Sakurai Solutions

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Chapter 1

Fundamental Concepts

1.1 Commutation Relations

Prove

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB$$

I think it's easier if we start with the right hand side. Expanding out the anti-commutation relations,

$$-AC(DB + BD) + A(CB + BC)D - C(DA + AD)B + (CA + AC)DB$$

Distributing and grouping terms,

$$= ABCD - CDAB = [AB, CD]$$

1.2 Pauli Matrices

Suppose a 2×2 matrix X (not necessarily Hermitian, nor unitary) is written as

$$X = a_0 + \vec{\sigma} \cdot \vec{a}$$

where a_0 and $a_{1,2,3}$ are numbers.

1.2.1 Trace

How are a_0 and a_k $(k = 1, 2, 3)$ related to $tr(X)$ and $tr(\sigma_k X)$?

We can write $tr(X)$,

$$tr(X) = tr(a_0 I) + tr(\vec{\sigma} \cdot \vec{a})$$

We know that by definition, the Pauli matrices are traceless, so re-scaling them by a scalar factor does nothing to the trace.

$$= 2a_0 + 0 = 2a_0$$

If we multiply X by one of the Pauli matrices,

$$\sigma_k X = a_0 \sigma_k + a_1 \sigma_k \sigma_1 + a_2 \sigma_k \sigma_2 + a_3 \sigma_k \sigma_3$$

We know that when we take the trace, the first term will always die since that is just one of the Pauli matrices. We also know that non-k terms also die since

$$\sigma_a \sigma_b = \delta_{ab} I + i\epsilon_{abc}\sigma_c$$

which for $a \neq b$, returns another Pauli matrix. Thus, only the k term survives,

$$tr(\sigma_k X) = 2a_k$$

From these, we can show,

$$\begin{cases} a_0 = \dfrac{tr(X)}{2} \\ \\ a_k = \dfrac{tr(\sigma_k X)}{2} \end{cases}$$

1.2.2 Matrix Elements

Obtain a_0 and a_k in terms of the matrix elements X_{ij}.

As a reminder,

$$\begin{cases} \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{cases}$$

$$X = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}$$

We now want to multiply each Pauli matrix by X, i.e., $\sigma_k X$,

$$\sigma_1 X = \begin{pmatrix} X_{21} & X_{22} \\ X_{11} & X_{12} \end{pmatrix}$$

$$\sigma_2 X = \begin{pmatrix} -iX_{21} & -iX_{22} \\ iX_{11} & iX_{12} \end{pmatrix}$$

$$\sigma_3 X = \begin{pmatrix} X_{11} & X_{12} \\ -X_{21} & -X_{22} \end{pmatrix}$$

Using the results from the previous part,

$$\begin{cases} a_0 = \frac{1}{2}(X_{11} + X_{22}) \\ \\ a_1 = \frac{1}{2}(X_{21} + X_{12}) \\ \\ a_2 = \frac{1}{2}(-iX_{21} + iX_{12}) \\ \\ a_3 = \frac{1}{2}(X_{11} - X_{22}) \end{cases}$$

1.3 Invariant Determinant

Show that the determinant of a 2×2 matrix $\vec{\sigma} \cdot \vec{a}$ is invariant under

$$\vec{\sigma} \cdot \vec{a} \to \vec{\sigma} \cdot \vec{a}' = \exp\left(\frac{i\vec{\sigma} \cdot \hat{n}\phi}{2}\right) \vec{\sigma} \cdot \vec{a} \exp\left(\frac{-i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)$$

Find a'_k in terms of a_k when $\hat{n}$ is in the positive z-direction and interpret your result.

Let's go ahead and take the determinant of both sides. What really matters is the right side, so let's look at that one. We know we can break up the determinant,

$$= \det\left(\exp\left(\frac{i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right) \det(\vec{\sigma} \cdot \vec{a}) \det\left(\exp\left(\frac{-i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right)$$

Each determinant is just a scalar, so we can rearrange them for free,

$$= \det\left(\exp\left(\frac{i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right) \det\left(\exp\left(\frac{-i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right) \det(\vec{\sigma} \cdot \vec{a})$$

$$= \det\left(\exp\left(\frac{i\vec{\sigma} \cdot \hat{n}\phi}{2}\right) \exp\left(\frac{-i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right) \det(\vec{\sigma} \cdot \vec{a})$$

$$= \det(\vec{\sigma} \cdot \vec{a})$$

For the second part,

$$\hat{n} = \hat{z} = (0, 0, 1)$$

Substituting this in and writing out explicitly,

$$\vec{\sigma} \cdot \vec{a}' = \exp\left(\frac{i\vec{\sigma}_z\phi}{2}\right) \vec{\sigma} \cdot \vec{a} \exp\left(\frac{-i\vec{\sigma}_z\phi}{2}\right)$$

$$= \begin{pmatrix} \exp\left(\frac{i\phi}{2}\right) & 0 \\ 0 & \exp\left(\frac{-i\phi}{2}\right) \end{pmatrix} \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} \begin{pmatrix} \exp\left(\frac{-i\phi}{2}\right) & 0 \\ 0 & \exp\left(\frac{i\phi}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} a_3 & (a_1 - ia_2)\exp(i\phi) \\ (a_1 + ia_2)\exp(-i\phi) & -a_3 \end{pmatrix}$$

Using the results from question 2,

$$\begin{cases} a_0 = \dfrac{tr(X)}{2} \\ a_k = \dfrac{tr(\sigma_k X)}{2} \end{cases}$$

where $X = a_0 + \vec{\sigma} \cdot \vec{a}$.

$$a_0' = \frac{a_3 - a_3}{2} = 0$$

$$a_1' = \frac{1}{2}[(a_1 + ia_2)\exp(-i\phi) + (a_1 - ia_2)\exp(i\phi)] = a_1\cos(\phi) + a_2\sin(\phi)$$

$$a_2' = \frac{1}{2}[-i(a_1 + ia_2)\exp(-i\phi) + i(a_1 - ia_2)\exp(i\phi)] = -a_1\sin(\phi) + a_2\cos(\phi)$$

$$a_3' = \frac{1}{2}(a_3 + a_3) = a_3$$

which we recognize as rotation about the z-axis.

1.4 Bra-Ket Algebra

Using the rules of bra-ket algebra, prove or evaluate the following:

1.4.1 $tr(XY) = tr(YX)$, where X and Y are operators

We can rearrange the trace,

$$tr(XY) = tr(X)tr(Y) = tr(Y)tr(X) = tr(YX)$$

1.4.2 $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators

---

Let's act XY on some ket, $|\alpha\rangle$,

$$(XY)|\alpha\rangle$$

In bra-space,

$$\langle\alpha|(XY)^\dagger$$

Alternatively,

$$XY|\alpha\rangle = X(Y|\alpha\rangle)$$

In bra-space,

$$\langle\alpha|Y^\dagger X^\dagger$$

Comparing these two cases,

$$(XY)^\dagger = Y^\dagger X^\dagger$$

1.4.3 $\exp[if(A)] = ?$ in ket-bra form, where A is a Hermitian operator whose eigenvalues are known

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Let's act the function on a vector,

$$\exp(if(A))|\alpha\rangle = [\cos(f(A)) + i\sin(f(A))]|\alpha\rangle$$

Since we know the eigenvalues,

$$= [\cos(f(\alpha)) + i\sin(f(\alpha))]|\alpha\rangle$$

$$\exp(if(A)) = \cos(f(A)) + i\sin(f(A))$$

1.4.4 $\sum_{a'} \psi^*_{a'}(\vec{x}')\psi_{a'}(\vec{x}'')$, where $\psi_{a'}(\vec{x}') = \langle \vec{x}' | a' \rangle$

Writing it out,

$$\sum_{a'} \langle a' | \vec{x}' \rangle \langle \vec{x}'' | a' \rangle = \delta_{\vec{x}', \vec{x}''}$$

1.5 Matrix Representation

1.5.1 General

Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Suppose $\langle a'|\alpha\rangle$, $\langle a''|\alpha\rangle$, ... and $\langle a'|\beta\rangle$, $\langle a''|\beta\rangle$, ... are all known, where $|a'\rangle$, $|a''\rangle$, ... form a complete set of base kets. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$ in that basis.

From the text,

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} \langle a'|\alpha\rangle\langle a'|\beta\rangle^* & \langle a'|\alpha\rangle\langle a''|\beta\rangle^* & \dots \\ \langle a''|\alpha\rangle\langle a'|\beta\rangle^* & \langle a''|\alpha\rangle\langle a''|\beta\rangle^* & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

1.5.2 Spin-$\frac{1}{2}$ systems

We now consider a spin $\frac{1}{2}$ system and let $|\alpha\rangle$ and $|\beta\rangle$ be $|s_z = \hbar/2\rangle$ and $|s_x = \hbar/2\rangle$, respectively. Write down explicitly the square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the usual (s_z diagonal) basis.

We expect to get a 2×2 matrix,

$$|s_z = \hbar/2\rangle\langle s_x = \hbar/2| = |+\rangle\frac{1}{\sqrt{2}}(\langle +| + \langle -|)$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$$

1.6 Adding Eigenkets

Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator A. Under what condition can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of A? Justify your answer.

If we act A on our eigenkets,

$$\begin{cases} A|i\rangle = a|i\rangle \\ A|j\rangle = a'|j\rangle \end{cases}$$

If $|i\rangle + |j\rangle$ is to be an eigenket,

$$A(|i\rangle + |j\rangle) = a''(|i\rangle + |j\rangle)$$

Alternatively, we can write this as

$$A(|i\rangle + |j\rangle) = A|i\rangle + A|j\rangle = a|i\rangle + a'|j\rangle$$

Comparing these results, this can only be true if either $|i\rangle = |j\rangle$ (the less interesting result), or $a = a'$, i.e., the eigenvalues are degenerate.

1.7 Ket Space

Consider a ket space spanned by the eigenkets $\{|a'\rangle\}$ of a Hermitian operator A. There is no degeneracy

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1.7.1 Null Operator

Prove that

$$\prod_{a'}(A - a')$$

is the null operator.

Let's act A on some unsuspecting eigenvector,

$$A\,|\Psi\rangle = a'\,|\Psi\rangle$$

$$A\,|\Psi\rangle - a'\,|\Psi\rangle = 0$$

$A - a' = 0$ for at least one case. Since we product over all a', if $A - a' = 0$ for one case, then the product over all of those is 0.

1.7.2 Projection Operator

What is the significance of

$$\prod_{a''\neq a'}\frac{(A - a'')}{(a' - a'')}$$

If we act the given on $|a'\rangle$,

$$\prod_{a''\neq a'}\frac{(A - a'')}{(a' - a'')}\,|a'\rangle = \prod_{a''\neq a'}\frac{(a' - a'')}{(a' - a'')}\,|a'\rangle = |a'\rangle$$

Let's act it on an eigenvector,

$$\prod_{a''\neq a'}\frac{(A - a'')}{(a' - a'')}\,|\Psi\rangle$$

We can insert identity and use the above relation,

$$= \prod_{a''\neq a'}\frac{(A - a'')}{(a' - a'')}\,|a'\rangle\,\langle a'|\Psi\rangle = |a'\rangle\,\langle a'|\Psi\rangle$$

As the title suggests, this is the projection operator of $|a'\rangle$.

1.7.3 Example Spin

Illustrate (a) and (b) using A set equal to S_z of a spin-1/2 system.

As a reminder,

$$S_z = \begin{bmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{bmatrix}$$

The eigenvalues are $\omega = \pm\frac{\hbar}{2}$. Showing the null vector, we multiply,

$$\left(S_z - \frac{\hbar}{2}\right)\left(S_z + \frac{\hbar}{2}\right) = \begin{bmatrix} 0 & 0 \\ 0 & -\hbar \end{bmatrix}\begin{bmatrix} \hbar & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Showing part b,

$$\frac{S_z + \frac{\hbar}{2}}{\hbar} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

1.8 Orthonormality

Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove

$$[S_i, S_j] = i\epsilon_{ijk}\hbar S_k, \quad \{S_i, S_j\} = \left(\frac{\hbar^2}{2}\right)\delta_{ij}$$

where

$$\begin{cases} S_x = \dfrac{\hbar}{2}(|+\rangle\langle-| + |-\rangle\langle+|) \\ S_y = \dfrac{i\hbar}{2}(-|+\rangle\langle-| + |-\rangle\langle+|) \\ S_z = \dfrac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|) \end{cases}$$

Setting $i = x$ and $j = y$ and brute forcing,

$$\begin{aligned}[S_x, S_y] = S_xS_y - S_yS_x = &\frac{i\hbar^2}{4}[(|+\rangle\langle-|) + (|-\rangle\langle+|)][-(|+\rangle\langle-|) + (|-\rangle\langle+|)] \\ &-\frac{i\hbar^2}{4}[-(|+\rangle\langle-|) + (|-\rangle\langle+|)][(|+\rangle\langle-|) + (|-\rangle\langle+|)]\end{aligned}$$

$$\begin{aligned}= \frac{i\hbar^2}{4}[&-(|+\rangle\langle-|)(|+\rangle\langle-|) + (|+\rangle\langle-|-\rangle\langle+|) - (|-\rangle\langle+|+\rangle\langle-|) + (|-\rangle\langle+|-\rangle\langle+|) \\ &+(|+\rangle\langle-|+\rangle\langle-|) + (|+\rangle\langle-|-\rangle\langle+|) - (|-\rangle\langle+|+\rangle\langle-|) - (|-\rangle\langle+|-\rangle\langle+|)]\end{aligned}$$

Using the orthonormality relationships,

$$\begin{cases}\langle+|+\rangle = \langle-|-\rangle = 1 \\ \langle+|-\rangle = \langle-|+\rangle = 0\end{cases}$$

$$= \frac{i\hbar^2}{2}[(|+\rangle\langle+|) - (|-\rangle\langle-|)] = i\hbar S_z$$

We do the same thing with the anti-commutation relation,

$$\begin{aligned}\{S_x, S_y\} = S_xS_y + S_yS_x = &\frac{i\hbar^2}{4}[(|+\rangle\langle-|) + (|-\rangle\langle+|)][-(|+\rangle\langle-|) + (|-\rangle\langle+|)] \\ &+\frac{i\hbar^2}{4}[-(|+\rangle\langle-|) + (|-\rangle\langle+|)][(|+\rangle\langle-|) + (|-\rangle\langle+|)]\end{aligned}$$

$$\begin{aligned}= \frac{i\hbar^2}{4}[&-(|+\rangle\langle-|+\rangle\langle-| + (|+\rangle\langle-|-\rangle\langle+|) - (|-\rangle\langle+|+\rangle\langle-|) + (|-\rangle\langle+|-\rangle\langle+|) \\ &-(|+\rangle\langle-|+\rangle\langle-|) - (|+\rangle\langle-|-\rangle\langle+|) + (|-\rangle\langle+|+\rangle\langle-|) + (|-\rangle\langle+|-\rangle\langle+|)]\end{aligned}$$

$$= 0$$

We can repeat this for all other combinations to prove the desired relations.

1.9 Rotation Operators

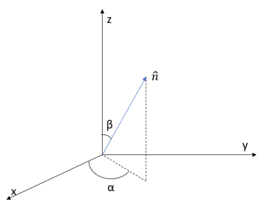

Figure 1.1: Angles

Construct $|\vec{S}\cdot\hat{n};+\rangle$ such that

$$\vec{S}\cdot\hat{n}\,|\vec{S}\cdot\hat{n};+\rangle = \left(\frac{\hbar}{2}\right)|\vec{S}\cdot\hat{n};+\rangle$$

where $\hat{n}$ is characterized by the angles shown in the figure. Express your answer as a linear combination of $|+\rangle$ and $|-\rangle$. [Note: The answer is

$$\cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)\exp(i\alpha)\,|-\rangle$$

But do not just verify that this answer satisfies the above eigenvalue equation. Rather, treat the problem as a straightforward eigenvalue problem. Also do not use rotation operators, which we will introduce later in this book]

The first thing we do is figure out $\vec{S}\cdot\hat{n}$,

$$\begin{cases}\vec{S} = \frac{\hbar}{2}(\sigma_x,\sigma_y,\sigma_z)\\ \\ \hat{n} = (\cos(\alpha)\sin(\beta),\sin(\alpha)\sin(\beta),\cos(\beta))\end{cases}$$

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\left[\begin{pmatrix}0 & \cos(\alpha)\sin(\beta)\\ \cos(\alpha)\sin(\beta) & 0\end{pmatrix} + \begin{pmatrix}0 & -i\sin(\alpha)\sin(\beta)\\ i\sin(\alpha)\sin(\beta) & 0\end{pmatrix} + \begin{pmatrix}\cos(\beta) & 0\\ 0 & \cos(\beta)\end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\begin{bmatrix}\cos(\beta) & \sin(\beta)(\cos(\alpha)-i\sin(\alpha))\\ \sin(\beta)(\cos(\alpha)+i\sin(\alpha)) & \cos(\beta)\end{bmatrix}$$

$$= \frac{\hbar}{2}\begin{bmatrix}\cos(\beta) & \sin(\beta)\exp(-i\alpha)\\ \sin(\beta)\exp(i\alpha) & \cos(\beta)\end{bmatrix}$$

If we now say that $|\vec{S}\cdot\hat{n};+\rangle$ is some arbitrary vector, we can solve the eigenvalue problem,

$$\vec{S}\cdot\hat{n}\,|\vec{S}\cdot\hat{n};+\rangle = \frac{\hbar}{2}|\vec{S}\cdot\hat{n};+\rangle$$

$$\begin{pmatrix}\cos(\beta) & \sin(\beta)\exp(-i\alpha)\\ \sin(\beta)\exp(i\alpha) & -\cos(\beta)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}x\cos(\beta)+y\sin(\beta)\exp(-i\alpha)\\ x\sin(\beta)\exp(i\alpha)-y\cos(\beta)\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix}$$

Because our eigenket needs to be normalized, $|x|^2 + |y|^2 = 1$. Looking at the first line of the matrix,

$$x\cos(\beta) + y\sin(\beta)\exp(-i\alpha) = x$$

$$|y| = \frac{(1-\cos(\beta))|x|}{\sin(\beta)}$$

Inserting this into the normalization condition,

$$|x|^2 + \frac{|x|^2 - 2|x|^2\cos(\beta) + |x|^2\cos^2(\beta)}{\sin^2(\beta)} = 1$$

$$|x|^2 = \frac{1+\cos(\beta)}{2}$$

$$x = \cos\left(\frac{\beta}{2}\right)$$

Plugging this into the second line,

$$y = \frac{\cos\left(\frac{\beta}{2}\right)(1-\cos(\beta))}{\sin(\beta)}\exp(i\alpha)$$

$$= \sin\left(\frac{\beta}{2}\right)\exp(i\alpha)$$

which gives the solution provided by Sakurai.

1.10 Energy Eigenvalues

The Hamiltonian operator for a two-state system is given by

$$\mathscr{H} = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)$$

where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$).

Schrodinger's equation,

$$\mathscr{H}|\Psi\rangle = E|\Psi\rangle$$

Let's say that we have some vector

$$|\Psi\rangle = b|1\rangle + c|2\rangle$$

$$\mathscr{H}|\Psi\rangle = ab|1\rangle + ac|1\rangle - ac|2\rangle + ab|2\rangle$$

$$\begin{cases} ab + ac = Eb \\ -ac + ab = Ec \end{cases}$$

Setting $a = 1$ for simplicity,

$$\frac{b+c}{b} = E$$

$$\frac{b-c}{c} = E$$

Alternatively, we could use matrix representation. Setting

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In this basis,

$$\mathscr{H} = \begin{bmatrix} a & a \\ a & -a \end{bmatrix}$$

Solving the characteristic equation,

$$\det(\mathscr{H} - \lambda I) = \det \begin{bmatrix} a-\lambda & a \\ a & -a-\lambda \end{bmatrix}$$

$$= \lambda^2 - 2a^2$$

Our eigenvalues are $\lambda = \pm\sqrt{2}a$. Solving for the eigenvectors,

$$|\sqrt{2}a\rangle = \frac{1}{4+2\sqrt{2}}\begin{bmatrix}1+\sqrt{2}\\1\end{bmatrix}$$

$$|-\sqrt{2}a\rangle = \frac{1}{4-2\sqrt{2}}\begin{bmatrix}1-\sqrt{2}\\1\end{bmatrix}$$

Taking this solution, we compare to the bra-ket method, setting $E = \pm\sqrt{2}$, $b = 1 \pm \sqrt{2}$, and $c = 1$ to see that our results are consistent.

1.11 Hamiltonian Eigenvalue Problem

A two-state system is characterized by the Hamiltonian

$$\mathscr{H} = \mathscr{H}_{11} \left|1\right\rangle \left\langle 1\right| + \mathscr{H}_{22} \left|2\right\rangle \left\langle 2\right| + \mathscr{H}_{12}[\left|1\right\rangle \left\langle 2\right| + \left|2\right\rangle \left\langle 1\right|]$$

where $\mathscr{H}_{11}, \mathscr{H}_{22}$, and $\mathscr{H}_{12}$ are real numbers with the dimension of energy, and $|1\rangle$ and $|2\rangle$ are eigenkets of some observable ($\neq \mathscr{H}$). Find the energy eigenkets and corresponding energy eigenvalues. Make sure that your answer makes good sense for $\mathscr{H}_{12} = 0$. (You need not solve this problem from scratch. The following fact may be used without proof:

$$(\vec{S} \cdot \hat{n}) \left|\hat{n}; +\right\rangle = \frac{\hbar}{2} \left|\hat{n}; +\right\rangle$$

with $|\hat{n}; +\rangle$ given by

$$\left|\hat{n}; +\right\rangle = \cos\left(\frac{\beta}{2}\right) \left|+\right\rangle + \exp(i\alpha) \sin\left(\frac{\beta}{2}\right) \left|-\right\rangle$$

where β and α are the polar and azimuthal angles, respectively, that characterize $\hat{n}$)

Like the previous problem, this can be solved in bra-ket notation, but I'm more comfortable with matrices.

$$\mathscr{H} = \begin{bmatrix} \mathscr{H}_{11} & \mathscr{H}_{12} \\ \mathscr{H}_{12} & \mathscr{H}_{22} \end{bmatrix}$$

Solving the characteristic equation gives

$$\lambda_1 = \frac{(\mathscr{H}_{11} + \mathscr{H}_{22}) + \sqrt{(\mathscr{H}_{11} + \mathscr{H}_{22})^2 - 4(\mathscr{H}_{11}\mathscr{H}_{22} - \mathscr{H}_{12}^2)}}{2}$$

$$\lambda_2 = \frac{(\mathscr{H}_{11} + \mathscr{H}_{22}) - \sqrt{(\mathscr{H}_{11} + \mathscr{H}_{22})^2 - 4(\mathscr{H}_{11}\mathscr{H}_{22} - \mathscr{H}_{12}^2)}}{2}$$

$$\left|\lambda_1\right\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x = \frac{\mathscr{H}_{12}}{\mathscr{H}_{12}^2 + (\mathscr{H}_{11} - \lambda_1)(\mathscr{H}_{22} - \lambda_1)}$$

$$y = \frac{\mathscr{H}_{11} - \lambda_1}{\mathscr{H}_{12}^2 + (\mathscr{H}_{11} - \lambda_1)(\mathscr{H}_{22} - \lambda_1)}$$

If $\mathscr{H}_{12} = 0$,

$$\mathscr{H} = \begin{bmatrix} \mathscr{H}_{11} & \\ & \mathscr{H}_{22} \end{bmatrix}$$

Our eigenvalues are $\lambda = \mathscr{H}_{11}, \mathscr{H}_{22}$,

$$|\mathscr{H}_{11}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\mathscr{H}_{22}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1.12 Spin-1/2 eigenvalue problem

A spin-1/2 system is known to be in an eigenstate of $\vec{S} \cdot \hat{n}$ with eigenvalue $\hbar/2$, where $\hat{n}$ is a unit vector lying in the xz-plane that makes an angle γ with the positive z-axis.

1.12.1 Measure S_x

Suppose S_x is measured. What is the probability of getting $+\hbar/2$?

We write

$$|S_x\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

$$|\vec{S} \cdot \hat{n}\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle$$

The probability is given

$$P\left(\frac{\hbar}{2}\right) = |\langle S_x|\vec{S} \cdot \hat{n}; +\rangle|^2$$

$$= \frac{1}{2}\left|\cos\left(\frac{\beta}{2}\right)\langle +|+\rangle + \sin\left(\frac{\beta}{2}\right)\langle -|-\rangle\right|^2$$

$$= \frac{1}{2}(1 + \sin(\gamma))$$

Since $\beta = \gamma$. Now we expect half the particles in the $\hbar/2$ state if $\hat{n}$ is aligned in the x-axis and all the particles in the $\hbar/2$ state if $\hat{n}$ is orthogonal to the x-axis, which we can show

$$\begin{cases} P(\gamma = 0) = \dfrac{1}{2} \\ P(\gamma = \pi/2) = 1 \\ P(\gamma = \pi) = \dfrac{1}{2} \end{cases}$$

1.12.2 Dispersion

Evaluate the dispersion in S_x, that is

$$\langle (S_x - \langle S_x \rangle)^2 \rangle$$

(For your own peace of mind check your answers for the special cases $\gamma = 0, \pi/2, and\ \pi$)

We need to calculate

$$\begin{cases} S_x = \dfrac{\hbar}{2}[(|+\rangle\langle-|) + (|-\rangle\langle+|)] \\ \\ S_x^2 = \dfrac{\hbar^2}{4}[(|+\rangle\langle+|) + (|-\rangle\langle-|)] \end{cases}$$

Our arbitrary vector

$$|\Psi\rangle = \cos\left(\frac{\gamma}{2}\right)|+\rangle + \sin\left(\frac{\gamma}{2}\right)|-\rangle$$

$$\langle S_x\rangle = \langle\Psi|S_x|\Psi\rangle = \frac{\hbar}{2}\sin(\gamma)$$

$$\langle S_x^2\rangle = \frac{\hbar^2}{4}$$

Combining these results,

$$\langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4}\cos^2(\gamma)$$

Checking,

$$\begin{cases} \gamma = 0; \Delta S_x = \dfrac{\hbar^2}{4} \\ \\ \gamma = \dfrac{\pi}{2}; \Delta S_x = 0 \\ \\ \gamma = \pi; \Delta S_x = \dfrac{\hbar^2}{4} \end{cases}$$

1.13 Stern-Gerlach

A beam of spin 1/2-atoms goes through a series of Stern-Gerlach-type measurements as follows

a. The first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms.

b. The second measurement accepts $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms, where s_n is the eigenvalue of the operator $\vec{S}\cdot\hat{n}$, with $\hat{n}$ making an angle β in the xz-plane with respect to the z-axis.

c. The third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms. What is the intensity of the final $s_z = -\hbar/2$ beam when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity? How must we orient the second measuring apparatus if we are to maximize the intensity of the final $s_z = -\hbar/2$ beam?

Writing the first measurement in bra-ket notation, we only want the $|+\rangle$ state particles to survive,

$$M = |+\rangle\langle+|$$

For the second measurement,

$$M' = -|\hat{n};+\rangle\langle\hat{n};+|$$

where

$$|\hat{n};+\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle$$

For the third measurement, we only want the $|-\rangle$ particles to survive,

$$M'' = |-\rangle\langle-|$$

The total will then be

$$M''M'M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos^2\left(\frac{\beta}{2}\right) & \cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right) \\ \cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right) & \sin^2\left(\frac{\beta}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ \cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right) & 0 \end{bmatrix}$$

which in bra-ket notation is

$$M_{tot} = \cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right)|-\rangle\langle+|$$

Applying this to be a beam,

$$M_{tot}(|+\rangle + |-\rangle) = \cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right)|-\rangle$$

The intensity is related tot he beam squared, so

$$I = \cos^2\left(\frac{\beta}{2}\right)\sin^2\left(\frac{\beta}{2}\right) = \frac{1}{4}\sin^2(\beta)$$

which is maximized when $\beta = \pi/2$, which gives an intensity of 1/4 the initial beam.

1.14 Eigenvectors

A certain observable in quantum mechanics has a 3×3 matrix representation as follows:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$

1.14.1 Eigenvalues

Find the normalized eigenvectors of this observable and the corresponding eigenvalues. Is there any degeneracy?

Solving the characteristic equation,

$$\det(\Omega - \lambda I) = -\lambda(\lambda^2 - 1)$$

Giving the eigenvalues $\lambda = 0, \pm 1$. Solving for the eigenvectors,

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}; \quad |1\rangle = \frac{1}{2}\begin{bmatrix}1\\ \sqrt{2}\\ 1\end{bmatrix}; \quad |-1\rangle = \frac{1}{2}\begin{bmatrix}1\\ -\sqrt{2}\\ 1\end{bmatrix}$$

There is no degeneracy

1.14.2 Spin-1 Particles

Give a physical example where all this is relevant

Looking this up, these are the eigenvalues and eigenvectors of $\hbar A$ for the spin-1 particle. I believe this is further explained in chapter 3.

1.15 Simultaneous Eigenkets

Let A and B be observables. Suppose the simultaneous eigenkets of A and B $\{|a', b'\rangle\}$ form a complete orthonormal set of base kets. Can we always conclude that

$$[A, B] = 0$$

If your answer is yes, prove the assertion. If your answer is no, give a counterexample.

We start by inserting identity on both sides,

$$[A, B] = \sum_{a',b'} \sum_{a'',b''} |a'', b''\rangle \langle a'', b''| (AB - BA) |a', b'\rangle \langle a', b'|$$

If we act the operators on our ket, we use the eigenvalue,

$$AB |a', b'\rangle = a'b' |a', b'\rangle$$

$$[A, B] = \sum_{a',b'} \sum_{a'',b''} |a'', b''\rangle \langle a'', b''| (a'b' - b'a') |a', b'\rangle \langle a', b'|$$

We know that $a'b' - b'a' = 0$ since these are not operators, so we can move them around for free. $[A, B] = 0$ if the simultaneous eigenkets of A and B form a complete orthonormal set of base kets.

1.16 Simultaneous Eigenkets

Two Hermitian operators anti-commute:

$$\{A, B\} = AB + BA = 0$$

Is it possible to have a simultaneous (that is, common) eigenket of A and B? Prove or illustrate your assertion.

Let's act some eigenket on our anti-commutator,

$$\langle a''|AB|a'\rangle + \langle a''|BA|a'\rangle$$

$$= a'' \langle a''|B|a'\rangle + a' \langle a''|B|a'\rangle = (a'' + a') \langle a''|B|a'\rangle$$

Since $(a'' + a') \neq 0$, this implies that $\langle a''|B|a'\rangle = 0$ for both $a'' = a'$ and $a'' \neq a'$, which implies they do not have simultaneous eigenkets.

1.17 Degenerate Eigenkets

Two observables A_1 and A_2, which do not involve time explicitly, are known not to commute,

$$[A_1, A_2] \neq 0$$

yet we also know that A_1 and A_2 both commute with the Hamiltonian:

$$[A_1, \mathscr{H}] = 0; \quad [A_2, \mathscr{H}] = 0$$

Prove that the energy eigenstates are, in general, degenerate. Are there exceptions? As an example, you may think of the central-force problem $\mathscr{H} = \vec{p}^2/2m + V(r)$, with $A_1 \to L_z$, $A_2 \to L_x$.

We'll start by assuming the Hamiltonian has no degeneracy,

$$\mathscr{H} |n\rangle = E |n\rangle$$

is unique since there is no degeneracy.
Using the fact that our operators commute with the Hamiltonian,

$$[A_1, \mathscr{H}] = 0$$

$$A_1 \mathscr{H} |n\rangle - \mathscr{H} A_1 |n\rangle = 0$$

$$E(A_1 |n\rangle) = \mathscr{H}(A_1 |n\rangle)$$

$$A_1 |n\rangle = a_1 |n\rangle$$

Similarly, we can show

$$A_2 |n\rangle = a_2 |n\rangle$$

If we now act the commutator on our vector,

$$[A_1, A_2] |n\rangle = (a_1 a_2 - a_2 a_1) |n\rangle = 0$$

We know this cannot be true since

$$[A_1, A_2] \neq 0$$

Thus, energy eigenstates must be degenerate.

1.18 Uncertainty

1.18.1 Schwarz Inequality

The simplest way to derive the Schwarz inequality goes as follows. First, observe

$$(\langle\alpha| + \lambda^* \langle\beta|) \cdot (|\alpha\rangle + \lambda |\beta\rangle) \geq 0$$

for any complex number λ; then choose λ in such a way that the preceding inequality reduces to the Schwarz inequality.

Reminder the Schwarz inequality,

$$\langle\alpha|\alpha\rangle \langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

Let's start with the suggestion, which expanded out,

$$= \langle\alpha|\alpha\rangle + \lambda \langle\alpha|\beta\rangle + \lambda^* \langle\beta|\alpha\rangle + \lambda^* \lambda \langle\beta|\beta\rangle \geq 0$$

Let's set

$$\lambda = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$$

In this case, we'll get

$$\langle\alpha|\alpha\rangle - \frac{\langle\beta|\alpha\rangle \langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle} - \frac{\langle\alpha|\beta\rangle \langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} + \frac{\langle\alpha|\beta\rangle \langle\beta|\alpha\rangle \langle\beta|\beta\rangle}{\langle\beta|\beta\rangle} \geq 0$$

We multiply through by $\langle\beta|\beta\rangle$,

$$\langle\alpha|\alpha\rangle \langle\beta|\beta\rangle - \langle\beta|\alpha\rangle \langle\alpha|\beta\rangle \geq 0$$

$$\langle\alpha|\alpha\rangle \langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

1.18.2 Equality

Show that the equality sign in the generalized uncertainty relation holds if the state in question satisfies

$$\Delta A |\alpha\rangle = \lambda \Delta B |\alpha\rangle$$

with λ purely imaginary.

We know

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq |\langle\Delta A\Delta B\rangle|^2$$

$$|\langle\Delta A\Delta B\rangle|^2 = \frac{1}{4}|\langle[A,B]\rangle|^2 + \frac{1}{4}|\langle\{A,B\}\rangle|^2$$

We know that the dispersion,

$$\begin{cases} \Delta A = A - \langle A\rangle \\ \Delta B = B - \langle B\rangle \end{cases}$$

Taking the generalized uncertainty relation, let's take each term in $|\langle\Delta A\Delta B\rangle|^2$ and try to figure out what they are.

$$[A,B] = AB - BA$$

$$= (\Delta A + \langle A\rangle)(\Delta B + \langle B\rangle) - (\Delta B + \langle B\rangle)(\Delta A + \langle A\rangle)$$

$$[A,B] = [\Delta A, \Delta B]$$

Now looking at the dispersion,

$$\langle[A,B]\rangle = \langle\alpha|\Delta A\Delta B - \Delta B\Delta A|\alpha\rangle$$

$$= \lambda^* \langle\alpha|(\Delta B)^2\alpha\rangle - \lambda\,\langle\alpha|(\Delta B)^2\alpha\rangle$$

Since λ is purely imaginary,

$$= -2\lambda\,\langle\alpha|(\Delta B)^2|\alpha\rangle = -2\lambda\langle(\Delta B)^2\rangle$$

Similarly,

$$\{A,B\} = \{\Delta A, \Delta B\}$$

$$\langle\{\Delta A, \Delta B\}\rangle = \lambda^* \langle\alpha|(\Delta B)^2|\alpha\rangle + \lambda\,\langle\alpha|(\Delta B)^2|\alpha\rangle = 0$$

Therefore,

$$|\langle\Delta A\Delta B\rangle|^2 = \frac{1}{4} * 4\lambda^2\langle(\Delta B)^2\rangle^2$$

$$= \lambda^2\langle(\Delta B)^2\rangle^2$$

1.18.3 Gaussian Wave Packet

Explicit calculations using the usual rules of wave mechanics show that the wave function for a Gaussian wave packet given by

$$\langle x'|\alpha\rangle = (2\pi d^2)^{-1/4} \exp\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}\right]$$

satisfies the minimum uncertainty relation

$$\sqrt{\langle(\Delta x)^2\rangle}\sqrt{\langle(\Delta p)^2\rangle} = \frac{\hbar}{2}$$

Prove that the requirement

$$\langle x'|\Delta x|\alpha\rangle = (\textit{imaginary number})\,\langle x'|\Delta p|\alpha\rangle$$

is indeed satisfied for such a Gaussian wave packet, in agreement with (b)

Turning these into integrals,

$$\langle x'|\Delta x|\alpha\rangle = \int \langle x'|x''\rangle \langle x''|x|\alpha\rangle \; dx'' - \int \langle x'|x''\rangle \langle x''|\langle x\rangle|\alpha\rangle \; dx''$$

$$= \int \delta(x'-x'')x'' \langle x''|\alpha\rangle \; dx'' - \int \delta(x'-x'')\langle x\rangle \langle x''|\alpha\rangle \; dx''$$

Similarly,

$$\langle x'|\Delta p|\alpha\rangle = \int \delta(x'-x'')\left(-i\hbar\frac{\partial}{\partial x}\right)\langle x''|\alpha\rangle \; dx'' - \int \delta(x'-x'')\langle p\rangle \langle x''|\alpha\rangle \; dx''$$

Inserting in the Gaussian wave packet,

$$\langle x'|\Delta x|\alpha\rangle = \int \delta(x'-x'')x''(2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right) \; dx''$$

$$- \int \delta(x'-x'')\langle x\rangle(2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right) \; dx''$$

Similarly,

$$\langle x'|\Delta p|\alpha\rangle = \int \delta(x'-x'')\left(-i\hbar\frac{\partial}{\partial x''}\right)\left[(2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right)\right] \; dx''$$

$$- \int \delta(x'-x'')\langle p\rangle\left[(2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right)\right] \; dx''$$

$$= \int \delta(x' - x'') \left(i\hbar \frac{x''}{2d^2} \right) \left[(2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p \rangle x''}{\hbar} - \frac{(x'' - \langle x \rangle)^2}{4d^2} \right) \right] \, dx''$$

$$- \int \delta(x' - x'') \left(i\hbar \frac{\langle x \rangle}{2d^2} \right) \left[(2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p \rangle x''}{\hbar} - \frac{(x'' - \langle x \rangle)^2}{4d^2} \right) \right] \, dx''$$

Showing

$$\langle x' | \Delta x | \alpha \rangle = -\frac{2id^2}{\hbar} \, \langle x' | \Delta p | \alpha \rangle$$

1.19 Spin Expectation Value

1.19.1 Expectation Value

Compute

$$\langle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$$

where the expectation value is taken for the $|S_z;+\rangle$ state. Using your result, check the generalized uncertainty relation

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2$$

with $A \to S_X$, $B \to S_y$.

As a reminder,

$$\begin{cases} S_x = \dfrac{\hbar}{2}[(|+\rangle\langle-|) + (|-\rangle\langle+|)] \\ S_y = -\dfrac{i\hbar}{2}[(|+\rangle\langle-|) - (|-\rangle\langle+|)] \\ |S_z;+\rangle = |+\rangle \end{cases}$$

From this, we can find

$$\begin{cases} S_x^2 = \dfrac{\hbar^2}{4}[(|+\rangle\langle+|) + (|-\rangle\langle-|)] \\ S_y^2 = \dfrac{\hbar^2}{4}[(|+\rangle\langle+|) + (|-\rangle\langle-|)] \end{cases}$$

$$[S_x, S_y] = i\hbar S_z = \frac{i\hbar^2}{2}[(|+\rangle\langle+|) - (|-\rangle\langle-|)]$$

For the expectation values,

$$\langle S_x^2\rangle = \langle S_z;+|S_x^2|S_z;+\rangle = \frac{\hbar^2}{4}\langle+|\,[(|+\rangle\langle+|) + (|-\rangle\langle-|)]\,|+\rangle = \frac{\hbar^2}{4}$$

$$\langle S_x\rangle = \frac{\hbar}{2}\langle+|\,[(|+\rangle\langle-|) + (|-\rangle\langle+|)]\,|+\rangle = 0$$

$$\langle(\Delta S_x)^2\rangle = \frac{\hbar^2}{4}$$

Similarly, we can show

$$\langle (\Delta S_y)^2 \rangle = \frac{\hbar^2}{4}$$

Let's check the generalized uncertainty relation,

$$\langle [S_x, S_y] \rangle = \langle i\hbar S_z \rangle = \frac{i\hbar^2}{2} \langle + | \left[(|+\rangle \langle +|) - (|-\rangle \langle -|) \right] |+\rangle = \frac{i\hbar^2}{2}$$

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle \geq \frac{1}{4} |\langle [S_x, S_y] \rangle|^2$$

$$\frac{\hbar^4}{16} \geq \frac{\hbar^4}{16}$$

1.19.2 Uncertainty Relation

Check the uncertainty relation with $A \to S_x$, $B \to S_y$ for the $|S_x;+\rangle$ state.

As a reminder,

$$|S_x; +\rangle = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle$$

Let's calculate expectation values,

$$\langle S_x^2 \rangle = \frac{\hbar^2}{8} [\langle +| + \langle -|][(|+\rangle \langle +|) + (|-\rangle \langle -|)][|+\rangle + |-\rangle] = \frac{\hbar^2}{4}$$

$$\langle S_x \rangle = \frac{\hbar}{4} [\langle +| + \langle -|][(|+\rangle \langle -|) + (|-\rangle \langle +|)][|+\rangle + |-\rangle] = \frac{\hbar}{2}$$

$$\langle (\Delta S_x)^2 \rangle = 0$$

We can convince ourselves that $\langle (\Delta S_y)^2 \rangle = 0$.
Checking the uncertainty relation,

$$\langle [S_x, S_y] \rangle = \langle i\hbar S_z \rangle = \frac{i\hbar^2}{4} [\langle +| + \langle -|][(|+\rangle \langle +|) - (|-\rangle \langle -|)][|+\rangle + |-\rangle] = 0$$

1.20 Uncertainty Product

Find the linear combination of $|+\rangle$ and $|-\rangle$ kets that maximizes the uncertainty product

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle$$

Verify explicitly that for the linear combination you found, the uncertainty relation for S_x and S_y is not violated

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In general,

$$\langle(\Delta S_x)^2\rangle = \frac{\hbar^2}{4}\langle\Psi|[(|+\rangle\langle+|) + (|-\rangle\langle-|)]|\Psi\rangle - \left(\frac{\hbar}{2}\langle\Psi|[(|+\rangle\langle-|) + (|-\rangle\langle+|)]|\Psi\rangle\right)^2$$

$$\langle(\Delta S_y)^2\rangle = \frac{\hbar^2}{4}\langle\Psi|[(|+\rangle\langle+|) + (|-\rangle\langle-|)]|\Psi\rangle - \left(\frac{-i\hbar}{2}\langle\Psi|[(|+\rangle\langle-|) - (|-\rangle\langle+|)]|\Psi\rangle\right)^2$$

We can set

$$|\Psi\rangle = a\,|+\rangle + (1-a^2)^{1/2}\exp(i\beta)\,|-\rangle$$

This is functionally equivalent to $|\vec{S}\cdot\hat{n};+\rangle$. Inserting this in,

$$\langle(\Delta S_x)^2\rangle = \frac{\hbar^2}{4}[1 - 4a^2(1-a^2)\cos^2(\beta)]$$

$$\langle(\Delta S_y)^2\rangle = \frac{\hbar^2}{4}[1 - 4a^2(1-a^2)\sin^2(\beta)]$$

Multiplying the two together,

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \frac{\hbar^4}{16}[1 - 4a^2(1-a^2) + 4a^4(1-a^2)^2\sin^2(2\beta)]$$

We want to set$\beta = \pi/4$,

$$= \frac{\hbar^4}{16}[1 - 4a^2(1-a^2) + 4a^4(1-a^2)^2]$$

This is maximized when $a^2 = 0$ or 1, which means

$$|\Psi\rangle = \begin{cases} \pm\,|+\rangle \\ \exp\left(-\dfrac{i\pi}{4}\right)|-\rangle \end{cases}$$

$\pm\,|+\rangle$ has already been done in the previous question,

$$\exp\left(-\frac{\pi}{4}\right)\langle-|\left[\frac{i\hbar^2}{2}(|+\rangle\langle+|) - \frac{i\hbar^2}{2}(|-\rangle\langle-|\right]|-\rangle\exp\left(\frac{i\pi}{4}\right) = -\frac{i\hbar^2}{2}$$

The uncertainty relation gives $\hbar^4/16$ on both sides.

1.21 Uncertainty Product

Evaluate the $x - p$ uncertainty product $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle$ for a one-dimensional particle confined between two rigid walls

$$V = \begin{cases} 0 & for\ 0 < x < a, \\ \infty & otherwise \end{cases}$$

Do this for both the ground and excited states.

For a particle in a box, the solution is

$$\Psi = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

We want to calculate the uncertainty product,

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$$

So let's look at each component individually,

$$\langle x^2\rangle = \int_{-\infty}^{\infty} \Psi^* x^2 \Psi \ dx = \int_0^a \frac{2}{a} x^2 \sin\left(\frac{n\pi x}{a}\right) \ dx$$

$$= a^2 \left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)$$

$$\frac{2}{a}\int_0^a x \sin^2\left(\frac{n\pi x}{a}\right) \ dx = \frac{a}{2}$$

$$\langle(\Delta x)^2\rangle = a^2 \left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right)$$

For the momentum,

$$p = -i\hbar\frac{\partial}{\partial x}$$

It is helpful to know,

$$p\,|\Psi\rangle = -i\hbar\sqrt{\frac{2}{a}}\frac{n\pi}{a}\cos\left(\frac{n\pi x}{a}\right)$$

$$p^2\,|\Psi\rangle = \hbar^2\sqrt{\frac{2}{a}}\frac{n^2\pi^2}{a^2}\sin\left(\frac{n\pi x}{a}\right)$$

We can now calculate,

$$\langle p^2 \rangle = \frac{\hbar^2 n^2 \pi^2}{a^2} \frac{2}{a} \int_0^a \sin^2\left(\frac{n\pi x}{a}\right)\, dx = \frac{\hbar^2 n^2 \pi^2}{a^2}$$

$$\langle p \rangle = -\frac{2i\hbar n\pi}{a^2} \int_0^a \sin\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi x}{a}\right)\, dx = 0$$

$$\langle (\Delta p)^2 \rangle = \frac{\hbar^2 n^2 \pi^2}{a^2}$$

Combining these,

$$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle = \hbar^2 n^2 \pi^2 \left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right)$$

$$= \frac{\hbar^2}{2}\left(\frac{n^2\pi^2}{6} - 1\right)$$

1.22 Balancing a Pencil

Estimate the rough order of magnitude of the length of time that an ice pick can be balanced on its point if the only limitation is that set by the Heisenberg uncertainty principle. Assume that the point is sharp and that the point and the surface on which it rests are hard. You may make approximations which do not alter the general order of magnitude of the result. Assume reasonable values for the dimensions and weight of the ice pick. Obtain an approximate numerical result and express it in seconds.

We can model this as an inverted pendulum,

$$\theta(t) = a \exp\left(\sqrt{\frac{g}{l}}t\right) + b \exp\left(-\sqrt{\frac{g}{l}}t\right)$$

From the uncertainty principle,

$$\Delta x = l\theta = (a+b)l$$

$$\Delta p = ml\frac{d\theta}{dt} = \sqrt{\frac{g}{l}}(a-b)ml = m\sqrt{gl}(a-b)$$

1.23 Degenerate Eigenkets

Consider a three-dimensional ket space. If a certain set of orthonormal kets-say, $|1\rangle$, $|2\rangle$, $|3\rangle$-are used as the base kets, the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

with a and b both real.

1.23.1 Degenerate spectrum

Obviously A exhibits a degenerate spectrum. Does B also exhibit a degenerate spectrum?

To determine if B is degenerate, let's solve the characteristic equation,

$$\det(B - \lambda I) = \det \begin{pmatrix} b-\lambda & 0 & 0 \\ 0 & -\lambda & -ib \\ 0 & ib & -\lambda \end{pmatrix}$$

$$= (b - \lambda)(\lambda^2 - b^2)$$

Since there are repeated eigenvalues, $lambda = \pm b$, there is degeneracy.

1.23.2 Commute

Show that A and B commute.

To show that A and B commute, we need to show that $AB = BA$,

$$AB = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}$$

1.23.3 Simultaneous eigenkets

Find a new set of orthonormal kets which are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

A has eigenvalues $\lambda = \pm a$. To find a,

$$\begin{pmatrix} 0 & & \\ & -2a & \\ & & -2a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|a\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly,

$$|-a\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

We choose imaginary components because we know the solution to $|b\rangle$, and we want the eigenvectors to be orthonormal.

$$|b\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

We can show that these are simultaneous eigenkets

$$\begin{cases} A\,|a\rangle = a\,|a\rangle \\ A\,|-a\rangle = -a\,|-a\rangle \\ A\,|b\rangle = -a\,|b\rangle \end{cases}$$

$$\begin{cases} B\,|a\rangle = b\,|a\rangle \\ B\,|-a\rangle = -b\,|-a\rangle \\ B\,|b\rangle = b\,|b\rangle \end{cases}$$

1.24 Spinors

1.24.1 Matrix representation

Prove that $(1/\sqrt{2})(1+i\sigma_x)$ acting on a two-component spinor can be regarded as the matrix representation of the rotation operator about the x-axis by angle $-\pi/2$. (The minus sign signifies that the rotation is clockwise).

The rotation matrix is given by

$$\cos\left(\frac{\phi}{2}\right) - i\vec{\sigma}\cdot\hat{n}\sin\left(\frac{\phi}{2}\right) \tag{1.24.1}$$

Clockwise rotation about the $x-axis$ by $-\pi/2$ implies that $\phi = -\pi/2$,

$$\cos\left(\frac{-\pi}{4}\right) - i\vec{\sigma}\cdot\hat{x}\sin\left(-\frac{\pi}{4}\right)$$

$$\frac{1}{\sqrt{2}}(1+i\sigma_x)$$

1.24.2 Matrix Representation

Construct the matrix representation of S_z when the eigenkets of S_y are used as base vectors.

We can write S_z as

$$S_z = \frac{\hbar}{2}\frac{1}{\sqrt{2}}(1-i\sigma_x)\sigma_z\frac{1}{\sqrt{2}}(1+i\sigma_x)$$

$$= \frac{\hbar}{4}\begin{bmatrix}1 & -i\\ -i & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix}$$

$$= \frac{\hbar}{2}\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$$

1.25 Operator

Some authors define an operator to be real when every member of its matrix elements $\langle b'|A|b\rangle$ is real in some representation ($\{|b'\rangle\}$ basis in this case). Is this concept representation independent, that is, do the matrix elements remain real even if some basis other than $\{|b'\rangle\}$ is used? Check your assertion using familiar operators such as S_y and S_z (see Problem 24) or x and p_x.

Given a basis $\{|c\rangle\}$, we can write

$$|c'\rangle = \sum_{b'} |b'\rangle \langle b'|c'\rangle$$

We can insert an operator,

$$\langle c'|A|c''\rangle = \sum_{b'} \sum_{b''} \langle c'|b'\rangle \langle b'|A|b''\rangle \langle b''|c''\rangle$$

Since each braket is a scalar, we can arrange them freely,

$$= \sum_{b',b''} \langle c'|b'\rangle \langle b''|c''\rangle \langle b'|A|b''\rangle$$

$\langle c'|b'\rangle \langle b''|c''\rangle$ needs to be real, but the individual components don't need to be.

1.26 Transformation Matrix

Construct the transformation matrix that connects the S_z diagonal basis to the S_x diagonal basis. Show that your result is consistent with the general relation

$$U = \sum_r |b^{(r)}\rangle \langle a^{(r)}|$$

In the S_z basis,

$$\begin{cases} |S_z;+\rangle = |+\rangle \\ |S_z;-\rangle = |-\rangle \end{cases}$$

$$\begin{cases} |S_x;+\rangle = \dfrac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \\ \\ |S_x;-\rangle = \dfrac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \end{cases}$$

In the S_x basis,

$$\begin{cases} |S_x;+\rangle' = |+\rangle \\ |S_x;-\rangle' = |-\rangle \end{cases}$$

We want to find a matrix U such that

$$\begin{cases} |S_x;+\rangle' = U\,|S_x;+\rangle \\ |S_x;-\rangle' = U\,|S_x;-\rangle \end{cases}$$

We want to solve

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} \dfrac{1}{\sqrt{2}} \\ \\ \dfrac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} \dfrac{1}{\sqrt{2}} \\ \\ -\dfrac{1}{\sqrt{2}} \end{pmatrix}$$

The first set of equations gives us

$$\begin{cases} 1 = \dfrac{1}{\sqrt{2}}(U_{11} + U_{12}) \\ \\ 0 = \dfrac{1}{\sqrt{2}}(U_{21} + U_{22}) \end{cases}$$

which implies $U_{21} = -U_{22}$. We can do the same thing with the second equation to show that $U_{11} = U_{12}$. The easiest matrix that satisfies all of these conditions is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

We can find the transformation matrix another way,

$$U = \sum_{r} |S_x\rangle' \langle S_x| = \frac{1}{\sqrt{2}} \left[|+\rangle \left(\langle +| + \langle -| \right) + |-\rangle \left(\langle +| - \langle -| \right) \right]$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

1.27 Transformation Matrix

1.27.1 Functions

Suppose that $f(A)$ is a function of a Hermitian operator A with the property $A|a'\rangle = a'|a'\rangle$. Evaluate $\langle b''|f(A)|b'\rangle$ when the transformation matrix from the a' basis to the b' basis is known.

We want to insert identity,

$$\langle b''|f(A)|b'\rangle = \sum_{a',a''} \langle b''|a''\rangle \langle a''|f(A)|a'\rangle \langle a'|b'\rangle$$

We know that

$$f(A)|a'\rangle = f(a')|a'\rangle$$

Our sum is now,

$$= \sum_{a',a''} f(a') \langle b''|a''\rangle \langle a''|a'\rangle \langle a'|b'\rangle$$

By orthogonality,

$$= \sum_{a'} f(a') \langle b''|a'\rangle \langle a'|b'\rangle$$

1.27.2 Continuous Spectra

Using the continuum analogue of the result obtained in (a), evaluate

$$\langle \vec{p}''|F(r)|\vec{p}'\rangle$$

Simplify your expression as far as you can. Note that r is $\sqrt{x^2+y^2+z^2}$, where x, y, and z are operators.

In the continuous spectrum,

$$\langle \vec{p}''|F(r)|\vec{p}'\rangle = \int F(r') \langle \vec{p}''|\vec{r}'\rangle \langle \vec{r}'|\vec{p}'\rangle \; d\vec{r}'$$

Using

$$\langle \vec{r}'|\vec{p}'\rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left(\frac{i\vec{p}' \cdot \vec{r}'}{\hbar}\right)$$

We insert this,

$$= \frac{1}{(2\pi\hbar)^3} \int F(\vec{r}') \exp\left(\frac{i(\vec{p}' - \vec{p}'') \cdot \vec{r}'}{\hbar}\right) \; d\vec{r}'$$

In spherical coordinates,

$$= \frac{2\pi}{(2\pi\hbar)^3} \int_{-1}^{1} \int_{0}^{\infty} r'^2 F(r') \exp\left(\frac{i(\vec{p}' - \vec{p}'')r' \cos(\theta)}{\hbar} \right) \, dr' d\cos(\theta)$$

$$= \frac{1}{2\pi^2\hbar^2 q} \int_0^{\infty} \sin\left(\frac{qr'}{\hbar}\right) F(r') \, dr'$$

with $q = |\vec{p}' - \vec{p}''|$.

1.28 Poisson Bracket

1.28.1 Classical Poisson Bracket

Let x and p_x be the coordinate and linear momentum in one dimension. Evaluate the classical Poisson bracket

$$[x, F(p_x)]_{classical}$$

By definition,

$$[x, F(p_x)]_{cl} = \frac{\partial x}{\partial x}\frac{\partial F(p_x)}{\partial p_x} - \frac{\partial x}{\partial p_x}\frac{\partial F(p_x)}{\partial x}$$

The second term dies so,

$$= \frac{\partial F(p_x)}{\partial p_x}$$

1.28.2 Commutator

Let x and p_x be the corresponding quantum-mechanical operators this time. Evaluate the commutator

$$\left[x, \exp\left(\frac{ip_x a}{\hbar}\right)\right]$$

Using the Dirac rule,

$$[,]_{cl} = \frac{[,]}{i\hbar}$$

which means we simple need to evaluate,

$$\frac{\partial \exp\left(\frac{ip_x a}{\hbar}\right)}{\partial p_x} = \frac{ia}{\hbar}\exp\left(\frac{ip_x a}{\hbar}\right)$$

$$\left[x, \exp\left(\frac{ip_x a}{\hbar}\right)\right] = -a\exp\left(\frac{ip_x a}{\hbar}\right)$$

1.28.3 Eigenvalues

Using the result obtained in (b), prove that

$$\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle, \quad (x\,|x'\rangle = x'\,|x'\rangle)$$

is an eigenstate of the coordinate operator x. What is the corresponding eigenvalue?

Using our vector,

$$|\Psi\rangle = \exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle$$

Expanding the commutation relation from above,

$$xF(p_x) - F(p_x)x = - = aF(p_x)$$

Acting x on our vector,

$$x|\Psi\rangle = x\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle$$

$$= -a\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle + \exp\left(\frac{ip_x a}{\hbar}\right)x'|x'\rangle$$

$$= (x' - a)|\Psi\rangle$$

1.29 Commutation Relations

1.29.1 Gottfried

On page 247, Gottfried (1966) states that

$$[x_i, G(\vec{p})] = i\hbar \frac{\partial G}{\partial p_i}, \quad [p_i, F(\vec{x})] = -i\hbar \frac{\partial F}{\partial x_i}$$

can be easily derived from the fundamental commutation relations for all functions of F and G that can be expressed as power series in their arguments. Verify this statement.

As we showed in the previous question,

$$[x_i, G(\vec{p})]_{cl} = \frac{\partial G}{\partial p_i}$$

Using Dirac's rule,

$$[x_i, G(\vec{p})] = i\hbar \frac{\partial G}{\partial p_i}$$

The same argument can be made for $[p_i, F(\vec{x})]$.

1.29.2 Example

Evaluate $[x^2, p^2]$. Compare your result with the classical Poisson bracket $[x^2, p^2]_{classical}$.

Acting the commutator on a general vector,

$$[x^2, p^2] \left|\Psi\right\rangle = -x^2\hbar^2 \frac{\partial^2 \Psi}{\partial x^2} + \hbar^2 \frac{\partial^2 (x^2 \Psi)}{\partial x^2}$$

Evaluating,

$$= \hbar^2 [2\Psi + 4x\Psi']$$

$$[x^2, p^2] = 2i\hbar\{x, p\}$$

$$[x^2, p^2]_{cl} = 4xp$$

We can make these two equal using Dirac's rule.

1.30 Translation Operator

The translation operator for a finite (spatial) displacement is given by

$$\mathscr{T}(\vec{l}) = \exp\left(\frac{-i\vec{p}\cdot\vec{l}}{\hbar}\right)$$

where $\vec{p}$ is the momentum operator.

1.30.1 Commutation

Evaluate

$$[x_i, \mathscr{T}(\vec{l})]$$

Using Dirac's rule as well as the result from question 28,

$$[x_i, \mathscr{T}(\vec{l})]_{cl} = \frac{[x_i, \mathscr{T}(\vec{l})]}{i\hbar}$$

$$= \sum_i \frac{\partial}{\partial p_i}\left(\exp\left(-\frac{i\vec{p}\cdot\vec{l}}{\hbar}\right)\right) = -\frac{i\vec{l}}{\hbar}\exp\left(-\frac{i\vec{p}\cdot\vec{l}}{\hbar}\right)$$

$$[x_i, \mathscr{T}(\vec{l})] = \sum_i l_i \mathscr{T}(\vec{l})$$

1.30.2 Translation

Using (a)(or otherwise), demonstrate how the expectation value $\langle x\rangle$ changes under translation.

Acting the translation operator on a vector gives

$$|\Psi'\rangle = \mathscr{T}(\vec{l})\,|\Psi\rangle$$

The expectation value

$$\langle\Psi'|x|\Psi\rangle = \langle\Psi|\mathscr{T}(\vec{l})^\dagger x \mathscr{T}(\vec{l})|\Psi\rangle$$

Using the commutation relation from before,

$$= \langle\Psi|\mathscr{T}^\dagger(\vec{l})\mathscr{T}(\vec{l})x|\Psi\rangle + \langle\Psi|\mathscr{T}^\dagger(\vec{l})l_i\mathscr{T}(\vec{l})|\Psi\rangle$$

$$= \langle\Psi|x|\Psi\rangle + \sum_i l_i$$

$$= \langle x\rangle + \vec{l}$$

1.31 Translation Operator

In the main text we discussed the effect of $\mathscr{T}(d\vec{x}')$ on the position and momentum eigenkets and on a more general state ket $|\alpha\rangle$. We can also study the behavior of expectation values $\langle x\rangle$ and $\langle p\rangle$ under infinitesimal translation. Using (1.6.25), (1.6.45), and $|\alpha\rangle \rightarrow \mathscr{T}(d\vec{x}')\,|\alpha\rangle$ only, prove $\langle x\rangle \rightarrow \langle x\rangle + d\vec{x}'$, $\langle p\rangle \rightarrow \langle p\rangle$ under infinitesimal translation.

1.6.25,

$$[\vec{x}, \mathscr{T}(d\vec{x}')] = d\vec{x}'$$

1.6.45,

$$[\vec{p}, \mathscr{T}(d\vec{x}')] = 0$$

The expectation value of x,

$$\langle\alpha'|x|\alpha'\rangle = \langle\alpha|\mathscr{T}^{\dagger}(d\vec{x}')x\mathscr{T}(d\vec{x}')|\alpha\rangle$$

$$= \langle\alpha|\mathscr{T}^{\dagger}(d\vec{x}')(\mathscr{T}(d\vec{x}')x + d\vec{x}')|\alpha\rangle$$

$$= \langle x\rangle + d\vec{x}'$$

Similarly,

$$\langle\alpha'|p|\alpha\rangle = \langle\alpha|\mathscr{T}^{\dagger}(d\vec{x}')\vec{p}\mathscr{T}(d\vec{x}')|\alpha\rangle$$

$$= \langle\alpha|\mathscr{T}^{\dagger}(d\vec{x}')\mathscr{T}(d\vec{x}')\vec{p}|\alpha\rangle$$

$$= \langle\alpha|\vec{p}|\alpha\rangle = \langle p\rangle$$

1.32 Gaussian Wave Packet

1.32.1 Expectation Values

Verify (1.7.39a) and (1.7.39b) for the expectation value of p and p^2 from the Gaussian wave packet of (1.7.35)

The Gaussian wave packet,

$$\langle x'|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \exp\left(ikx' - \frac{x'^2}{2d^2}\right)$$

The expectation value is

$$\langle p\rangle = \langle\alpha|x'\rangle \left(-i\hbar\frac{\partial}{\partial x}\right)\langle x'|\alpha\rangle$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi^{1/2}d} \exp\left(-ikx' - \frac{x'^2}{2d^2}\right)(-i\hbar)\left(\exp\left(ikx' - \frac{x'^2}{2d^2}\right)\left(ik - \frac{x'}{d^2}\right)\right)\, dx'$$

$$= \int_{-\infty}^{\infty} \frac{-i\hbar\left(ik - \frac{x'}{d^2}\right)}{\pi^{1/2}d} \exp\left(-\frac{x'^2}{d^2}\right)\, dx'$$

$$= \hbar k$$

For p^2,

$$\langle p^2\rangle = -\hbar^2\langle\alpha|x'\rangle \frac{\partial^2}{\partial x^2}\langle x'|\alpha\rangle$$

Since we know

$$\int_{-\infty}^{\infty} x\exp(-x^2)\, dx = 0$$

we can only write the terms that don't die,

$$= \int_{-\infty}^{\infty} \frac{\hbar^2 k^2}{\pi^{1/2}d} \exp\left(-\frac{x'^2}{d^2}\right) + \frac{\hbar^2}{\pi^{1/2}d^3} \exp\left(-\frac{x'^2}{d^2}\right) - \frac{\hbar^2 x'^2}{\pi^{1/2}d^5} \exp\left(-\frac{x'^2}{d^2}\right)\, dx$$

$$= \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$

1.32.2 Momentum space wave function

Evaluate the expectation value of p and p^2 using the momentum space wave function (1.7.42).

The wave packet in momentum space,

$$\langle p'|\alpha\rangle = \frac{d^{1/2}}{\hbar^{1/2}\pi^{1/4}} \exp\left(-\frac{(p'-\hbar k)^2 d^2}{2\hbar^2}\right)$$

For p,

$$\langle p\rangle = \langle\alpha|p\rangle\, p\, \langle p|\alpha\rangle$$

$$= \int_{-\infty}^{\infty} \frac{d}{\hbar\pi^{1/2}} p \exp\left(-\frac{(p'-\hbar k)^2 d^2}{\hbar^2}\right)\, dp$$

$$= \hbar k$$

Similarly,

$$\langle p^2\rangle = \langle\alpha|p\rangle\, p^2\, \langle p|\alpha\rangle$$

$$= \int_{-\infty}^{\infty} \frac{d}{\hbar\pi^{1/2}} p^2 \exp\left(-\frac{(p-\hbar k)^2 d^2}{\hbar^2}\right)\, dp$$

$$= \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$

1.33 Momentum Translation Operator

1.33.1 Proofs

Prove the following:

$$\begin{cases} \langle p'|x|\alpha\rangle = i\hbar \dfrac{\partial}{\partial p'} \langle p'|\alpha\rangle \\ \\ \langle \beta|x|\alpha\rangle = \int dp' \ \phi^*_\beta(p') i\hbar \dfrac{\partial}{\partial p'} \phi_\alpha(p') \end{cases}$$

$$\langle p'|x|\alpha\rangle = \langle p'|x|p''\rangle \langle p''|alpha\rangle = \langle p'|x|p'\rangle \langle p'|\alpha\rangle$$

$$= i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle$$

$$\langle \beta|x|\alpha\rangle = \langle \beta|p'\rangle \langle p'|x|p'\rangle \langle p'|\alpha\rangle$$

$$= \int dp' \phi^*_\beta(p') i\hbar \frac{\partial}{\partial p'} \phi_\alpha(p')$$

1.33.2 Physical significance

What is the physical significance of

$$\exp\left(\frac{ix\Xi}{\hbar}\right)$$

where x is the position operator and Ξ is some number with the dimension of momentum? Justify your answer

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This is the momentum translation operator.