we say the force is conservative.

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s} \tag{1.7}$$

$$W_{12} = T_2 - T_1 \tag{1.8}$$

$$T = \frac{1}{2}mv^2 \tag{1.9}$$

$$\vec{F} = -\nabla V(\vec{r}) \tag{1.10}$$

You should now be able to do Goldstein 1.1, 1.11, 1.12, and 1.13.

You should now be able to do Jose 1.1, 1.3, 1.7, 1.9, 1.10, 1.13, 1.14, 1.15, and 1.20.

1.2 Mechanics of Multiple Particle Systems

In the previous section, we looked at a system consisting of a single particle. Naturally, the next step is to have multiple particles. The first thing we do is label each particle with some subscript i. So the position of particle i is x_i, and its mass is m_i. Thankfully, other than that, not much changes, Newton's Laws don't radically morph into some new form.

1.2.1 Newton's Second Law and Multiple Particles

However, Newton's second law (1.5) does change slightly. We can break up the force a particle feels into either external or internal forces. External forces are, as the name suggests, forces from outside the system. Internal forces are forces from other particles in the system. If we sum over all particles, we find that only the external force remains since the internal forces cancel out by Newton's third law.

$$\sum_j \vec{F}_j + \vec{F}_i^{(e)} = \dot{\vec{p}}_i \tag{1.11}$$

1.2.2 Center of Mass

We start by introducing a concept central to multiple particle systems. If we have a system consisting of N particles, the total mass of that system is given by

$$M = \sum_{i=1}^{N} m_i$$

Averaging the distances and masses of each particle, we can find the center of mass (1.12). One important concept that jumps out of the center of mass is that we don't really need to care about the individual particles in a system, we just need to know what happens to the center of mass. You may have seen a variation on this problem in your undergraduate class. Imagine you have a ball flying through the air. Suddenly, it breaks apart in mid flight. The individual pieces will fly off in different directions, but the center of mass will follow the trajectory the ball would have followed if it had remained whole.

$$\vec{R} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{x}_i \tag{1.12}$$

We can see this by looking at the total momentum of a system. If we sum up all the momenta, we can rewrite the total momentum of a system (1.13) in terms of the center of mass.

$$\vec{P} = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i$$

$$\vec{P} = M\dot{\vec{R}} \tag{1.13}$$

Just as we had conservation of momentum and angular momentum in single-particle systems, we have the same conservation laws for multiple particle systems.

1.2.3 Collisions

There are two types of collisions that we will be dealing with. The first is elastic collisions. An example of elastic collisions would be two pool balls bouncing off of each other. Here, both energy and momentum are conserved.

The other type of collision we will deal with is inelastic collision. An example would be a hockey puck colliding with an octopus and both of them sliding together. Kinetic energy is not conserved, but momentum is. We can convince ourselves that something weird is going on here by thinking about the system in reverse. We have an object moving along, and it spontaneously splits into two pieces. To do that, we need to introduce some energy into the system.

1.2.4 Rigid Bodies

We will not be going into rigid bodies in this section.

You should now be able to do Goldstein 1.2, 1.3, 1.14, 1.17, and 1.23. You should now be able to do Jose 1.2, 1.4, 1.5, 1.6, 1.8, 1.16, 1.17, and 1.18.

1.3 Constraints

A constraint is some condition that we need to take into account that restricts the motion of a particle. For example, if we go back to the question of throwing a ball, the motion we find is only valid for $y > 0$. The motion of the ball is limited to it being above ground.

1.3.1 Holonomic Constraints

The most common type of constraint that we will be dealing with are holonomic constraints. If we can express the constraint in the form of equation (1.14), it is holonomic. Furthermore, if the equation depends explicitly on time, it is rheonomous. Otherwise, it is scleronomous. A bead moving on a wire would be holonomic and scleronomous. However, if the wire is moving as well according to some time-dependant function, the constraint would be holonomic and rheonomous. An example of a non-holonomic constraint is rolling without slipping.

$$f(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...t) = 0 \tag{1.14}$$

1.3.2 Phase Portraits

This is a problem we'll leave for future Ben.

You should now be able to do Goldstein 1.4, 1.5, and 1.6.

You should now be able to do Jose 1.11, 1.12, and 1.26.