# Jackson Solutions

Benjamin D. Suh

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# Contents

# Chapter 1

# Introduction to Electrostatics

## 1.1 Properties of Conductors

**Use Gauss's theorem (1.18) [and (1.21) if necessary] to prove the following:**

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### 1.1.1 Charges Placed on Conductor

**Any excess charge placed on a conductor must lie entirely on its surface. (A conductor by definition contains charges capable of moving freely under the action of applied electric fields).**

We know that the electric field inside a conductor is zero. Looking at Gauss's Law (1.18), the divergence of zero must also be zero, so the charge density is zero. Any excess charge must move to the surface in order to maintain the absence of an electric field inside the conductor.

### 1.1.2 Conductors in Electric Fields

**A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to charges placed inside it.**

In the first case, we put a charge outside then apply Gauss's law (1.18) to the inside of the conductor. Since there is no charge contained in our Gaussian surface, $\vec{E} = 0$.

In the second case, we place a charge inside the conductor then create a Gaussian surface enclosing the conductor. Since there is a charge enclosed, $\vec{E} \neq 0$.

### 1.1.3 Electric Field Strength at the Surface

**The electric field at the surface of a conductor is normal to the surface and has a magnitude $\sigma/\epsilon_0$, where $\sigma$ is the charge density per unit area on the surface.**

We'll take a symmetric surface without loss of generality.

$$E \cdot A = \frac{q_{enc}}{\epsilon_0}$$

$$E = \frac{q_{enc}}{\epsilon_0 A} = \frac{\sigma}{\epsilon_0}$$

## 1.2 Dirac Delta Function and Gaussian Function

**The Dirac delta function in three dimensions can be taken as the improper limit as $\alpha \to 0$ of the Gaussian function**

$$D(\alpha; x, y, z) = (2\pi)^{-3/2}\alpha^{-3}exp\left[-\frac{1}{2\alpha^2}(x^2+y^2+z^2)\right]$$

**Consider a general orthogonal coordinate system specified by the surfaces u=constant, v=constant, w=constant, with length elements du/U, dv/V, dw/W in the three perpendicular directions. Show that**

$$\delta(\vec{x}-\vec{x}') = \delta(u-u')\ delta(v-v')\ delta(w-w')\cdot UVW$$

**by considering the limit of the Gaussian above. Note that as $\alpha \to 0$ only the infinitesimal length element need be used for the distance between the points in the exponent.**

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From the statement of the problem, we expect

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-x')\ \delta(y-y')\ \delta(z-z')\ dx\ dy\ dz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lim_{\alpha\to 0} D(\alpha; x, y, z)\ dx\ dy\ dz$$

Now, we run into some problems going from $\{x, y, z\} \to \{u, v, w\}$. Since we don't have a map, we can't make a conversion, which would be the most straightforward method. In any case, we can still get to the desired result. In the $\{u, v, w\}$ basis, we can write the right side as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lim_{\alpha\to 0} D(\alpha; x, y, z)\frac{du\ dv\ dw}{UVW} = 1$$

The solution presents itself if we make the substitution

$$\lim_{\alpha\to 0}\frac{D(\alpha; u, v, w)}{UVW} = F(u, v, w)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v, w)\ du\ dv\ dw = 1$$

Comparing to the integral of the Dirac delta function (1.10),

$$F(u, v, w) = \delta(u-u')\ \delta(v-v')\ \delta(w-w')$$

$$\lim_{\alpha\to 0}\frac{D(\alpha; u, v, w)}{UVW} = \delta(u-u')\ \delta(v-v')\ \delta(w-w')$$

$$\lim_{\alpha\to 0} D(\alpha; u, v, w) = \delta(u-u')\ \delta(v-v')\ \delta(w-w')\cdot UVW$$

Comparing to what we expect to get,

$$\delta(\vec{x}-\vec{x}') = \delta(u-u')\ \delta(v-v')\ \delta(w-w')\cdot UVW$$

Note that because these two are the same other than a renaming of variables,

$$\lim_{\alpha\to 0} D(\alpha; u, v, w) = \lim_{\alpha\to 0}(\alpha; x, y, z)$$

## 1.3 Charge Distributions

**Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(\vec{x})$.**

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### 1.3.1 Spherical Coordinates, Spherical Shell

**In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.**

For a spherical shell charge distribution, the general form of the charge distribution is given by

$$\rho(\vec{x}) = c\ \delta(r - R)$$

We obtain this from noting that our charge distribution has no angular dependence. In addition, there should only be a charge on the shell, which is a distance $R$ from the origin. In order to find the value of the constant $c$, we integrate over all space to get the total charge $Q$.

$$\int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} \rho(\vec{x}) r^2\ dr\ d(cos(\theta))\ d\phi = Q$$

$$4\pi c \int_0^{\infty} r^2\ \delta(r - R)\ dr = Q$$

$$4\pi c R^2 = Q$$

$$c = \frac{Q}{4\pi R^2}$$

Our charge distribution is given by

$$\rho(\vec{x}) = \frac{Q}{4\pi R^2} \delta(r - R)$$

### 1.3.2 Cylindrical Coordinates, Cylindrical Surface

**In cylindrical coordinates, a charge $\lambda$ per unit length uniformly distributed over a cylindrical surface of radius $b$.**

For a cylindrical shell distribution, the general form of the charge distribution is given by

$$\rho(\vec{x}) = c\ \delta(r - b)$$

We obtain this from noting that there is no angular dependence or z-dependence. Now, solving for c in the same method as before,

$$\int_0^{L} \int_0^{2\pi} \int_0^{\infty} \rho(\vec{x}) r\ r\ d\theta\ dz = \lambda L$$

Note that we integrate from 0 to L in the z-direction. We do this since we will have infinite charge as the cylinder's length goes to infinity. It cancels out anyways, so we can ignore that term.

$$2\pi c \int_0^\infty r\ \delta(r-b)\ dr = \lambda$$

$$2\pi cb = \lambda$$

$$c = \frac{\lambda}{2\pi b}$$

The charge distribution is given by

$$\rho(\vec{x}) = \frac{\lambda}{2\pi b}\delta(r-b)$$

### 1.3.3 Cylindrical Coordinates, Flat Disc

**In cylindrical coordinates, a charge $Q$ spread uniformly over a flat circular disc of negligible thickness and radius $R$.**

For a flat disc in cylindrical coordinates, the general form of the charge distribution is given by

$$\rho(\vec{x}) = c\ \delta(z)\ \Theta(R-r)$$

We do this by noting that a cylinder is a stack of infinitesimal disks, so we just pick one. For ease of calculation, we pick z=0. Solving for c,

$$\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^\infty \rho(\vec{x}) r\ dr\ d\theta\ dz = Q$$

$$2\pi c \int_{-\infty}^{\infty}\int_0^\infty \delta(z)\Theta(R-r) r\ dr\ dz = Q$$

$$2\pi c \int_0^R r\ dr = Q$$

$$2\pi c \frac{R^2}{2} = Q$$

$$c = \frac{Q}{\pi R^2}$$

The charge distribution is given by

$$\rho(\vec{x}) = \frac{Q}{\pi R^2}\delta(z)\ \Theta(R-r)$$

### 1.3.4 Spherical Coordinates, Flat Disc

**The same as part(c), but using spherical coordinates.**

There's a subtlety that we should note. The volume of the sphere goes by $r^3$, but the charge enclosed increases by $r^2$. Thus, our charge density should have a $r^{-1}$ dependence to make units match. Thus, the general charge distribution is given by

$$\rho(\vec{x}) = cr^{-1}\ \delta(cos(\theta))\ \Theta(R-r)$$

Solving for $c$,

$$\int_0^{2\pi}\int_{-1}^{1}\int_0^{\infty} \rho(\vec{x})r^2\ dr\ d(cos(\theta))\ d\phi = Q$$

$$2\pi c\int_{-1}^{1}\int_0^{\infty} \delta(cos(\theta))\ \Theta(R-r)r\ dr\ d(cos(\theta))\ d\phi = Q$$

$$2\pi c\int_0^{R} r\ dr = Q$$

$$2\pi c\frac{R^2}{2} = Q$$

$$c = \frac{Q}{\pi R^2}$$

The charge distribution is given by

$$\rho(\vec{x}) = \frac{Q}{\pi R^2 r}\delta(cos(\theta))\ \Theta(R-r)$$

## 1.4 Gauss's Law

**Each of three charged spheres of radius $a$, one conducting, one having a uniform charge density within its volume, and one having a spherically symmetric charge density that varies radially as $r^n$ (n¿-3), has a total charge $Q$. Use Gauss's theorem to obtain the electric fields both inside and outside each sphere**.

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Since each sphere has the same total charge, and since we remember that outside the sphere, we can treat it as a point charge $Q$, the electric field outside of each sphere is going to be the same regardless of charge distribution (for this problem. If we had some weird configuration (like a taegeugdo symbol), we wouldn't be able to use this result. Actually, I don't even think we could use Gauss's Law for that. In any case,).

$$4\pi r^2 \ \vec{E}_{out} = \frac{Q}{\epsilon_0}\hat{r}$$

$$\vec{E}_{out}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

### 1.4.1 Conductor

In a conductor, all of the charge will redistribute to the surface. The charge distribution is

$$\rho(r) = \frac{Q}{4\pi a^2}\delta(r-a)$$

Since Gaussian surface inside the sphere will never contain any charge, the electric field inside the conductor will be 0.

$$\vec{E}_{in}(\vec{x}) = 0$$

### 1.4.2 Uniform Charge Density

The charge density takes the form,

$$\rho(r) = c\ \Theta(a-r)$$

Using the methods developed in the previous problem, we can show that the charge density is

$$\rho(r) = \frac{3Q}{4\pi a^3}\Theta(a-r)$$

Using Gauss's Law (1.18),

$$4\pi r^2 \ \vec{E}_{in} = \frac{4}{3}\pi r^3 \frac{\rho(r)}{\epsilon_0}\hat{r}$$

$$4\pi r^2 \ \vec{E}_{in} = \frac{4}{3}\pi r^3 \frac{3Q}{4\pi\epsilon_0 a^3}\hat{r}$$

$$\vec{E}_{in}(\vec{x}) = \frac{Qr}{4\pi\epsilon_0 a^3}\hat{r}$$

### 1.4.3 Spherically Symmetric Charge Density

The charge density takes the form

$$\rho(r) = c\ r^n$$

An the explicit form of the charge density is

$$\rho(r) = \frac{(n+3)Q}{4\pi a^{n+3}} r^n\ \Theta(a-r)$$

Applying Gauss's Law (1.18),

$$4\pi r^2 \vec{E}_{in} = \frac{1}{\epsilon_0}\int_0^r 4\pi r^2 \rho(r)\hat{r}\ dr$$

$$4^2 \vec{E}_{in} = \frac{(n+3)Q}{4\pi\epsilon_0 a^{n+3}}\int_0^r r^{n+2}\hat{r}\ dr$$

$$r^2 \vec{E}_{in} = \frac{(n+3)Q}{4\pi\epsilon_0 a^{n+3}} r^{n+3} n + 3\hat{r}$$

$$\vec{E}(\vec{x}) = \frac{Q\ r^{n+1}}{4\pi\epsilon_0 a^{n+3}}\hat{r}$$

## 1.5 Charge Distribution of a Hydrogen Atom

**The time-averaged potential of a neutral hydrogen atom is given by**

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right)$$

**where $q$ is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, $a_0$ being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.**

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Since we have no angular dependence, Poisson's equation (1.35) will have only a radial component.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = -\frac{\rho}{\epsilon_0}$$

Plugging in the given potential and solving for $\rho$,

$$\rho = -\frac{q\alpha^3 e^{-\alpha r}}{8\pi\epsilon_0}$$

However, we see that at $r = 0$, $\Phi$ explodes. Near the origin, because the exponential term dies off faster than the polynomial, our potential becomes

$$\Phi = \frac{q}{4\pi\epsilon_0 r}$$

This is the same as the potential due to a point charge at the origin, so we have to ad a $q\ \delta(r)$ into our final charge density (alternatively, use equation (1.37) to solve Poisson's equation). Our final charge density is given by

$$\rho(r) = q\ \delta(r) - \frac{q\alpha^3 e^{-\alpha r}}{8\pi\epsilon_0}$$

Thinking about it, this makes sense physically as it represents a charge at the center (nucleus) surrounded by an electron cloud that drops off as we move away from the origin.

## 1.6 Capacitance

**A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance (in SI units it is measured in farads). Using Gauss's Law, calculate the capacitance of**

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### 1.6.1 Two Flat, Conducting Sheets

**two large, flat, conducting sheets of area $A$, separated by a small distance $d$**

In our examples from section 1.2, we found the electric field due to two plates,

$$E = \frac{Q}{\epsilon_0 A}$$

Using equation (1.22),

$$V = \int_0^d \frac{Q}{\epsilon_0 A} dz$$

$$V = \frac{Qd}{\epsilon_0 A}$$

If we remember from undergraduate EM, capacitance is given by $C = \dfrac{Q}{V}$, so the capacitance of this system is

$$C = \frac{A\epsilon_0}{d}$$

### 1.6.2 Two Concentric, Conducting Spheres

**two concentric conducting spheres with radii $a$, $b$ ($b > a$)**

We are interested in the electric field outside of the inner sphere. Since the spheres are hollow and we are looking at the region between the spheres, we can ignore the charge on the outer sphere.

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$

Solving for the potential (1.22),

$$V = \int_a^b \frac{Q}{4\pi\epsilon_0 r^2} dr$$

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

The capacitance is

$$4\pi\epsilon_0 \frac{ab}{b-a}$$

### 1.6.3 Two Concentric, Conducting Cylinders

**two concentric conducting cylinders of length $L$, large compared to their radii $a$, $b$ ($b > a$)**

In section 1.2, we found the electric field due to a wire, but as long as we are outside the wire, it shouldn't matter how thick the wire is (similar to how we can view a sphere as a point particle in the previous part).

$$E = \frac{Q}{2\pi\epsilon_0 r L}$$

Using equation (1.22),

$$V = \frac{Q}{2\pi\epsilon_0 L}\int_a^b \frac{1}{r}dr$$

$$V = \frac{Q}{2\pi\epsilon_0 L} ln\left(\frac{b}{a}\right)$$

The capacitance is given by

$$C = \frac{2\pi\epsilon_0 L}{ln\left(\frac{b}{a}\right)}$$

### 1.6.4 Calculator Work

**What is the inner diameter of the outer conductor in an air-filled coaxial cable whose center conductor is a cylindrical wire of diameter 1mm and whose capacitance is $3\times10^{-11}$F/m? $3\times10^{-12}$F/m?**

Using the result from part 1.6.3 and solving for $b$,

$$b = a\ exp\left(2\pi\epsilon_0 \frac{L}{C}\right)$$

For the first case, b=3mm. For the second, b=77km, which is a fairly sizable increase.

## 1.7 Capacitance of Two Parallel Cylinders

**Two long, cylindrical conductors of radii $a_1$ and $a_2$ are parallel and separated by a distance $d$, which is large compared with either radius. Show that the capacitance per unit length is given approximately by**

$$C \approx \pi\epsilon_0 \left( ln\frac{d}{a} \right)^{-1}$$

**where $a$ is the geometrical mean of the two radii.**

**Approximately what gauge wire (state diameter in millimeters) would be necessary to make a two-wire transmission line with a capacitance of $1.2\times10^{-11}$F/m if the separation of the wires was 0.5cm? 1.5cm? 5.0cm?**

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Let's place the capacitors as shown in figure (1.1) with one cylinder at the origin (which we'll say holds a negative charge) and the other a distance $d$ away (which we'll say holds a positive charge).

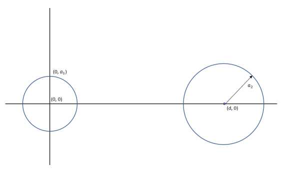


Figure 1.1: Cross-section view

The electric fields outside of each cylinder are

$$\vec{E}_1 = \frac{Q}{2\pi\epsilon_0 r L}\hat{r}$$

$$\vec{E}_2 = -\frac{Q}{2\pi\epsilon_0 r' L}\hat{r}'$$

In Cartesian coordinates,

$$r^2 = x^2 + y^2; \quad \hat{r} = \frac{1}{r}(x\ \hat{x} + y\ \hat{y})$$

$$r'^2 = (x-d)^2 + y^2; \quad \hat{r}' = \frac{1}{r'}((x-d)\ \hat{x} + y\ \hat{y})$$

We find the magnitude of the electric field between the two cylinders by adding the x-components of the electric field between them. Note that we only care about the x-component because we want to pick a point lying on the line directly between them. Furthermore, this means that we can set $y = 0$.

$$E = \frac{Q}{2\pi\epsilon_0 L}\left( \frac{x}{x^2} - \frac{x-d}{(x-d)^2} \right)$$

Using equation (1.22),

$$V = \frac{Q}{2\pi\epsilon_0 L}\int_{a_1}^{d-a_2} \frac{1}{x} - \frac{1}{x-d}\ dx$$

$$= \frac{Q}{2\pi\epsilon_0 L} ln\left(\frac{d-a_2}{a_1} \cdot \frac{d-a_1}{a_2}\right)$$

By saying that $d$ is large compared to $a_1$ and $a_2$,

$$\approx \frac{Q}{2\pi\epsilon_0 L} ln\left(\frac{d^2}{\sqrt{a_1 a_2}}\right)$$

I actually didn't know this term before doing this problem, but geometric mean is $a = \sqrt{a_1 a_2}$.

$$V \approx \frac{Q}{\pi\epsilon_0 L} ln\left(\frac{d}{a}\right)$$

We get what Jackson says we should get,

$$\frac{C}{L} = \pi\epsilon_0 \left(ln\left(\frac{d}{a}\right)\right)^{-1}$$

Plugging things into a calculator, we find that a distance of 0.5cm corresponds to a radius of 1mm (18 gauge). A distance of 1.5cm corresponds to a radius of 3mm (9 gauge). A distance of 5.0cm corresponds to a radius of 10mm (00 gauge).

## 1.8 Energy in a Capacitor

**For the three capacitor geometries in Problem 1.6, calculate the total electrostatic energy and express it alternatively in terms of the equal and opposite charges $Q$ and $-Q$ placed on the conductors and the potential difference between them.**

**Sketch the energy density of the electrostatic field in each case as a function of the appropriate linear coordinate.**

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Since the charge is located solely at the surface, we only need to worry about the potential on the surface since our charge density will include a Dirac delta function at the surface. Thus, the potential energy (1.32) simplifies to

$$W = \frac{1}{2}QV$$

For the second part, I don't want to sketch, so I'm just going to give the energy density (1.34).

### 1.8.1 Two Flat, Conducting Sheets

As a reminder, we found

$$E = \frac{Q}{A\epsilon_0}$$

$$V = \frac{Qd}{A\epsilon_0}$$

$$Q = \frac{AV\epsilon_0}{d}$$

For a fixed charge,

$$W = \frac{1}{2}\frac{Q^2 d}{A\epsilon_0}$$

For a fixed potential difference,

$$W = \frac{\epsilon_0 AV^2}{2d}$$

The energy density is

$$w = \frac{Q^2}{2\epsilon_0 A^2}$$

### 1.8.2 Two Concentric, Conducting Spheres

As a reminder, we found

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$Q = 4\pi\epsilon_0 V\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}$$

For a fixed charge,

$$W = \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

For a fixed potential difference,

$$W = 2\pi\epsilon_0 V^2\left(\frac{ab}{b-a}\right)$$

The energy density is

$$w = \frac{Q^2}{32\pi^2\epsilon_0 r^4}$$

### 1.8.3 Two Concentric, Conducting Cylinders

As a reminder, we found

$$E = \frac{Q}{2\pi\epsilon_0 rL}$$

$$V = \frac{Q}{2\pi\epsilon_0 L} ln\left(\frac{b}{a}\right)$$

$$Q = \frac{2\pi\epsilon_0 LV}{ln\left(\frac{b}{a}\right)}$$

For a fixed charge,

$$W = \frac{Q^2}{4\pi\epsilon_0 L} ln\left(\frac{b}{a}\right)$$

For a fixed potential difference,

$$W = \frac{\pi\epsilon_0 L V^2}{ln\left(\dfrac{b}{a}\right)}$$

The energy density is

$$w = \frac{Q^2}{8\pi\epsilon_0 r^2 L^2}$$

## 1.9 Force between Capacitors

**Calculate the attractive force between conductors in the parallel plate capacitor (Problem 1.6a) and the parallel cylinder capacitor (Problem 1.7) for (a). fixed charges on each conductor (b). fixed potential difference between conductors**

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### 1.9.1 Parallel Plate Capacitors

To calculate the force between the conductors, we want to use equation (1.4).

$$F = \int \rho(\vec{x}) E(\vec{x})\ d^3x$$

We use

$$\rho(\vec{x}) = \frac{Q}{A}\delta(z)$$

$$E = \frac{Q}{2A\epsilon_0}$$

For a fixed charge,

$$F = \frac{Q^2}{2A\epsilon_0}$$

If we have a fixed potential difference, we use the result from problem 1.6.

$$V = \frac{Qd}{A\epsilon_0}$$

$$Q = \frac{\epsilon_0\ V\ A}{d}$$

Substituting this in,

$$F = \frac{A\epsilon_0 V^2}{2d^2}$$

### 1.9.2 Parallel Cylinder Capacitor

From problem 1.7, the electric field due to a single cylinder is

$$E = \frac{\lambda}{2\pi\epsilon_0 r L}$$

The charge density is

$$\rho(\vec{x}) = \lambda\ \delta(y)\delta(x-d)$$

Using equation (1.4),

$$F = \frac{\lambda^2}{2\pi\epsilon_0 d}$$

We can write the charge density in terms of potential,

$$\lambda = -\frac{\pi\epsilon_0 V}{ln\left(\frac{d}{a}\right)}$$

For a fixed potential difference, the force between the two conductors is

$$F = \frac{\pi\epsilon_0 V}{2d\ ln^2\left(\frac{d}{a}\right)}$$

## 1.10 Mean Value Theorem

**Prove the mean value theorem: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point.**

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We'll start with $G(\vec{x}, \vec{x}') = 1/R$ and substitute this into the general solution to Poisson's equation (1.43),

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')}{R}\ d^3x' + \frac{1}{4\pi R}\oint_S \frac{\partial \Phi}{\partial n'}\ da' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\ da'$$

That's a lot of terms, so let's look at each one individually. The first term disappears since we are in charge-free space. Using equation (1.22) and Gauss's Law (1.16), the second term can be written as

$$\frac{1}{4\pi R}\oint_S \frac{\partial \Phi}{\partial n'}\ da' = \frac{1}{4\pi R}\oint_S \vec{E}\cdot\hat{n}'\ da' = \frac{1}{4\pi\epsilon_0 R}\int_V \rho(\vec{x}')\ d^3x'$$

which also disappears because we are in charge-free space. For the final term,

$$-\frac{1}{4\pi}\oint_S \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\ da' = -\frac{1}{4\pi}\oint_S \Phi\frac{\partial}{\partial R}\left(\frac{1}{R}\right)\ da' = \frac{1}{4\pi R^2}\oint_S \Phi\ da'$$

We're left with

$$\Phi(\vec{x}) = \frac{1}{4\pi R^2}\oint_S \Phi(\vec{x})\ da'$$

which is what we're looking for. On the left, we have the value of the electrostatic potential at any point. On the right, we have the average of the potential over the surface of a sphere.

## 1.11 Electric Field Discontinuity, Curved Conductor

**Use Gauss's theorem to prove that at the surface of a curved charged conductor, the normal derivative of the electric field is given by**

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

**where $R_1$ and $R_2$ are the principal radii of curvature of the surface.**

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We'll start by creating a Gaussian pillbox just above the surface. Since the pillbox is fully outside the surface, it contains no charge, so from Gauss's Law (1.16),

$$\oint_S \vec{E} \cdot \hat{n} \; da = 0$$

As the sides of the pillbox get smaller,

$$E_{top} \oint_{top} da - E_{bot} \oint_{bot} da = 0$$

$$E_{top} \oint_{top} da = E_{bot} \oint_{bot} da$$

If we say that the top plate is some distance $\Delta r$ above the surface and the bottom plate is at the surface,

$$E(\vec{x} + \Delta r \cdot \hat{n}) \; (R_1 + \Delta r)(R_2 + \Delta r) = E(\vec{x}) \; R_1 R_2$$

From the definition of a derivative,

$$\frac{\partial E}{\partial n} = \lim_{\Delta r \to 0} \frac{E(\vec{x} + \Delta r \cdot \hat{n}) - E(\vec{x})}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \frac{E(\vec{x}) \dfrac{R_1 R_2}{(R_1 + \Delta r)(R_2 + \Delta r)} - E(\vec{x})}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \frac{R_1 R_2 - (R_1 + \Delta r)(R_2 + \Delta r)}{\Delta r (R_1 + \Delta r)(R_2 + \Delta r)} E(\vec{x})$$

$$= \frac{-R_1 - R_2}{R_1 R_2} E(\vec{x})$$

Which is the solution we are looking for

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

## 1.12 Green's Reciprocation Theorem

**Prove Green's reciprocation theorem: If $\Phi$ is the potential due to a volume-charge density $\rho$ within a volume $V$ and a surface-charge density $\sigma$ on the conducting surface $S$ bounding the volume $V$, while $\Phi'$ s the potential due to another charge distribution $\rho'$ and $\sigma'$, then**

$$\int_V \rho\Phi' \; d^3x + \oint_S \Phi'\sigma \; da = \int_V \Phi\rho' \; d^3x + \oint_S \Phi\sigma' \; da$$

---

We'll start by using Green's theorem (1.39) with $\phi = \Phi$ and $\psi = \Phi'$.

$$\int_V \Phi\nabla^2\Phi' - \Phi'\nabla^2\Phi \; d^3x = \oint_S \Phi\frac{\partial\Phi'}{\partial r} - \Phi'\frac{\partial\Phi}{\partial r} \; da$$

From problem 1.1, we saw

$$E = \frac{\sigma}{\epsilon_0}$$

Using equation (1.22),

$$\frac{\partial\Phi}{\partial r} = \frac{\sigma}{\epsilon_0}$$

Using this result in conjuntion with Poisson's equation(1.35),

$$\int_V \Phi\left(-\frac{\rho'}{\epsilon_0}\right) \; d^3x - \int_V \Phi'\left(\frac{-\rho}{\epsilon_0}\right) \; d^3x = \oint_S \Phi\left(\frac{\sigma'}{\epsilon_0}\right) \; da - \oint_S \Phi'\left(\frac{\sigma}{\epsilon_0}\right) \; da$$

$$\int_V \rho\Phi' \; d^3x + \oint_S \Phi'\sigma \; da = \int_V \Phi\rho' \; d^3x + \oint_S \Phi\sigma' \; da$$

## 1.13 Green's Reciprocation Theorem Example

**Two infinite grounded parallel conducting planes are separated by a distance $d$. A point charge $q$ is placed between the planes. Use the reciprocation theorem of Green to prove that the total induced charge on one of the planes is equal to $(-q)$ times the fractional perpendicular distance of the point charge from the other plane. (Hint: As your comparison electrostatic problem with the same surfaces choose one whose charge densities and potential are known and simple).**

---

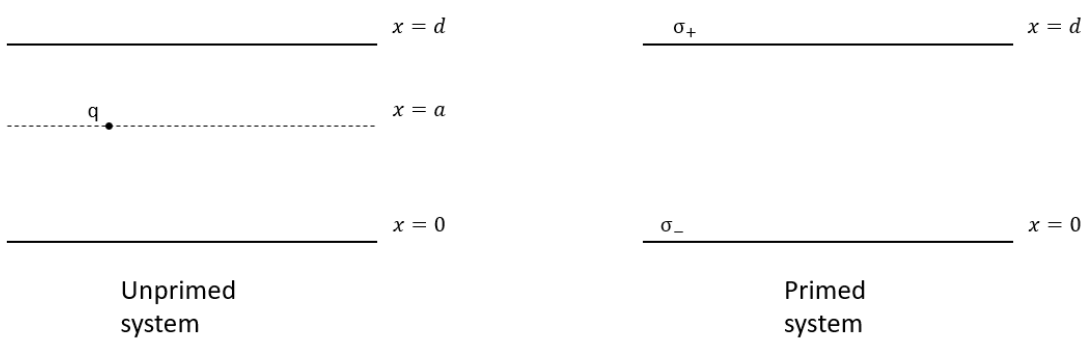


Figure 1.2: Two ways to draw this configuration

We can treat this system as either a charge between two conducting plates or as two plates with some unknown induced surface charge density. We'll call the first system the unprimed system and the second, the primed.

In the unprimed system, we create a Gaussian pillbox around our charge. Applying equation (1.22),

$$\Phi = \frac{qx}{A\epsilon_0}$$

We also set

$$\rho = q\ \delta(x-a)$$

$$\sigma = 0$$

In the primed system, we use the result from section 1.2,

$$\Phi' = \frac{\sigma x}{\epsilon_0}$$

We also have

$$\rho' = 0$$

$$\sigma' = \begin{cases} \sigma_+, & x = d; \\ \sigma_-, & x = 0; \end{cases}$$

Plugging these into Green's reciprocation theorem,

$$\int \frac{\sigma x}{\epsilon_0} q \ \delta(x-a) \ d^3x + 0 = 0 + \oint_S \frac{qx}{A\epsilon_0} \sigma' \ da$$

$$q\frac{\sigma a}{\epsilon_0} = \frac{qd}{A\epsilon_0}\sigma_+ A$$

$$\sigma_+ = \sigma\frac{a}{d}$$

Since the two systems have a similar area of interest and because the magnitude of the induced charge on each plate should be the same,

$$q'_{bot} = -q\frac{a}{d}$$

## 1.14 Green Functions

**Consider the electrostatic Green functions of Section 1.10 for Dirichlet and Neumann boundary conditions on the surface $S$ bounding the volume $V$. Apply Green's theorem(1.39) with integration variable $\vec{y}$ and $\phi = G(\vec{x}, \vec{y})$, $\psi = G(\vec{x}', \vec{y})$, with $\nabla_y^2 G(\vec{z}, \vec{y}) = -4\pi\delta(\vec{y} - \vec{z})$. Find an expression for the difference $[G(\vec{x}, \vec{x}') - G(\vec{x}', \vec{x})]$ n terms of an integral over the boundary surface $S$.**

---

Let's start by writing Green's theorem (1.39) with the given conditions.

$$\int_V G(\vec{x}, \vec{y})\ \nabla^2 G(\vec{x}', \vec{y})\ d^3x - \int_V G(\vec{x}', \vec{y})\ \nabla^2 G(\vec{x}, \vec{y})\ d^3x = \oint_S \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\ da$$

Using equation (1.42),

$$-4\pi\ G(\vec{x}, \vec{x}') + 4\pi\ G(\vec{x}', \vec{x}) = \oint_S G(\vec{x}, \vec{y}) \frac{\partial G(\vec{x}', \vec{y})}{\partial n}\ da - \oint_S G(\vec{x}', \vec{y}) \frac{\partial G(\vec{x}, \vec{y})}{\partial n}\ da$$

$$G(\vec{x}, \vec{x}') - G(\vec{x}, \vec{x}') = -\frac{1}{4\pi}\left[\oint_S G(\vec{x}, \vec{y}) \frac{\partial G(\vec{x}', \vec{y})}{\partial n} - G(\vec{x}', \vec{y}) \frac{\partial G(\vec{x}, \vec{y})}{\partial n}\ da\right]$$

### 1.14.1 Symmetry of Dirichlet Green Function

**For Dirichlet boundary conditions on the potential and the associated boundary condition on the Green function, show that $G_D(\vec{x}, \vec{x}')$ must be symmetric in $\vec{x}$ and $\vec{x}'$.**

Using the Dirichlet boundary condition for Green functions (1.44) in the preamble, we see that the right side of that equation disappears since the Green function is zero on the boundary.

$$G_D(\vec{x}, \vec{x}') = G_D(\vec{x}', \vec{x})$$

### 1.14.2 Non-symmetry of Neumann Green Function

**For Neumann boundary conditions, use the boundary condition (1.46) for $G_N(\vec{x}, \vec{x}')$ to show that $G_N(\vec{x}, \vec{x}')$ is not symmetric in general, but that $G_N(\vec{x}, \vec{x}') - F(\vec{x})$ is symmetric in $\vec{x}$ and $\vec{x}'$, where**

$$F(\vec{x}) = \frac{1}{S} \oint_S G_N(\vec{x}, \vec{y})\ da_y$$

Plugging into the preamble,

$$G_N(\vec{x}, \vec{x}') - G_N(\vec{x}, \vec{x}') = -\frac{1}{4\pi}\left[\oint_S G_N(\vec{x}, \vec{y}) \left(\frac{-4\pi}{S}\right)\ da - \oint G_N(\vec{x}', \vec{y}) \left(\frac{-4\pi}{S}\right)\ da\right]$$

$$= \frac{1}{S}\oint_S G_N(\vec{x},\vec{y})\ da - \frac{1}{S}\oint_S G_N(\vec{x}',\vec{y})\ da$$

$$G_N(\vec{x},\vec{x}') - \frac{1}{S}\oint_S G_N(\vec{x},\vec{y})\ da = G_N(\vec{x}',\vec{x}) - \frac{1}{S}G_N(\vec{x}',\vec{y})\ da$$

Neumann Green function is not symmetric, but if we were to set a particular $F(\vec{x})$, the second term would die, and it would be symmetric.

$$F(\vec{x}) = \frac{1}{S}\oint_S G_N(\vec{x},\vec{y})\ da$$

### 1.14.3 Invariance of Potential

**Show that the addition of $F(\vec{x})$ to the Green function does not affect the potential $\Phi(\vec{x})$. See problem 3.26 for an example of the Neumann Green function.**

Starting with the general solution to Poisson's equation (1.43), if we mak the given substitution, we would introduce an additional

$$\frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')F(\vec{x})\ d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}F(\vec{x})\ da'$$

$$= \frac{F(\vec{x})}{4\pi}\left[\int_V \frac{\rho(\vec{x}')}{\epsilon_0}\ d^3x' + \oint_S \frac{\partial\Phi}{\partial n'}\ da'\right]$$

Using Gauss's Law (1.18) on the first term and equation (1.22) on the second,

$$= \frac{1}{4\pi}\left[\oint_S \vec{E}\cdot\hat{n}'\ da' - \oint_S \vec{E}\cdot\hat{n}'\ da'\right]$$

Which, rather nicely, goes to zero.

# Chapter 2

# Boundary-Value Problems in Electrostatics I

## 2.1 Method of Images: Infinite Plane Conductor

**A point charge $q$ is brought to a position a distance $d$ away from an infinite plane conductor held at zero potential. Using the method of images, find:**

---

### 2.1.1 Surface-charge Density

**the surface-charge density induced on the plane**

For most of these, we actually solved in section 2.1. The charge density is given by equation (2.3).

$$\sigma = -\frac{qd}{2\pi\ (x^2 + y^2 + d^2)^{3/2}}$$

### 2.1.2 Force

**the force between the plane and the charge by using Coulomb's law for the force between the charge and its image**

Given by equation (2.4)

$$\vec{F}(\vec{x}) = -\frac{q^2}{16\pi\epsilon_0 d^2}\hat{z}$$

### 2.1.3 Alternate Force Derivation

**the total force acting on the plane by integrating $\sigma^2/2\epsilon_0$ over the whole plane**

The problem wants us to integrate $\sigma^2$, so we'll do just that

$$\vec{F} = \int \frac{\sigma^2}{2\epsilon_0}\hat{z}\ da = \frac{q^2 d^2}{2\epsilon_0(2\pi)^2}\int \frac{1}{(x^2+y^2+d^2)^3}\hat{z}\ da$$

We now turn this integral into cylindrical coordinates to make the math a little easier,

$$= \frac{q^2 d^2}{2\epsilon_0(2\pi)^2}\int_0^\infty \frac{2\pi\rho}{(\rho^2+d^2)^3}\hat{z}\ d\rho$$

$$\vec{F}(\vec{x}) = \frac{q^2}{16\pi\epsilon_0 d^2}\hat{z}$$

We end up getting the opposite of what we saw in part (b) since we are looking at the force of the charge on the plane rather than the force of the plane on the charge.

### 2.1.4 Work to Remove Charge

**the work necessary to remove the charge $q$ from its position to infinity**

We know we should get a positive value since we need to do work on the charge to pull it away from the plate.

$$W = \int_d^\infty F\ dz = \frac{q^2}{16\pi\epsilon_0}\int_d^\infty \frac{1}{z^2}\ dz$$

$$W = \frac{q^2}{16\pi\epsilon_0 d}$$

### 2.1.5 Potential Energy

**the potential energy between the charge $q$ and its image [compare the answer to part d and discuss]**

Since we have two point charges, we can use our equation for the potential energy between discrete charges (1.30).

$$W = -\frac{q^2}{8\pi\epsilon_0 d}$$

Our answer here is twice as large since we are adding the energy from an image charge, which is not real. If we want the true potential energy of the system, we need to ignore contributions from image charges.

### 2.1.6 Calculator Work

**Find the answer to part d in electron volts for an electron originally one angstrom from the surface**

W=3.6V. N.B., always work out these numbers for yourself since I'm working with a hand-crank calculator. Actually, I make no guarantees for any of these problems. I'll try to be as correct as possible, but I make no promises. Sorry about that.

## 2.2 Method of Images: Grounded, Conducting Sphere

**Using the method of images, discuss the problem of a point charge $q$ inside a hollow, grounded, conducting sphere of inner radius $a$. Find**

---

### 2.2.1 Potential

**the potential inside the sphere;**

As stated in section 2.1.2, we expect the derivation to be the same regardless of whether we are inside the sphere or not. Thus, we can use the potential we found in that section (2.14) with the same image charge (2.5) and image position (2.6),

$$\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'\left|\vec{x}-\frac{a^2}{x'^2}\vec{x}'\right|}\right)$$

### 2.2.2 Surface-charge density

**the induced surface-charge density;**

This time, we will have a positive induced surface-charge density on the outside of the sphere and a negative induced surface-charge density on the inside since our positive charge is now inside the conductor rather than outside as it was in the text. In any case, we can still use the solution we found (2.10),

$$\sigma = \frac{q}{4\pi a^2}\left(\frac{a}{x'}\right)\frac{1-\frac{a^2}{x'^2}}{\left(1-\frac{a^2}{x'^2}-2\frac{a}{x'}\,cos(\gamma)\right)^{3/2}}$$

### 2.2.3 Force

**the magnitude and direction of the force acting on $q$**

Again, equal and opposite. We can use equation (2.11),

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2 a}{x'^3}\left(1-\frac{a^2}{x'^2}\right)^{-2}\hat{x}$$

### 2.2.4 Additional Surface Potential/Charge

**Is there any change in the solution if the sphere is kept at a fixed potential $V$? If the sphere has a total charge $Q$ on its inner and outer surfaces?**

If we add a fixed potential or surface charge, we will need to look at sections 2.1.3 and 2.1.4.

## 2.3 Method of Images: Line Charge

**A straight-line charge with constant linear charge density $\lambda$ is located perpendicular to the $x - y$ plane in the first quadrant at $(x_0, y_0$. The intersecting planes $x = 0$, $y \geq 0$ and $y = 0$, $x \geq 0$ are conducting boundary surfaces held at zero potential. Consider the potential, fields, and surface charges in the first quadrant.**

---

We can think about the equivalent two-dimensional problem (a point charge in the first quadrant), then perform successive mirror image problems to get to the solution. Knowing this, we'll say we have four line charges: $\lambda$ at $(x_0, y_0)$, $\lambda'$ at $(-x_0, y_0)$, $\lambda''$ at $(-x_0, -y_0)$, and $\lambda'''$ at $(x_0, -y_0)$.

### 2.3.1 Potential

**The well-known potential for an isolated line charge at $(x_0, y_0)$ is $\Phi(x, y) = (\lambda/4\pi\epsilon_0)ln(R^2/r^2)$, where $r^2 = (x - x_0)^2 + (y - y_0)^2$ and $R$ is a constant. Determine the expression for the potential of the line charge in the presence of the intersecting planes. Verify explicitly that the potential and the tangential electric field vanish on the boundary surfaces.**

The first thing we should do is figure out the values of $\lambda'$, $\lambda''$, and $\lambda'''$. In order to do this, we need to set the potential at each boundary to zero, rendering the part where we explicitly show the potential is zero at the boundary rather superfluous. If you skip this part and use your intuition to find the potential, you may have to show that part, but we won't do that here. In any case, our general potential is given by,

$$\Phi(x, y) = \frac{\lambda}{4\pi\epsilon_0} ln\left(\frac{R^2}{(x - x_0)^2 + (y - y_0)^2}\right) + \frac{\lambda'}{4\pi\epsilon_0} ln\left(\frac{R^2}{(x + x_0)^2 + (y - y_0)^2}\right)$$

$$+\frac{\lambda''}{4\pi\epsilon_0} ln\left(\frac{R^2}{(x + x_0)^2 + (y + y_0)^2}\right) + \frac{\lambda'''}{4\pi\epsilon_0} ln\left(\frac{R^2}{(x - x_0)^2 + (y + y_0)^2}\right)$$

Using the condition, $\Phi(0, y) = 0$,

$$ln\left(\left(\frac{R^2}{x_0^2 + (y - y_0)^2}\right)^{\lambda+\lambda'} \left(\frac{R^2}{x_0^2 + (y + y_0)^2}\right)^{\lambda''+\lambda'''}\right) = 0$$

Either part could go to 1 and this relation would work, so we have the two conditions

$$\lambda + \lambda' = 0$$

$$\lambda'' + \lambda''' = 0$$

If we repeat this process for the boundary $\Phi(x, 0) = 0$,

$$\lambda + \lambda''' = 0$$

$$\lambda' + \lambda'' = 0$$

Solving these systems of equations,

$$\lambda' = -\lambda$$

$$\lambda'' = \lambda$$

$$\lambda''' = -\lambda$$

Plugging these back into our potential and simplifying,

$$\Phi(x,y) = -\frac{\lambda}{4\pi\epsilon_0} ln[(x-x_0)^2 + (y-y_0)^2] + \frac{\lambda}{4\pi\epsilon_0} ln[(x+x_0)^2 + (y-y_0)^2]$$
$$-\frac{\lambda}{4\pi\epsilon_0} ln[(x-x_0)^2 + (y+y_0)^2] + \frac{\lambda}{4\pi\epsilon_0} ln[(x-x_0)^2 + (y-y_0)^2]$$

To show that the normal components of the electric fields are zero at the boundaries, we take the normal derivative (1.22),

$$E_x = -\frac{\partial\Phi}{\partial x} = \frac{\lambda}{2\pi\epsilon_0}\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2} - \frac{\lambda}{2\pi\epsilon_0}\frac{x+x_0}{(x+x_0)^2+(y-y_0)^2}$$
$$+\frac{\lambda}{2\pi\epsilon_0}\frac{x+x_0}{(x+x_0)^2+(y+y_0)^2} - \frac{\lambda}{2\pi\epsilon_0}\frac{x-x_0}{(x-x_0)^2+(y+y_0)^2}$$

$$E_y = -\frac{\partial\Phi}{\partial y} = \frac{\lambda}{2\pi\epsilon_0}\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2} - \frac{\lambda}{2\pi\epsilon_0}\frac{y-y_0}{(x+x_0)^2+(y-y_0)^2}$$
$$+\frac{\lambda}{2\pi\epsilon_0}\frac{y+y_0}{(x+x_0)^2+(y+y_0)^2} - \frac{\lambda}{2\pi\epsilon_0}\frac{y+y_0}{(x-x_0)^2+(y+y_0)^2}$$

If we evaluate $E_x$ and $E_y$ at $y = 0$ and $x = 0$ respectively, we'll see that the electric field is zero at the boundary.

### 2.3.2 Surface-charge density

**Determine the surface charge density $\sigma$ on the plane $y = 0$, $x \geq 0$.**

We'll use the electric fields we calculated in the previous section.

$$\sigma = \epsilon_0 E_y|_{y=0} = -\frac{\lambda y_0}{\pi}\left(\frac{1}{(x-x_0)^2+y_0^2} - \frac{1}{(x+x_0)^2+y_0^2}\right)$$

### 2.3.3 Total Induced Charge

**Show that the total charge (per unit length in $z$) on the plane $y = 0$, $x \geq 0$ is**

$$Q_x = -\frac{2}{\pi}\lambda\ tan^{-1}\left(\frac{x_0}{y_0}\right)$$

**What is the total charge on the plane $x = 0$?**

Let's integrate the surface charge density from the previous part. Actually, we really just look this integral up.

$$Q_x = \int_0^\infty \sigma \; dx = -\frac{\lambda y_0}{\pi} \int_0^\infty \frac{1}{(x-x_0)^2 + y_0^2} - \frac{1}{(x+x_0)^2 + y_0^2} \; dx$$

$$Q_x = -\frac{2\lambda}{\pi} tan^{-1}\left(\frac{x_0}{y_0}\right)$$

Using the power of symmetry, we know that we should get the same thing for $Q_y$ just with $x_0$ and $y_0$ flipped since our charge density is

$$\sigma = \epsilon_0 E_x|_{x=0} = -\frac{\lambda x_0}{\pi}\left(\frac{1}{(y-y_0)^2 + x_0^2} - \frac{1}{(y+y_0)^2 + x_0^2}\right)$$

$$Q_y = -\frac{2\lambda}{\pi} tan^{-1}\left(\frac{y_0}{x_0}\right)$$

### 2.3.4 Quadrupole Approximation

**Show that far from the origin [$\rho >> \rho_0$, where $\rho = \sqrt{(x^2+y^2)}$ and $\rho_0 = \sqrt{(x_0^2+y_0^2)}$] the leading term in the potential is**

$$\Phi \to \Phi_{asym} = \frac{4\lambda}{\pi\epsilon_0}\frac{(x_0 y_0)(xy)}{\rho^4}$$

**Interpret**

We'll want to take the potential we found in part (a) and convert it to cylindrical coordinates,

$$\Phi = -\frac{\lambda}{4\pi\epsilon_0} ln\left(1 + \frac{\rho_0^2}{\rho^2} - \frac{2\rho\rho_0 \; cos(\theta-\theta_0)}{\rho^2}\right) + \frac{\lambda}{4\pi\epsilon_0} ln\left(1 + \frac{\rho_0^2}{\rho^2} + \frac{2\rho\rho_0 \; cos(\theta+\theta_0)}{\rho^2}\right)$$

$$-\frac{\lambda}{4\pi\epsilon_0} ln\left(1 + \frac{\rho_0^2}{\rho^2} + \frac{2\rho\rho_0 \; cos(\theta+\theta_0)}{\rho^2}\right) + \frac{\lambda}{4\pi\epsilon_0} ln\left(1 + \frac{\rho_0^2}{\rho^2} - \frac{2\rho\rho_0 \; cos(\theta+\theta_0)}{\rho^2}\right)$$

Using the approximation $ln(1+x) = x - \frac{x^2}{2}$. Note that we want the first two terms since we notice a $1/\rho^4$ term in our solution. After some algebra,

$$\Phi = -\frac{\lambda}{\pi\epsilon_0}\left(\frac{\rho_0}{\rho}\right)^2 \left(cos^2(\theta+\theta_0) - cos^2(\theta-\theta_0)\right)$$

Using the angle addition formulas,

$$= -\frac{\lambda}{\pi\epsilon_0}\left(\frac{\rho_0}{\rho}\right)^2 (-4cos(\theta) \; cos(\theta_0) \; sin(\theta) \; sin(\theta_0))$$

Using the relation,

$$cos(\theta) = \frac{x}{\rho}; \quad cos(\theta_0) = \frac{x_0}{\rho}$$

$$sin(\theta) = \frac{y}{\rho}; \quad sin(\theta_0) = \frac{y_0}{\rho}$$

$$\Phi = \frac{4\lambda}{\pi\epsilon_0}\frac{(x_0y_0)(xy)}{\rho^4}$$

As the title suggests, we have a quadrupole.

## 2.4 Method of Images: Charged Sphere

**A point charge is placed a distance $d > R$ from the center of an equally charged, isolated, conducting sphere of radius $R$.**

---

### 2.4.1 Force

**Inside of what distance from the surface of the sphere is the point charge attracted rather than repelled by the charged sphere?**

We can calculate the force by using Coulomb's Law (1.1). This give us

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\left(1 - \frac{R^3(2d^2 - R^2)}{d\ (d^2 - R^2)^2}\right)$$

Setting this to zero,

$$1 - \frac{R^3\ (2d^2 - R^2)}{d\ (d^2 - R^2)} = 0$$

This gives us an expression for the distance

$$\left(\frac{d}{R}\right)^5 - 2\left(\frac{d}{R}\right)^3 - 2\left(\frac{d}{R}\right)^2 + \left(\frac{d}{R}\right) + 1 = 0$$

### 2.4.2 Leading Force Term

**What is the limiting value of the force of attraction when the point charge is located a distance $a(= d - R)$ from the surface of the sphere, if $a << R$?**

If we substitute the expression $d = a - R$ into the force equation above,

$$F = \frac{q^2}{4\pi\epsilon_0}\frac{1}{a^2 + 2aR + R^2}\left(\frac{(a + R)(a^2 + 2aR)^2 - R^3(2a^2 + 4aR + R^2)}{(a + R)(a^2 + 2aR)^2}\right)$$

To find the limiting value, we only need to pick out the leading terms in $R$, which are in bold above. This leaves,

$$F = -\frac{q^2}{16\pi\epsilon_0 a^2}$$

### 2.4.3 Different Charges

**What are the results for parts a and b if the charge on the sphere is twice (half) as large as the point charge, but still the same sign?**

Part (b) sees no changes since the leading term is not tied to the charge $Q$.

For part (a), if $Q$ is doubled, we need to solve

$$2 - \frac{R^3(2d^2 - R^2}{d(d^2 - R^2)^2} = 0$$

This gives the expression

$$2\left(\frac{d}{R}\right)^5 - 4\left(\frac{d}{R}\right)^3 - 2\left(\frac{d}{R}\right)^2 + 2\left(\frac{d}{R}\right) + 1 = 0$$

If $Q$ is halved, we need to solve

$$\frac{1}{2} - \frac{R^3(2d^2 - R^2)}{d(d^2 - R^2)^2} = 0$$

This gives the expression

$$\frac{1}{2}\left(\frac{d}{R}\right)^5 - \left(\frac{d}{R}\right)^3 - 2\left(\frac{d}{R}\right)^2 + \frac{1}{2}\left(\frac{d}{R}\right) + 1 = 0$$

## 2.5 Method of Images: Work

### 2.5.1 Grounded, Conducting Sphere

**Show that the work done to remove the charge $q$ from a distance $r > a$ to infinity against the force of a grounded conducting sphere is**

$$W = \frac{q^2 a}{8\pi\epsilon_0(r^2 - a^2)}$$

**Relate this result to the electrostatic potential and the energy discussion of Jackson section 1.11.**

---

We'll need to use the force between the point charge and a grounded, conducting sphere (2.11). Note that we can re-orient our sphere in any way we choose, so let's orient our particle along the y-axis.

$$W = \int F\ dy = \frac{q^2 a}{4\pi\epsilon_0}\int_r^\infty \frac{y}{(y^2 - a^2)^2}\ dy$$

$$W = \frac{q^2 a}{8\pi\epsilon_0(r^2 - a^2)}$$

If we were to calculate the work using equation (1.32), we would get

$$W = -\frac{aq^2}{4\pi\epsilon_0(r^2 - a^2)}$$

We get twice the correct energy due to counting the image charge.

### 2.5.2 Repeat the calculation of the work done to remove the charge $q$ against the force of an isolated charged conducting sphere. Show that the work done is

$$W = \frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{2(r^2 - a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right]$$

**Relate the work to the electrostatic potential and the energy discussion of Jackson section 1.11.**

---

$$W = \int F\ dy = \frac{qQ}{4\pi\epsilon_0}\int_r^\infty \frac{1}{y^2}\ dy - \frac{q^2 a^3}{4\pi\epsilon_0}\int_r^\infty \frac{2y^2 - a^2}{y^3(y^2 - a^2)^2}\ dy$$

$$W = \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2 a}{2(r^2 - a^2)} + \frac{q^2 a}{2r^2} + \frac{qQ}{r}\right)$$

Using equation (1.32),

$$W = \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2 a}{r^2 - a^2} + \frac{q^2 a}{r^2} + \frac{qQ}{r}\right)$$

We get close to double to correct energy. $Q$ has no image charge, which means that term is not doubled.

## 2.6 Skipped. Iterative Methods

## 2.7 Potential with Dirichlet Boundary Conditions

**Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).**

---

This is not entirely dissimilar to the problem introduced in section 2.2 only now we're looking at the two-dimensional system.

### 2.7.1 Write down the appropriate Green function $G(\vec{x}, \vec{x}')$

Since we are dealing with an infinite plane, we can use the Green function for an infinite plane (2.2),

$$G(\vec{x}, \vec{x}') = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}$$

### 2.7.2 Potential

If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius $a$ centered at the origin, and $\Phi = 0$ outside that circle, find an integral expression for the potential at the point $P$ specified in terms of cylindrical coordinates $(\rho, \phi, z)$.

We'll want to use the general Dirichlet solution to Poisson's equation (1.45). Since we are in charge-free space, this reduces to

$$\Phi(\vec{x}) = -\frac{1}{4\pi} \oint_S \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial(-z')} \, da'$$

Converting to cylindrical coordinates,

$$= \frac{V}{4\pi} \int_0^a \int_0^{2\pi} \frac{\partial}{\partial z'} \left( \frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \, cos(\theta - \theta') + (z-z')^2}} - \frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \, cos(\theta - \theta') + (z+z')^2}} \right) \rho' \, d\theta' d\rho'$$

Since we are centered at the origin, $z' = 0$,

$$\Phi(\vec{x}) = \frac{Vz}{2\pi} \int_0^a \int_0^{2\pi} \frac{\rho'}{(\rho^2 + \rho'^2 - 2\rho\rho' \, cos(\theta - \theta') + z^2)^{3/2}} \, d\theta' d\rho'$$

Again, just as with the hemispheres, we see that we don't get a solution in closed form. Instead, we have to turn to approximations.

### 2.7.3 Potential Along the Axis of the Circle

**Show that, along the axis of the circle $(\rho = 0)$, the potential is given by**

$$\Phi = V \left( 1 - \frac{z}{\sqrt{a^2 + z^2}} \right)$$

Setting $\rho = 0$, our potential reduces to

$$\Phi(\vec{x}) = \frac{Vz}{2\pi} \int_0^a \int_0^{2\pi} \frac{\rho'}{(\rho'^2 + z^2)^{3/2}} \, d\theta' d\rho'$$

### 2.7.4 Large Distances

**Show that at large distances ($\rho^2 + z^2 >> a^2$) the potential can be expanded in a power series in $(\rho^2 + z^2)^{-1}$, and that the leading terms are**

$$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2+z^2)} + \frac{5(3\rho^2a^2+a^4)}{8(\rho^2+z^2)^2} + ...\right]$$

**Verify that the results of parts c and d are consistent with each other in their common range of validity.** We can rewrite the potential as

$$\Phi(\vec{x}) = \frac{Vz}{2\pi(\rho^2+z^2)^{3/2}}\int_0^a\int_0^{2\pi}\frac{\rho'}{\left(1+\dfrac{\rho'^2 - 2\rho\rho'\ cos(\theta')}{\rho^2+z^2}\right)^{3/2}}\,d\theta' d\rho'$$

Note that we say $\theta - \theta' \approx \theta'$. Using the binomial approximation, keeping three terms since we looked at the solution,

$$\approx \frac{Vz}{2\pi(\rho^2+z^2)^{3/2}}\int_0^a\int_0^{2\pi}\rho'\left(1 - \frac{3}{2}\frac{\rho'^2 - 2\rho\rho'\ cos(\theta')}{\rho^2+z^2} + \frac{15}{8}\frac{\rho'^4 - 4\rho\rho'^3\ cos(\theta') + 4\rho^2\rho'^2\ cos^2(\theta')}{(\rho^2+z^2)^2}\right)\,d\theta' d\rho'$$

$$\Phi(\vec{x}) = \frac{Va^2z}{2(\rho^2+z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2+z^2)} + \frac{5}{8}\frac{3\rho^2a^2+a^4}{(\rho^2+z^2)^2} + ...\right]$$

To verify that this agrees with the solution from part (c), we set $\rho = 0$,

$$\Phi(\vec{x}) = \frac{Va^2}{2z^2}\left[1 - \frac{3a^2}{4z^2} + \frac{5a^4}{8z^4}\right]$$

$$= V\left[1 - \left(1 - \frac{a^2}{2z^2} + \frac{3a^4}{8z^4} - \frac{5a^6}{16z^6}\right)\right]$$

We recognize the expression in the parenthesis as a binomial expression,

$$= V\left[1 - \left(1 - \frac{a^2}{z^2}\right)^{-1/2}\right]$$

$$= V\left(1 - \frac{z}{(a^2+z^2)^{1/2}}\right)$$

## 2.8 Two Line Charges Potential

**A two-dimensional potential problem is defined by two straight parallel line charges separated by a distance $R$ with equal and opposite linear charge densities $\lambda$ and $-\lambda$.**

---

### 2.8.1 Equipotential

**Show by direct construction that the surface of constant potential $V$ is a circular cylinder (circle in the transverse dimensions) and find the coordinates of the axis of the cylinder and its radius in terms of $R$, $\lambda$, and $V$.**

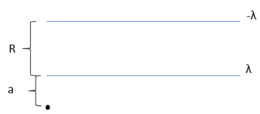


Figure 2.1: x-y plane view

We'll define the origin as the center of the cylinder. The positive line charge lies a distance $a$ above the x-axis and the negative line charge lies a distance $R$ above that as shown in figure (2.1). Part (a) of problem 2.3 gives the potential due to a line charge, so we use that,

$$\Phi = \frac{\lambda}{4\pi\epsilon_0}[-ln(x^2 + (y-a)^2) + ln(x^2 + ((y-(a+R))^2)]$$

Converting to cylindrical coordinates,

$$= \frac{\lambda}{4\pi\epsilon_0}[-ln(\rho^2 + a^2 - 2a\rho\ sin(\theta)) + ln(\rho^2 + (a+R)^2 - 2\rho(a+R)\ sin(\theta))]$$

We want to find the surface of constant potential, so let's take the derivative of the potential according to $\theta$ and set it equal to zero.

$$\frac{\partial\Phi}{\partial\theta} = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{2a\rho\ cos(\theta)}{\rho^2 + a^2 - 2a\rho\ cos(\theta)} - \frac{2\rho(a+R)\ cos(\theta)}{\rho^2 + (a+R)^2 - 2\rho(a+R)\ sin(\theta)}\right) = 0$$

Solving for $\rho$,

$$\rho^2 = a(a+R)$$

Since $\rho$ is a constant, this implies that our surface of equipotential is a circle (which, when extended to three dimensions, becomes a circular cylinder). Now let's find $a$. We go back to our general potential, and set $\theta$ to 0 since we know the potential doesn't care about the angle. At the boundary,

$$V = \frac{\lambda}{4\pi\epsilon_0}ln\left(\frac{\rho^2 + (a+R)^2}{\rho^2 + a^2}\right)$$

$$= \frac{\lambda}{4\pi\epsilon_0}ln\left(\frac{a^2 + aR + (a+R)^2}{a^2 + aR + a^2}\right)$$

Solving for $a$,

$$a = \frac{R}{exp\left(\frac{4\pi\epsilon_0 V}{\lambda}\right) - 1}$$

We'll set $x = \dfrac{2\pi\epsilon_0 V}{\lambda}$ for ease of calculation. Substituting this back into our expression for $\rho$,

$$\rho^2 = \frac{R^2}{(e^{2x}-1)^2} + \frac{R^2}{e^{2x}-1} = \frac{R^2 e^{2x}}{(e^{2x}-1)^2}$$

$$\rho = \frac{R\ e^x}{e^{2x}-1} = \frac{R}{2\ sinh(x)}$$

### 2.8.2 Capacitance

**Use the results of part a to show that the capacitance per unit length $C$ of two right-circular cylindrical conductors, with radii $a$ and $b$, separated by a distanced $d > a + b$, is**

$$C = \frac{2\pi\epsilon_0}{cosh^{-1}\left(\dfrac{d^2 - a^2 - b^2}{2ab}\right)}$$

It's a little difficult to get from the configuration given to the capacitance, so we'll start with the capacitance and try to show that this gives the correct configuration. We'll start by calculating the quantity $\dfrac{d^2 - a^2 - b^2}{2ab}$. We define

$$x_{\pm} = \frac{2\pi\epsilon_0 V}{\pm\lambda}$$

From part a,

$$d = a_- - a_+ = \frac{R}{e^{2x_-}-1} - \frac{R}{e^{2x_+}-1}$$

$$d^2 = \frac{R^2(e^{2x_+} - e^{2x_-})^2}{(e^{2x_-}-1)^2(e^{2x_+}-1)^2}$$

We can find the radii of the cylinders by using the equation for $\rho$ we found in the previous part.

$$a = \frac{R\ e^{x_+}}{e^{2x_+}-1}$$

$$b = \frac{R\ e^{x_-}}{e^{2x_-}-1}$$

Combining all of these terms,

$$\frac{d^2 - a^2 - b^2}{2ab} = \frac{(e^{2x_+} - e^{2x_-})^2 - e^{2x_+}(e^{2x_-}-1)^2 - e^{2x_-}(e^{2x_+}-1)^2}{2e^{x_+}e^{x_-}(e^{2x_-}-1)(e^{2x_+}-1)}$$

$$= -\frac{e^{x_+ - x_-} + e^{-(x_+ - x_-)}}{2}$$

$$= -cosh(x_+ - x_-)$$

Substituting back in our expression for $x$,

$$\frac{d^2 - a^2 - b^2}{2ab} = -cosh\left(\frac{2\pi\epsilon_0(V_+ - V_-)}{\lambda}\right)$$

If we define the capacitance as

$$C = \frac{\lambda}{V_+ - V_-}$$

$$C = \frac{2\pi\epsilon_0}{cosh^{-1}\left(\frac{d^2 - a^2 - b^2}{2ab}\right)}$$

### 2.8.3 Limits

**Verify that the result for $C$ agrees with the answer in Problem 1.7 in the appropriate limit and determine the next nonvanishing order correction in powers of $a/d$ and $b/d$.**

From problem 1.7, we found

$$\frac{C}{L} = \pi\epsilon_0 \left(ln\left(\frac{d}{\sqrt{ab}}\right)\right)^{-1}$$

Setting $z = \frac{d^2 - a^2 - b^2}{2ab}$,

$$cosh^{-1}(z) = ln\left(z + \sqrt{z^2 - 1}\right)$$

Since $z >>!$, we can use the binomial approximation,

$$ln\left(z + \sqrt{z^2 - 1}\right) = ln\left(2z - \frac{1}{2z}\right)$$

$$\approx ln(2z)$$

From our solution to part (b),

$$C \approx \frac{2\pi\epsilon_0}{ln(2z)}$$

Using the limit $d >> a, b$, $z \approx \frac{d^2}{ab}$,

$$C \approx \frac{2\pi\epsilon_0}{ln\left(\frac{d^2}{ab}\right)} = \frac{\pi\epsilon_0}{ln\left(\frac{d}{\sqrt{ab}}\right)}$$

To go to the next nonvanishing order, we keep the $\frac{1}{2z}$ term in the $ln$ expansion,

$$C \approx \frac{2\pi\epsilon_0}{ln\left(\frac{d^2}{ab} - \frac{ab}{d^2}\right)}$$

### 2.8.4 Two Cylinders Inside Each Other

**Repeat the calculation of the capacitance per unit length for two cylinders inside each other ($d < |b - a|$). Check the result for concentric cylinders ($d = 0$).**

I posit that the capacitance is given by

$$C = \frac{2\pi\epsilon_0}{cosh^{-1}\left(\frac{a^2 + b^2 - d^2}{2ab}\right)}$$

You can follow the steps laid out in part (b) to confirm this. If we then follow the same limits in part (c) and set $d = 0$,

$$C \approx \frac{2\pi\epsilon_0}{ln\left(\frac{a}{b}\right)}$$

## 2.9 Method of Images: Sphere in an Electric Field

**An insulated, spherical, conducting shell of radius $a$ is in a uniform electric field $E_0$. If the sphere is cut into two hemispheres by a plane perpendicular to the field, find the force required to prevent the hemispheres from separating**

---

### 2.9.1 Uncharged Shell

**if the shell is uncharged;**

We have a surface charge density on the sphere, given by equation (2.18). To find the force needed to keep the two halves together, we use equation (1.4),

$$F = \int \sigma E \ d^3x$$

Using the discontinuity in the electric field (1.26) and introducing a factor of 1/2 so we avoid counting the force of the hemisphere acting on itself,

$$F = \frac{1}{2\epsilon_0} \int \sigma^2 \ da$$

$$= \frac{a^2}{2} \int_0^{2\pi} \int_0^1 9\epsilon_0 E_0^2 cos^3(\theta) \ d(cos(\theta))d\phi$$

$$F = \frac{9\pi\epsilon_0 E_0^2 a^2}{4}$$

### 2.9.2 Charged Shell

**if the total charge on the shell is $Q$**

If we introduce a charge $Q$ on the sphere, it will spread out and add an additional term to our induced surface charge density,

$$\sigma = 3\epsilon_0 E_0 \ cos(\theta) + \frac{Q}{4\pi a^2}$$

Our force now becomes

$$F = \frac{1}{2\epsilon_0} \int \sigma^2 \ da = \frac{a^2\pi}{\epsilon_0} \int_{-1/2}^{1/2} 9\epsilon_0^2 E_0^2 \ cos^3(\theta) + \frac{3\epsilon_0 E_0 Q \ cos^2(\theta)}{2\pi a^2} + \frac{Q^2 \ cos(\theta)}{16\pi^2 a^4} \ d(cos(\theta))$$

$$F = \frac{9\pi\epsilon_0 E_0^2 a^2}{4} + \frac{Q^2}{32\pi\epsilon_0 a^2}$$

## 2.10 Sheet with Boss

**A large parallel plate capacitor is made up of two plane conducting sheets with separation $D$, one of which has a small hemispherical boss of radius $a$ on its inner surface ($D >> a$). The conductor with the boss is kept at zero potential, and the other conductor is at a potential such that far from the boss the electric field between the plates is $E_0$**

---

### 2.10.1 Surface Charge Density

On the hemisphere, we can use equation (2.16) for the potential, and (2.18) for the charge density.

$$Phi = -E_0 \left( r - \frac{a^3}{r^2} \right) cos(\theta)$$

$$\sigma = 3\epsilon_0 E_0 \ cos(\theta)$$

Meanwhile, on the plane, we can use the same solution. We just need to convert from spherical to cylindrical coordinates

$$\Phi = -E_0 \left( z - \frac{a^3 z}{(\rho^2 + z^2)^{3/2}} \right)$$

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial z}\bigg|_{z=0} = \epsilon_0 E_0 \left( 1 - \frac{a^3}{\rho^3} \right)$$

### 2.10.2 Total Charge

**Show that the total charge on the boss has the magnitude $4\pi\epsilon_0 E_0 a^2$**

We want to use the surface charge density on the boss,

$$Q = \int 3\epsilon_0 E_0 \ cos(\theta) \ da$$

$$= 6\pi\epsilon_0 E_0 a^2 \int_0^1 cos(\theta) \ d(cos(\theta))$$

$$Q = 3\pi\epsilon_0 E_0 a^2$$

### 2.10.3 Surface Charge Induced by a Point Charge

**If, instead of the other conducting sheet at a different potential, a point charge $q$ is placed directly above the hemispherical boss at a distance $d$ from its center, show that the charge induced on the boss is**

$$q' = -q\left[1 - \frac{d^2 - a^2}{d\sqrt{d^2 + a^2}}\right]$$

If we introduce a charge near the hemisphere, this is similar to a sphere in an electric field (2.16),

$$\Phi = -dfracq4\pi\epsilon_0 \frac{1}{\sqrt{r^2 + d^2 + 2rd\ cos(\theta)}} + \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2 + d^2 - 2rd\ cos(\theta)}}$$

$$+\frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{\frac{d^2r^2}{a^2} + a^2 + 2rd\ cos(\theta)}} - \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{\frac{d^2r^2}{a^2} + a^2 - 2rd\ cos(\theta)}}$$

From this, we can calculate the induced surface charge density,

$$\sigma = -\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = \frac{q(d^2 - a^2)}{4\pi a}\left(\frac{1}{(a^2 + d^2 + 2ad\ cos(\theta))^{3/2}} - \frac{1}{(a^2 + d^2 - 2ad\ cos(\theta))^{3/2}}\right)$$

Integrating over to find the total induced charge,

$$q' = \int \sigma\ da$$

$$= 2\pi a^2 \int_0^1 \frac{q(d^2 - a^2)}{4\pi a}\left(\frac{1}{(a^2 + d^2 + 2ad\ cos(\theta))^{3/2}} - \frac{1}{(a^2 + d^2 - 2ad\ cos(\theta))^{3/2}}\right)\ d(cos(\theta))$$

$$q' = -q\left(1 - \frac{d^2 - a^2}{d(d^2 + a^2)^{1/2}}\right)$$

## 2.11 Method of Images: Line Charge and Conducting Cylinder

**A line charge with linear charge density $\tau$ is placed parallel to, and a distance $R$ away from, the axis of a conducting cylinder of radius $b$ held at fixed voltage such that the potential vanishes at infinity. Find**

---

### 2.11.1 Image Line Charge

**the magnitude and position of the image charge(s)**

We recognize this as a method of images problem, so let's create an image charge $\tau'$ inside the circle a distance $R'$ from the origin. Something, something symmetry so we can look at the cross-section then expand to three dimensions. Using the potential due to a line charge provided in problem 2.3, the general potential will be

$$\Phi_{tot} = \frac{\tau}{4\pi\epsilon_0} ln\left(\frac{L}{\rho^2 + R^2 - 2\rho R\ cos(\theta)}\right) + \frac{\tau'}{4\pi\epsilon_0} ln\left(\frac{L}{\rho^2 + R'^2 - 2\rho R'\ cos(\theta)}\right)$$

Let's now apply the first boundary condition. As$\rho$ goes to infinity, we expect the potential to vanish,

$$\frac{\tau}{4\pi\epsilon_0} ln\left(\frac{L}{\rho^2}\right) + \frac{\tau'}{4\pi\epsilon_0} ln\left(\frac{L}{\rho^2}\right) = 0$$

This implies that $\tau' = -\tau$, the image charge has the same magnitude but opposite direction, resulting in a system with no net charge. We can now write our potential as

$$\Phi = \frac{\tau}{4\pi\epsilon_0} ln\left(\frac{\rho^2 + R'^2 - 2\rho R'\ cos(\theta)}{\rho^2 + R^2 - 2\rho R\ cos(\theta)}\right)$$

We now apply the second boundary condition. We expect the potential to be $V$ at $\rho = b$,

$$V = \frac{\tau}{4\pi\epsilon_0} ln\left(\frac{b^2 + R'^2 - 2bR'\ cos(\theta)}{b^2 + R^2 - 2bR\ cos(\theta)}\right)$$

$$b^2 + R'^2 - (b^2 + R^2)\ exp\left(\frac{4\pi\epsilon_0 V}{\tau}\right) = 2b\ cos(\theta)\left(R' - R\ exp\left(\frac{4\pi\epsilon_0 V}{\tau}\right)\right)$$

We want a solution for all $\theta$, so we can try arbitrary angles. Let's first try $\theta = \frac{\pi}{2}$. The right side goes to zero, so the left side can be solved by

$$b^2 + R'^2 = (b^2 + R^2)\ exp\left(\frac{4\pi\epsilon_0 V}{\tau}\right)$$

Since our potential should hold for all angles, we know the right side must also equal zero,

$$R' = R\ exp\left(\frac{4\pi\epsilon_0 V}{\tau}\right)$$

Solving this system of equations,

$$RR'^2 - \left(b^2 + R^2\right)\ R' + Rb^2 = 0$$

$$R' = \frac{b^2}{R}$$

### 2.11.2 Potential

**the potential at any point (expressed in polar coordinates with the origin at the axis of the cylinder and the direction from the origin to the line charge as the axis of the cylinder and the direction from the origin to the line charge as the x axis), including the asymptotic form far from the cylinder**

We can use the potential found in the previous section. Remember still that we have no $\phi$ dependence since our line charges are infinitely long.

$$\Phi = \frac{\tau}{4\pi\epsilon_0} ln\left(\frac{\rho^2 + \frac{b^4}{R^2} - 2\frac{\rho b^2\ cos(\theta)}{R}}{\rho^2 + R^2 - 2\rho R\ cos(\theta)}\right)$$

If we want to find the asymptotic form, we want to use the approximation $ln(1+x) \approx x$,

$$= \frac{\tau}{4\pi\epsilon_0} ln\left(1 - 1 + \frac{\rho^2 + \frac{b^4}{R^2} - 2\frac{\rho b^2\ cos(\theta)}{R}}{\rho^2 + R^2 - 2\rho R\ cos(\theta)}\right)$$

$$\approx \frac{\tau}{4\pi\epsilon_0}\left(\frac{\frac{b^4 - R^4}{R^2} + \frac{2\rho\ cos(\theta)}{R}(R^2 - b^2)}{\rho^2 + R^2 - 2\rho R\ cos(\theta)}\right)$$

Since we are very far away, $\rho >> b$ and $\rho >> R$, we need only keep the term with the highest order in $\rho$.

$$\Phi = \frac{\tau}{2\pi\epsilon_0}\frac{(R^2 - b^2)}{\rho R} cos(\theta)$$

### 2.11.3 Induced Surface Charge Density

**the induced surface-charge density**

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial \rho}\bigg|_{\rho=b} = \frac{\tau}{4\pi}\frac{\partial}{\partial \rho}\left[ ln\left( \frac{\rho^2 + R^2 - 2\rho R\ cos(\theta)}{\rho^2 + \frac{b^4}{R^2} - \frac{2\rho b^2\ cos(\theta)}{R}} \right)\right]\bigg|_{\rho=b}$$

$$= \frac{\tau}{4\pi}\left[ \frac{2\rho - 2R\ cos(\theta)}{\rho^2 + R^2 - 2\rho R\ cos(\theta)} - \frac{2\rho - \frac{2b^2}{R}cos(\theta)}{\rho^2 + \frac{b^4}{R^2} - \frac{2\rho b^2}{R}cos(\theta)} \right]\bigg|_{\rho=b}$$

$$\sigma = \frac{\tau}{2\pi b}\left[ \frac{1 - \left(\frac{R}{b}\right)^2}{1 + \left(\frac{R}{b}\right)^2 - \frac{2R}{b}cos(\theta)} \right]$$

### 2.11.4 Force per Unit Length

**the force per unit length on the line charge**

We calculated the electric field due to an infinite line charge back in section 1.1,

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 a}\hat{\rho}$$

In our case, this becomes the field due to the image charge,

$$\vec{E} = -\frac{\tau}{2\pi\epsilon_0\left(R - \frac{b^2}{R}\right)}\hat{\rho}$$

The force per unit length can be found using equation (1.4),

$$\vec{F} = -\frac{\tau^2}{2\pi\epsilon_0(R^2 - b^2)}\hat{\rho}$$

## 2.12 Polar Separation of Variables: Potential Inside a Cylinder

**Starting with the series solution (2.32) for the two-dimensional potential problem with the potential specified on the surface of a cylinder of radius $b$, evaluate the coefficients formally, substitute them into the series, and sum it to obtain the potential inside the cylinder in the form of Poisson's integral:**

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\ \cos(\phi'-\phi)}d\phi'$$

**What modification is necessary if the potential is desired in the region of space bounded by the cylinder and infinity?**

---

The general potential is given by

$$\Phi = a_0 + b_0\ln(\rho) + \sum_{n=1}^{\infty} a_n\rho^n\sin(n\phi+\alpha_n) + \sum_{n=1}^{\infty} b_n\rho^{-n}\sin(n\phi+\beta_n)$$

Since we are looking at the interior of the cylinder, we set the $b$ terms to zero since our region of interest includes the origin. For the sake of symmetry, we write the potential in terms of $\frac{\rho}{b}$ rather than in terms of $\rho$.

$$\Phi = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n\left(\frac{\rho}{b}\right)^n\cos(n\phi) + B_n\left(\frac{\rho}{b}\right)^n\sin(n\phi)\right]$$

We recognize this as a Fourier series (2.24), so the coefficients will be given by equations (2.25) and (2.26).

$$A_0 = \frac{1}{\pi}\int_0^{2\pi}\Phi(b,\phi')\ d\phi'$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi}\Phi(b,\ \phi')\cos(n\phi')\ d\phi'$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi}\Phi(b,\ \phi')\sin(n\phi')\ d\phi'$$

Substituting this back into our potential,

$$\Phi = \frac{1}{2\pi}\Phi(b,\ \phi')\left[1+2\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n(\cos(n\phi)\cos(n\phi')+\sin(n\phi)\sin(n\phi'))\right]\ d\phi'$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n \cos[n(\phi-\phi')]\right]\, d\phi'$$

If we want this to match what Jackson gets, we need to use complex analysis.

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n \cos[n(\phi-\phi')] = Re\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n e^{in(\phi'-\phi)}\right]$$

$$= Re\left[-1 + 2\sum_{n=0}^{\infty}\left(\frac{\rho}{b}e^{i(\phi'-\phi)}\right)^n\right]$$

You can look up the sum,

$$= Re\left[-1 + 2\frac{1}{1-\frac{\rho}{b}e^{i(\phi'-\phi)}}\right]$$

Simplifying,

$$= Re\left(\frac{1-\left(\frac{\rho}{b}\right)^2 + \frac{2i\rho\sin(\phi'-\phi)}{b}}{1+\left(\frac{\rho}{b}\right)^2 - \frac{2\rho\cos(\phi'-\phi)}{b}}\right)$$

$$= \frac{1-\left(\frac{\rho}{b}\right)^2}{1+\left(\frac{\rho}{b}\right)^2 - \frac{2\rho\cos(\phi'-\phi)}{b}}$$

$$= \frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos(\phi'-\phi)}$$

The potential inside the cylinder is given by

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}\, d\phi'$$

If instead we look at the region outside the cylinder, the $b_0$ and $a_n$ terms die giving a general potential of

$$\Phi(\rho,\phi') = \int_0^{2\pi}\Phi(b,\phi')\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{b}{\rho}\right)^n \cos[n(\phi-\phi')]\right] d\phi'$$

$$\Phi(\rho,\phi') = \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{\rho^2-b^2}{\rho^2+b^2-2\rho b\cos(\phi'-\phi)}\, d\phi'$$

## 2.13 Polar Separation of Variables: Two Half-Cylinders at Different Potentials

### 2.13.1 Potential Inside

**Two halves of a long hollow conducting cylinder of inner radius $b$ are separated by small lengthwise gaps on each side, and are kept at different potentials $V_1$ and $V_2$. Show that the potential inside is given by**

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos(\phi)\right)$$

**where $\phi$ is measured from a plane perpendicular to the plane through the gap.**

---

Because we have a sufficiently long cylinder, we can use the series expansion (2.32). We assume we are far enough away from the edges that those effects are negligible. Since we're looking at the potential inside the cylinder, we want to suppress the $\ln(\rho)$ and $\rho^{-n}$ terms,

$$\Phi = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n)$$

Writing in terms of $\dfrac{\rho}{b}$ for symmetry,

$$= \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n\left(\frac{\rho}{b}\right)^n\cos(n\phi) + B_n\left(\frac{\rho}{b}\right)^n\sin(n\phi)\right]$$

This is a Fourier series (2.24), so the coefficients are given by equations (2.25) and (2.26).

$$A_0 = \frac{1}{\pi}\left(\int_{-\pi}^{0} V_1\ d\phi' + \int_0^{\pi} V_2\ d\phi'\right) = V_1 + V_2$$

$$A_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} V_1\cos(n\phi')\ d\phi' + \int_0^{\pi} V_2\cos(n\phi')\ d\phi'\right) = 0$$

$$B_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} V_1\sin(n\phi')\ d\phi' + \int_0^{\pi} V_2\sin(n\phi')\ d\phi'\right) = \begin{cases} 0; & n\ even; \\ \dfrac{2(V_1-V_2)}{n\pi}; & n\ odd \end{cases}$$

Substituting back into the general potential,

$$\Phi(\rho,\ \phi) = \frac{V_1+V_2}{2} + \frac{2(V_1-V_2)}{\pi}\sum_{n\ odd}\frac{1}{n}\left(\frac{\rho}{b}\right)^n\sin(n\phi)$$

Let's evaluate the sum. For simplicity, we set $x = \rho/b$,

$$\sum_{n\ odd} \frac{1}{n} x^n \sin(n\phi) = Im\left(\sum_{n\ odd} \frac{1}{n} x^n e^{in\phi}\right)$$

$$= Im\left(\sum_{n\ odd} \frac{Z^n}{n}\right)$$

where we have defined $Z = x\ e^{i\phi}$. Following Jackson section 2.10,

$$= \frac{1}{2}\tan^{-1}\left[2\frac{Im(Z)}{1-|Z|^2}\right] = \frac{1}{2}\tan^{-1}\left(\frac{2x\sin(\phi)}{1-x^2}\right)$$

Our potential is

$$\Phi(\rho,\ \phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2\left(\frac{\rho}{b}\right)\sin(\phi)}{1-\left(\frac{\rho}{b}\right)^2}\right)$$

$$= \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho\sin(\phi)}{b^2-\rho^2}\right)$$

To match Jackson's solution, we simply need to rotate our cylinder by a factor of $\pi/2$.

$$\Phi(\rho,\ \phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos(\phi)\right)$$

### 2.13.2 Surface-charge Density

Calculate the surface-charge density on each half of the cylinder.

---

We use equation (1.26),

$$\sigma = \epsilon_0 \frac{\partial\Phi}{\partial\rho}\bigg|_{\rho=b} = \epsilon_0 \frac{V_1-V_2}{\pi}\frac{(2b^3+2b\rho^2)\cos(\phi)}{(b^2+\rho^2)^2+4b^2\rho^2\cos^2(\phi)}\bigg|_{\rho=b}$$

$$\sigma = \epsilon_0 \frac{V_1-V_2}{\pi b\cos(\phi)}$$

## 2.14 Polar Separation of Variables: Four Quarter-Cylinders at Alternating Potentials

**A variant of the preceding two-dimensional problem is a long hollow conducting cylinder of radius $b$ that is divided into equal quarters, alternate segments being held at potential $+V$ and $-V$.**

---

### 2.14.1 Potential Inside

Solve by means of the series solution (2.32) and show that the potential inside the cylinder is

$$\Phi(\rho,\ \phi) = \frac{4V}{\pi} \sum_{n=0}^{\infty} \left(\frac{\rho}{b}\right)^{4n+2} \frac{\sin[(4n+2)\phi]}{2n+1}$$

---

Since we are looking at the potential inside the cylinder, we want to suppress the $\ln(\rho)$ and $\rho^{-n}$ terms.

$$\Phi = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left[A_n \left(\frac{\rho}{b}\right)^n \cos(n\phi) + B_n \left(\frac{\rho}{b}\right)^n \sin(n\phi)\right]$$

We recognize this as a Fourier series (2.24), which means the coefficients are given by equation (2.25) and (2.26),

$$A_0 = \frac{1}{\pi}\left[\int_0^{\pi/2} V\ d\phi' - \int_{\pi/2}^{\pi} V\ d\phi' + \int_{\pi}^{3\pi/2} V\ d\phi' - \int_{3\pi/2}^{2\pi} V\ d\phi'\right] = 0$$

$$A_n = \frac{1}{\pi}\left[\int_0^{\pi/2} V\cos(n\phi')\ d\phi' - \int_{\pi/2}^{\pi} V\cos(n\phi')\ d\phi' + \int_{\pi}^{3\pi/2} V\cos(n\phi')\ d\phi' - \int_{3\pi/2}^{2\pi} V\cos(n\phi')\ d\phi'\right] = 0$$

$$B_n = \frac{1}{\pi}\left[\int_0^{\pi/2} V\sin(n\phi')\ d\phi' - \int_{\pi/2}^{\pi} V\sin(n\phi')\ d\phi' - \int_{3\pi/2}^{2\pi} V\sin(n\phi')\ d\phi'\right]$$
$$= \frac{2V}{n\pi}\left[1 - 2\cos\left(\frac{n\pi}{2}\right) + \cos(n\pi)\right]$$

By plugging in integer values of $n$, we find that the $B_n$ terms only survive when $n$=2, 6, 10, ...

$$B_n = 4\frac{2V}{n\pi}$$

Summing these terms,

$$\Phi(\rho, \phi) = \frac{4V}{\pi} \sum_{n=0}^{\infty} \left(\frac{\rho}{b}\right)^{4n+2} \frac{\sin[(4n+2)\phi]}{2n+1}$$

### 2.14.2 Complex Analysis

**Sum the series and show that**

$$\Phi(\rho,\ \phi) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2\rho^2 b^2 \sin(2\phi)}{b^4-\rho^4}\right)$$

---

We can write the potential in complex terms as

$$\Phi = Im\left(\frac{8V}{\pi}\sum_{n\ odd}^{\infty}\left(\frac{\rho}{b}e^{i\phi}\right)^{4n+2}\frac{1}{4n+2}\right)$$

Making the substitution $m = 2n+1$ and $Z = \frac{\rho}{b}e^{i\phi}$,

$$Im\left(\frac{8V}{\pi}\sum_{m\ odd}\frac{Z^{2m}}{2m}\right)$$

Following Jackson,

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2\rho^2 b^2 \sin(2\phi)}{b^4-\rho^4}\right)$$

### 2.14.3 Equipotential

**Find the equipotential**

---

If we want to find the equipotential points, we need to find where the derivative is 0,

$$\frac{\partial\Phi}{\partial\rho} = \frac{2V}{\pi}\frac{1}{1+\left(\frac{2\rho^2b^2\sin(2\phi)}{b^4-\rho^4}\right)^2}\frac{4\rho b^2(b^4-\rho^4)+4\rho^3(2\rho^2b^2)}{(b^4-\rho^4)^2}\sin(2\phi) = 0$$

$$4\rho b^6 - 4\rho^5 b^2 + 8\rho^5 b^2 = 0$$

$$\rho = b$$

As expected, the potential is constant on the surface (sub some polarity switching).

## 2.15 Green Function Corresponding to Two-Dimensional Potential

### 2.15.1 Green Function Expansion

**Show that the Green function $G(x,\ y;\ x',\ y')$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0 \leq x \leq 1$, $0 \leq y \leq 1$, has an expansion**

$$G(x,\ y;\ x',\ y') = 2\sum_{n=1}^{\infty} g_n(y,\ y')\ \sin(n\pi x)\ \sin(n\pi x')$$

**where $g_n(y,\ y')$ satisfies**

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right)\ g_n(y,\ y') = -4\pi\delta(y'-y)\ \textit{and}\ g_n(y,\ 0) = g_n(y,\ 1) = 0$$

---

We'll start by looking at the condition for Green functions (1.42).

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right) G(x,\ y;\ x',\ y') = -4\pi\delta(x-x')\delta(y-y')$$

Substituting in the Green function given in the problem,

$$\left(-n^2\pi^2 + \frac{\partial^2}{\partial y'^2}\right)\left(2\sum_{n=1}^{\infty} g_n(y,\ y')\sin(n\pi x)\ \sin(n\pi x')\right) = -4\pi\delta(x-x')\delta(y-y')$$

On the right-hand side, we want to use the orthogonality condition for $\sin(x)$,

$$= -4\pi\left(2\sum_{n=1}^{\infty}\sin(n\pi x)\ \sin(n\pi x')\right)\delta(y-y')$$

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) g_n(y,\ y') = -4\pi\delta(y'-y)$$

We get the final conditions, $g_n(y,\ 0) = 0$ and $g_n(y,\ 1) = 0$, from requiring that the potential (and therefore the Green function) vanish at $y = 0$ and $y = 1$.

### 2.15.2 Green Function Explicit Form

**Taking for $g_n(y,\ y')$ appropriate linear combinations of $\sinh(n\pi y')$ and $\cosh(n\pi y')$ in the two regions, $y' < y$ and $y' > y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function, show that the explicit form of $G$ is**

$$G(x,\ y;\ x',\ y') = 8\sum_{n=1}^{\infty}\frac{1}{n\ \sinh(n\pi)}\sin(n\pi x)\ \sin(n\pi x')\ \sinh(n\pi y_<)\ \sinh[n\pi(1-y_>)]$$

**where $y_<(y_>)$ is the smaller (larger) of $y$ and $y'$.**

---

We want $g_n(y,\ y')$ to be a linear combination of sinh and cosh, so we'll create two solutions (one for when $y' < y$ and another for when $y' > y$),

$$g_n(y,\ y') = \begin{cases} a_< \sinh(n\pi y') + b_< \cosh(n\pi y'), & y' < y; \\ a_> \sinh(n\pi y') + b_> \cosh(n\pi y'), & y' > y \end{cases}$$

For ease of reference, the top equation is designated $g_<$ and the bottom, $g_>$. We know the potential must disappear at $y' = 0$ and at $y' = 1$. For the first case, we have $g_< = 0$ at $y' = 0$. For the second case, we have $g_> = 0$ at $y' = 1$. From this, $g_n$ simplifies to

$$g_< = a_< \sinh(n\pi y')$$

$$g_> = a_> \sinh[n\pi(1 - y')]$$

Now, let's apply the boundary conditions for $y' = y$. Since the Green function must be continuous at the boundary (1.27),

$$g_< = g_>|_{y=y'}$$

$$a_< \sinh(n\pi y) = a_> \sinh[n\pi(1 - y)]$$

Since the normal derivative of the Green function has a discontinuity (1.26),

$$\frac{\partial g_<}{\partial y'} - \frac{\partial g_>}{\partial y'} = 4\pi$$

$$a_< n\pi\ \cosh(n\pi y) + a_> n\pi\ \cosh[n\pi(1 - y)] = 4\pi$$

Solving for the constants,

$$a_< = \frac{4 \sinh[n\pi(1 - y)]}{n\ \sinh(n\pi)}$$

$$a_> = \frac{4 \sinh(n\pi y)}{n\ \sinh(n\pi)}$$

Thus,

$$g_< = \frac{4}{n\ \sinh(n\pi)} \sinh(n\pi y') \sinh[n\pi(1 - y)]$$

$$g_> = \frac{4}{n\ \sinh(n\pi)} \sinh(n\pi y) \sinh[n\pi(1 - y')]$$

If we want to combine these two equations, we note that they follow the same form. We then pick out the $y'$ term from each equation,

$$g_n = \frac{4\ \sinh(n\pi y_<)\ \sinh[n\pi(1 - y_>)]}{n\ \sinh(n\pi)}$$

Substituting this back into the given Green function, we get the desired form.

## 2.16 Two-Dimensional Potential, Square Area

**A two-dimensional potential exists on a unit square area** $(0 \leq x \leq 1,\ 0 \leq y \leq 1)$ **bounded by "surfaces" held at zero potential. Over the entire square there is a uniform charge density of unit strength (per unit length in z). Using the Green function of Problem 2.15, show that this solution can be written as**

$$\Phi(x,y) = \frac{4}{\pi^3 \epsilon_0} \sum_{m=0}^{\infty} \frac{\sin[(2m+1)\pi x]}{(2m+1)^3} \left\{ 1 - \frac{\cosh\left[(2m+1)\,\pi\left(y - \frac{1}{2}\right)\right]}{\cosh\left[(2m+1)\,\frac{\pi}{2}\right]} \right\}$$

---

If we want to find the potential everywhere, we need to use the general Dirichlet solution to Poisson's equation (1.45). We use Dirichlet Green function since we have the potential defined on the surface. Since the potential is defined to be 0 at the surfaces, equation (1.45) simplifies to

$$\Phi = \frac{1}{4\pi\epsilon_0} \int_V \rho\, G_D\ d^3x'$$

Using the Green function that we derived in the previous problem and setting $\rho = 1$,

$$= \frac{2}{\pi\epsilon_0} \sum_{n=1}^{\infty} \int_0^1 \int_0^1 \frac{\sin(n\pi x)\ \sin(n\pi x')}{n\ \sinh(n\pi)} \sinh(n\pi y_<)\ \sinh[n\pi(1-y_>)]\ dx'dy'$$

$$= \frac{2}{\pi\epsilon_0} \frac{\sin(n\pi x)}{n\ \sinh(n\pi)} \frac{1-\cos(n\pi)}{n\pi} \left( \int_0^y \sinh(n\pi y')\ \sinh[n\pi(1-y)]\ dy' + \int_y^1 \sinh(n\pi y)\ \sinh[n\pi(1-y')]\ dy' \right)$$

$$= \sum_{n=1}^{\infty} \frac{2}{\pi\epsilon_0} \frac{\sin(n\pi x)}{n\ \sinh(n\pi)} \frac{1-\cos(n\pi)}{n\pi} \frac{\sinh(n\pi)}{n\pi} \left( 1 - \frac{\cosh\left[n\pi\left(y-\frac{1}{2}\right)\right]}{\cosh\left(\frac{n\pi}{2}\right)} \right)$$

This returns not zero when $n$ is odd, i.e., $n = 2m+1$. Making this substitution,

$$\Phi(x,y) = \sum_{m=1}^{\infty} \frac{4}{\pi^3\epsilon_0} \frac{\sin[(2m+1)\pi x]}{(2m+1)^3} \left( 1 - \frac{\cosh\left[(2m+1)\pi\left(y-\frac{1}{2}\right)\right]}{\cosh\left(\frac{(2m+1)\pi}{2}\right)} \right]$$

## 2.17 Two-Dimensional Green Function

### 2.17.1 Green Function

**Construct the free-space Green function $G(x,\ y;\ x',\ y')$ for two-dimensional electrostatics by integrating $1/R$ with respect to $(z'-z)$ between the limits $\pm Z$, where $Z$ is taken to be very large. Show that apart from an inessential constant, the Green function can be written alternately as**

$$G(x,\ y;\ x',\ y') = -\ln[(x-x')^2+(y-y')^2] = -\ln[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')]$$

---

Let's do what Jackson suggests we do,

$$G(x,\ y;\ x',\ y') = \int_{-Z}^{Z} \frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]}\ d(z'-z)$$

$$= \ln\left[\left((x-x')^2+(y-y')^2+Z^2\right)^{1/2}+Z\right] - \ln\left[\left((x-x')^2+(y-y')^2+Z^2\right)^{1/2}-Z\right]$$

We can then use the binomial approximation to reduce this,

$$\approx \ln\left[Z\left(1+\frac{(x-x')^2+(y-y')^2}{2Z^2}\right)+Z\right] - \ln\left[Z\left(1+\frac{(x-x')^2+(y-y')^2}{2Z^2}\right)-Z\right]$$

$$= \ln\left[2Z+\frac{(x-x')^2+(y-y')^2}{2Z}\right] - \ln\left[\frac{(x-x')^2+(y-y')^2}{2Z}\right]$$

Since $Z >> (x-x')^2+(y-y')^2$, we can ignore the fraction in the first term,

$$\approx \ln(2Z) - \ln\left[\frac{(x-x')^2+(y-y')^2}{2Z}\right]$$

$$\approx \ln(2Z) + \ln(2Z) - \ln[(x-x')^2+(y-y')^2]$$

$$= \ln(4Z^2) - \ln[(x-x')^2+(y-y')^2]$$

Making the substitution,

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \end{cases}$$

It follows,

$$(-x')^2+(y-y')^2 = \rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')$$

$$G(\rho,\ \phi;\ \rho',\ \phi') = -\ln[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')]$$

### 2.17.2 Fourier Series

**Show explicitly by separation of variables in polar coordinates that the Green function can be expressed as a Fourier series in the azimuthal coordinate,**

$$G = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{im(\phi-\phi')} g_m(\rho,\ \rho')$$

**where the radial Green functions satisfy**

$$\frac{1}{\rho'}\frac{\partial}{\partial \rho'}\left(\rho'\frac{\partial g_m}{\partial \rho'}\right) - \frac{m^2}{\rho'^2} g_m = -4\pi\frac{\delta(\rho-\rho')}{\rho}$$

**note that $g_m(\rho,\ \rho')$for fixed $\rho$ is a different linear combination of the solutions of the homogeneous radial equation for $\rho' < \rho$ and for $\rho' > \rho$, with a discontinuity of slope at $\rho' = \rho$ determined by the source delta function.**

---

The homogeneous radial equation is

$$\frac{\rho}{R}\frac{d}{dR}\left(\rho\frac{dR}{d\rho}\right) = \nu^2$$

The Laplacian of the Green function (1.42) in polar coordinates is given by

$$\nabla^2 G = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial G}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial \phi^2} = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')$$

If as Jackson suggests, we write the Green function as

$$G = \sum_{-\infty}^{\infty} g(\rho,\ \rho')e^{-im\phi'}a(\phi)$$

We can also write the delta function on the right side as

$$\sum_{-\infty}^{\infty} e^{im(\phi-\phi')} = 2\pi\ \delta(\phi-\phi')$$

Plugging this in,

$$\frac{1}{\rho'}\frac{\partial}{\partial \rho'}\left(\rho'\frac{\partial}{\partial \rho'}g(\rho,\rho')e^{-im\phi'}a(\phi)\right) + \frac{1}{\rho'^2}m^2 a(\phi)g(\rho,\rho')e^{-im\phi'} = -\frac{4\pi}{\rho}\delta(\rho-\rho')\frac{1}{2\pi}\sum_{-\infty}^{\infty}e^{im\phi}e^{-im\phi'}$$

$$a(\phi)e^{-im\phi'}\left[\frac{1}{\rho'}\frac{\partial}{\partial \rho'}\left(\rho'\frac{\partial g(\rho,\rho')}{\partial \rho'}\right) + \frac{m^2}{\rho'^2}g(\rho,\rho')\right] = -\frac{4\pi}{\rho}\delta(\rho-\rho')\frac{1}{2\pi}\sum_{-\infty}^{\infty}e^{im\phi}e^{-im\phi'}$$

Matching solutions,

$$\frac{1}{\rho'}\frac{\partial}{\partial \rho'}\left(\rho'\frac{\partial g(\rho,\rho')}{\partial \rho'}\right)+\frac{m^2}{\rho'^2}g(\rho,\rho')=-\frac{4\pi}{\rho}\delta(\rho-\rho')$$

$$a(\phi)e^{-im\phi'}=\frac{1}{2\pi}\sum_{-\infty}^{\infty}e^{im\phi}e^{-im\phi'}$$

$$a(\phi)=\frac{1}{2\pi}e^{im\phi}$$

$$G=\frac{1}{2\pi}\sum_{-\infty}^{\infty}e^{im(\phi-\phi')}g(\rho,\rho')$$

### 2.17.3 Completing the Solution

**Complete the solution and show that the free-space Green function has the expansion**

$$G(\rho,\ \phi;\ \rho',\ \phi')=-\ln(\rho_>^2)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos[m(\phi-\phi')]$$

**where $\rho_<(\rho_>)$ is the smaller (larger) of $\rho$ and $\rho'$.**

---

We recognize the separation of variables is the same one outlined in section 2.6.

$$g(\rho,\ \rho')=a\rho'^m+b\rho'^{-m}$$

$$g(\rho,\rho')=a_0+b_0\ln(\rho'),\quad for\ m=0$$

For $m>0$,

$$\begin{cases} g_> = a\rho'^m+b\rho'^{-m}, & \rho'>\rho \\ g_< = c\rho'^m+d\rho'^{-m}, & \rho'<\rho \end{cases}$$

Looking at $\rho'=0$ and $\rho'=\infty$, we want $d$ and $a$ to go to zero respectively,

$$\begin{cases} g_> = b\rho'^{-m} \\ g_< = c\rho'^m \end{cases}$$

Using the condition that Green function is continuous at the bounds (1.27),

$$b\rho^{-m}=c\rho^m$$

Using the condition that the normal derivative of the Green function is discontinuous at the boundary (1.26),

$$\frac{\partial g_>}{\partial \rho'} - \frac{\partial g_<}{\partial \rho'} = -\frac{4\pi}{\rho'}\bigg|_{\rho'=\rho}$$

$$-mb\rho^{-m-1} - mc\rho^{m-1} = -\frac{4\pi}{\rho}$$

Solving the resulting system of equations,

$$b = \frac{4\pi\rho^m}{m}$$

$$c = \frac{4\pi}{m\rho^m}$$

$$\begin{cases} g_> = 4\pi\dfrac{\rho^m}{m\rho'^m} \\ \\ g_< = 4\pi\dfrac{\rho'^m}{m\rho^m} \end{cases}$$

$$g = \frac{4\pi}{m}\left(\frac{\rho_<}{\rho_>}\right)^m$$

When $m = 0$, we only need to look at the case when $\rho' > \rho$ since the other case would include $\rho' = 0$, which does not fly.

$$g_0 = b_0 \ln(\rho_>)$$

To determine the value of $b_0$, we use the discontinuity in the normal derivative of the Green function (1.26),

$$\frac{\partial g_0}{\partial \rho'} = -\frac{4\pi}{\rho'}\bigg|_{\rho'=\rho}$$

$$\frac{b_0}{\rho} = -\frac{4\pi}{\rho}$$

$$b_0 = -4\pi$$

Using the form of the Green function we derived in the previous section,

$$G = \frac{1}{2\pi}\left[-4\pi\ln(\rho_>) + 4\pi\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m e^{im(\phi-\phi')}\right]$$

When we take the real component of $e^{im(\phi-\phi')}$,

$$G(\rho,\ \phi;\ \rho',\ \phi') = -\ln(\rho_>^2) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos[m(\phi-\phi')]$$

## 2.18 Green Function: Interior of a Cylinder

### 2.18.1 Green Function

**By finding appropriate solutions of the radial equation in part b of Problem 2.17, find the Green function for the interior Dirichlet problem of a cylinder of radius** $b[g_m(\rho, \rho' = b) = 0]$**. First find the series expansion akin to the free-space Green function of Problem 2.17. Then show that it can be written in closed form as**

$$G = \ln\left[\frac{\rho^2\rho'^2 + b^4 - 2\rho\rho' b^2\cos(\phi - \phi')}{b^2(\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi'))}\right]$$

**or**

$$G = \ln\left[\frac{(b^2 - \rho^2)(b^2 - \rho'^2) + b^2|\vec{\rho} - \vec{\rho}'|^2}{b^2|\vec{\rho} - \vec{\rho}'|^2}\right]$$

---

From solving the polar separation of variables problem (2.32), the radial component of the potential is

$$\begin{cases} g_> = \alpha\rho'^m + \beta\rho'^{-m}, & \rho' > \rho \\ \\ g_< = c\rho'^m + d\rho'^{-m}, & \rho' < \rho \end{cases}$$

Since we're looking at the interior of the cylinder, only $d$ is zero (if we are at the origin, we have to use the $g_<$ solution). Using the condition that the Green function is continuous at the boundary (1.27),

$$c\rho^m = \alpha\rho^m + \beta\rho^{-m}$$

Using the condition that the normal derivative of the Green function has a discontinuity (1.26),

$$\frac{\partial g_>}{\partial \rho'} - \frac{\partial g_<}{\partial \rho'} = -\frac{4\pi}{\rho'}\Big|_{\rho'=\rho}$$

$$\alpha m\rho^{m-1} - \beta m\rho^{-m-1} - cm\rho^{m-1} = -\frac{4\pi}{\rho}$$

At the $\rho' = b$ boundary, $g_>$ goes to zero,

$$\alpha b^m + \beta b^{-m} = 0$$

Solving for constants,

$$\begin{cases} \alpha = -\dfrac{2\pi}{m}\rho^m b^{-2m} \\ \\ \beta = \dfrac{2\pi}{m}\rho^m \\ \\ c = \dfrac{2\pi}{m}\left[-\left(\dfrac{\rho}{b^2}\right)^m + \rho^{-m}\right] \end{cases}$$

$$\begin{cases} g_> = \dfrac{2\pi}{m}\left[\left(\dfrac{\rho}{\rho'}\right)^m - \left(\dfrac{\rho\rho'}{b^2}\right)^m\right] \\ g_< = \dfrac{2\pi}{m}\left[\left(\dfrac{\rho'}{\rho}\right)^m - \left(\dfrac{\rho\rho'}{b^2}\right)^m\right] \end{cases}$$

$$g = \frac{2\pi}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^m - \left(\frac{\rho\rho'}{b^2}\right)^m\right]$$

At $m = 0$, we want the components to vanish at the boundary $\rho' = b$,

$$g_0 = a_0 + b_0 \ln(\rho')$$

$$a_0 = b_0 \ln\left(\frac{1}{b}\right)$$

To determine the value of $b_0$, we use the discontinuity in the normal derivative of the Green function (1.26),

$$\frac{\partial g_0}{\partial \rho'} = -\frac{4\pi}{\rho'}\Big|_{\rho'=\rho}$$

$$b_0 = -4\pi$$

Using the form of the Green function from problem 2.17,

$$G = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{im(\phi-\phi')} g(\rho,\rho')$$

As we did in problem 2.17, we take the real component,

$$G = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{im(\phi-\phi')} g_m(\rho,\rho')$$

$$= \log\left(\frac{b^2}{\rho_>^2}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^m - \left(\frac{\rho\rho'}{b^2}\right)^m\right]\cos[m(\phi-\phi')]$$

$$= \ln\left(\frac{b^2}{\rho_>^2}\right) - \ln\left[1 + \left(\frac{\rho_<}{\rho_>}\right)^2 - 2\left(\frac{\rho_<}{\rho_>}\right)\cos(\phi-\phi')\right] + \ln\left[1 + \left(\frac{\rho\rho'}{b^2}\right)^2 - 2\left(\frac{\rho\rho'}{b^2}\right)\cos(\phi-\phi')\right]$$

$$= \ln\left[\frac{b^4 + \rho^2\rho'^2 - 2\rho\rho' b^2\cos(\phi-\phi')}{b^2(\rho^2+\rho'^2 - 2\rho\rho'\cos(\phi-\phi'))}\right]$$

### 2.18.2 Poisson's Integral

**Show that the solution of the Laplace equation with the potential given as $\Phi(b, \phi)$ on the cylinder can be expressed as Poisson's integral of Problem 2.12.**

---

Poisson's integral is given by

$$\Phi(\rho, \phi) = \int_0^{2\pi} \Phi(b, \phi') \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\phi' - \phi)}\, d\phi'$$

We start by looking at the general solution to Poisson's equation (1.43). If we are in charge-free space,

$$\Phi(\rho, \phi) = -\frac{1}{4\pi} \oint_S \Phi(b, \phi') \frac{\partial G}{\partial \rho'}\Big|_{\rho'=b}\, d\rho' d\phi'$$

$$= -\frac{1}{4\pi} \int_0^{2\pi} \Phi(b, \phi') b\, \frac{\partial G}{\partial \rho'}\Big|_{\rho'=b}\, d\phi'$$

$$\frac{\partial G}{\partial \rho'} = -\frac{2}{b} \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2\rho b\cos(\phi - \phi')}$$

$$\Phi = \frac{1}{2\pi} \int_0^{2\pi} \Phi(b, \phi') \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2\rho b\cos(\phi' - \phi)}\, d\phi'$$

### 2.18.3 Exterior

**What changes are necessary for the Green function for the exterior problem ($b < \rho < \infty$), for both the Fourier expansion and closed form? [Note that the exterior Green function is not rigorously correct because it does not vanish for $\rho$ or $\rho' \to \infty$. For situations in which the potential falls off fast enough as $\rho \to \infty$, no mistake is made in its use.]**

---

On the exterior of the cylinder, we can convince ourselves that $g$ is given by

$$g \to 4\pi \ln\left(\frac{\rho_<}{b}\right) + \frac{4\pi}{m}\left[\left(\frac{\rho_>}{\rho_<}\right)^m - \left(\frac{b^2}{\rho\rho'}\right)^m\right]$$

Plugging this into the equation for the Green function, we find that we actually get the same Green function both inside and outside the cylinder.

$$G = \ln\left[\frac{\rho^2\rho'^2 + b^4 - 2\rho\rho' b^2\cos(\phi - \phi')}{b^2(\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi'))}\right]$$

## 2.19 Green Function: Cylinder

**Show that the two-dimensional Green function for Dirichlet boundary conditions for the annular region, $b \le \rho \le c$ (concentric cylinders) has the expansion**

$$G = \frac{\ln(\rho_<^2/b^2)\ln(c^2/\rho_>^2)}{\ln(c^2/b^2)} + 2\sum_{m=1}^{\infty} \frac{\cos[m(\phi-\phi')]}{m[1-(b/c)^{2m}]}(\rho_<^m - b^{2m}/\rho_<^m)(1/\rho_>^m - \rho_>^m/c^{2m})$$

---

We'll start with the separation of variables (2.32) and look at the $m = 0$ case first,

$$g_0 = \alpha_0 + \beta_0 \ln(\rho)$$

Let's look at the condition at the boundaries. At $\rho = b$,

$$\alpha_0 + \beta_0 \ln(b) = 0$$

At $\rho = c$,

$$\gamma_0 + \delta_0 \ln(c) = 0$$

The Green function must be continuous at the boundary (1.27),

$$\beta_0 \ln\left(\frac{\rho}{b}\right) = \delta_0 \ln\left(\frac{\rho}{c}\right)$$

The derivative of the Green function must be discontinuous at the boundary (1.26),

$$\frac{\partial g_<}{\partial \rho'} - \frac{\partial g_>}{\partial \rho'} = -\frac{4\pi}{\rho'}\bigg|_{\rho'=\rho}$$

$$\delta_0 - \beta_0 = -4\pi$$

Solving for $\beta_0$ and $\delta_0$,

$$\begin{cases} \beta_0 = -4\pi \dfrac{\ln\left(\frac{\rho}{c}\right)}{\ln\left(\frac{c}{b}\right)} \\[2em] \delta_0 = -4\pi \dfrac{\ln\left(\frac{\rho}{b}\right)}{\ln\left(\frac{c}{b}\right)} \end{cases}$$

$$g_0 = -4\pi \frac{\ln\left(\frac{\rho_<}{b}\right)\ln\left(\frac{\rho_>}{c}\right)}{\ln\left(\frac{c}{b}\right)}$$

$$g_0 = 2\pi \frac{\ln\left(\frac{\rho_<^2}{b^2}\right)\ln\left(\frac{c^2}{\rho_>^2}\right)}{\ln\left(\frac{c^2}{b^2}\right)}$$

When $m$ is not zero,

$$\begin{cases} g_> = \alpha\rho'^m + \beta\rho'^m, & \rho' > \rho \\ g_< = \gamma\rho'^m + \delta\rho'^{-m}, & \rho' < \rho \end{cases}$$

At the bounds, we have

$$g(\rho, c) = \alpha c^m + \beta c^{-m} = 0$$

$$g(\rho, b) = \gamma b^m + \delta b^{-m} = 0$$

The Green function must be continuous at the bounds (1.27),

$$\alpha(\rho^m - c^{2m}\rho^{-m}) = \gamma(\rho^m - b^{2m}\rho^{-m})$$

The normal derivative of the Green function is discontinuous at the boundary (1.26),

$$\alpha m(\rho^m + c^{2m}\rho^{-m}) = \gamma m(\rho^m + b^{2m}\rho^{-m}) - 4\pi$$

Solving for the constants,

$$\alpha = -\frac{2\pi}{m}\left(\frac{\rho^m - b^{2m}\rho^{-m}}{c^{2m} - b^{2m}}\right)$$

$$\beta = -\alpha c^{2m}$$

Substituting these into the Green function,

$$\gamma = -\frac{2\pi}{m}\left(\frac{\rho^m - c^{2m}\rho^{-m}}{c^{2m} - b^{2m}}\right)$$

$$\delta = -\gamma b^{2m}$$

$$g_> = -\frac{2\pi}{m}\frac{(\rho^m - b^{2m}\rho^{-m})(\rho'^m - c^{2m}\rho^{-m})}{c^{2m} - b^{2m}}$$

$$g_< = -\frac{2\pi}{m}\frac{(\rho'^m - b^{2m}\rho'^{-m})(\rho^m - c^{2m}\rho^{-m})}{c^{2m} - b^{2m}}$$

$$g = \frac{2\pi}{m} \frac{\left(\rho_<^m - \frac{b^{2m}}{\rho_<^m}\right)\left(\frac{1}{\rho_>^m} - \frac{\rho_>^m}{c^{2m}}\right)}{1 - \left(\frac{b}{c}\right)^{2m}}$$

From problem 2.17, we can write the Green function as

$$G = \frac{1}{2\pi} \sum_{-\infty}^{\infty} g_m(\rho, \rho') e^{im(\phi - \phi')}$$

$$G = \frac{\ln\left(\frac{\rho_<^2}{b^2}\right) \ln\left(\frac{c^2}{\rho_>^2}\right)}{\ln\left(\frac{c^2}{b^2}\right)} + 2 \sum_{m=1}^{\infty} \frac{\cos[m(\phi - \phi')]}{m\left[1 - \left(\frac{b}{c}\right)^{2m}\right]} \left(\rho_<^m - \frac{b^{2m}}{\rho_<^m}\right)\left(\frac{1}{\rho_>^m} - \frac{\rho_>^m}{c^{2m}}\right)$$

## 2.20 Two-Dimensional Quadrupole

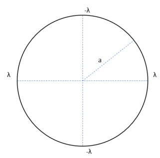

Figure 2.2:

**Two-dimensional electric quadrupole focusing fields for particle accelerators can be modeled by a set of four symmetrically placed line charges, with linear charge densities $\pm\lambda$, as shown in figure (2.2). The charge density in two dimensions can be expressed as**

$$\sigma(\rho,\phi) = \frac{\lambda}{a}\sum_{n=0}^{3}(-1)^n\delta(\rho-a)\delta\left(\phi-\frac{n\pi}{2}\right)$$

---

### 2.20.1 Potential

**Using the Green function expansion from Problem 2.17c, show that the electrostatic potential is**

$$\Phi(\rho,\phi) = \frac{\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{\rho_<}{\rho_>}\right)^{4k+2}\cos[(4k+2)\phi]$$

---

We need to use the general potential with Dirichlet boundary conditions (1.45),

$$\Phi(\rho,\phi) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G_D(\vec{x},\vec{x}')\ d^3x'$$

From problem 2.17, our Green function is

$$G_D(\rho,\phi;\rho',\phi') = -\ln(\rho_>^2) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos[m(\phi-\phi')]$$

Substituting this in,

$$\Phi = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^a \frac{\lambda}{a}\sum_{n=0}^{3}(-1)^n\delta(\rho'-a)\ \delta\left(\phi'-\frac{n\pi}{2}\right)G_D\ \rho'\ d\rho' d\phi'$$

$$= \frac{\lambda}{4\pi\epsilon_0}\sum_{n=0}^{3}(-1)^n\left[-\ln(\rho_>^2) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos\left[m\left(\phi-\frac{n\pi}{2}\right)\right]\right]$$

By taking the sums,

$$\sum_{n=0}^{3}(-1)^n(-\ln(\rho_>^2)) = 0$$

$$\sum_{n=0}^{3}(-1)^n \cos\left[m\left(\phi - \frac{n\pi}{2}\right)\right] = 2\cos(m\phi); \quad for\ m = 4k+2$$

Combining all of this,

$$\Phi = \frac{\lambda}{\pi\epsilon_0}\sum_{k=0}\frac{1}{2k+1}\left(\frac{\rho_<}{\rho_>}\right)^{4k+2}\cos[(4k+2)\phi]$$

### 2.20.2 Complex Analysis

**Relate the solution of part a to the real part of the complex function**

$$w(z) = \frac{2\lambda}{4\pi\epsilon_0}\ln\left[\frac{(z-ia)(z+ia)}{(z-a)(z+a)}\right]$$

**where $z = x + iy = \rho e^{i\phi}$. Comment on the connection to Problem 2.3**

---

Using the Euler identity, we can write the potential as

$$\Phi(\rho,\phi) = Re\left[\frac{\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{\rho_<}{\rho_>}\right)^{4k+2}e^{i(4k+2)\phi}\right]$$

$$= Re\left[\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{1+\left(\frac{\rho_< e^{i\phi}}{\rho_>}\right)^2}{1-\left(\frac{\rho_< e^{\phi}}{\rho_>}\right)^2}\right)\right]$$

Inside the cylinder, $\rho_< = \rho$ and $\rho_> = a$,

$$\Phi = Re\left[\frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{a^2+\rho^2 e^{2i\phi}}{a^2-\rho^2 e^{2i\phi}}\right)\right]$$

$$= Re\left[\frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{(\rho e^{i\phi}-ia)(\rho e^{i\phi}+ia)}{(\rho e^{i\phi}-a)(\rho e^{i\phi}+a)}\right)\right]$$

Making the substitution $z = \rho e^{i\phi}$,

$$= Re\left[\frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{(z-ia)(z+ia)}{(z-a)(z+a)}\right)\right]$$

Using the $w(z)$ given by Jackson,

$$\Phi = Re(w(z))$$

### 2.20.3 Electric Field

**Find expressions for the Cartesian components of the electric field near the origin, expressed in terms of $x$ and $y$. Keep the $k = 0$ and $k = 1$ terms in the expansion. For $y = 0$ what is the relative magnitude of the $k = 1$ ($2^6$ pole) contribution to $E_x$ compared to the $k = 0$ ($2^2$ pole or quadrupole) term?**

---

As usual, we can find the electric field by taking the derivative (1.26),

$$E_x = -\left(\cos(\phi)\frac{\partial\Phi}{\partial\rho} - \frac{\sin(\phi)}{\rho}\frac{\partial\Phi}{\partial\phi}\right)$$

$$E_y = -\left(\sin(\phi)\frac{\partial\Phi}{\partial\rho} + \frac{\cos(\phi)}{\rho}\frac{\partial\Phi}{\partial\phi}\right)$$

Inside the cylinder, we set $\rho_< = \rho$ and $\rho_> = a$. The potential is given by

$$\Phi = \frac{\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{\rho}{a}\right)^{4k+2}\cos[(4k+2)\phi]$$

$$\frac{\partial\Phi}{\partial\phi} = \frac{2\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty}\left(\frac{\rho}{a}\right)^{4k+1}\frac{1}{a}\cos[(4k+2)\phi]$$

$$\frac{\partial\Phi}{\partial\phi} = -\frac{2\lambda}{\pi\epsilon_0}\left(\frac{\rho}{a}\right)^{4k+2}\sin[(4k+2)\phi]$$

Plugging these in,

$$E_x = -\frac{2\lambda}{a\pi\epsilon_0}\sum_{k=0}^{\infty}\left(\frac{\rho}{a}\right)^{4k+1}\cos[(4k+1)\phi]$$

$$E_y = -\frac{2\lambda}{a\pi\epsilon_0}\sum_{k=0}^{\infty}\left(\frac{\rho}{a}\right)^{4k+1}\sin[(4k+3)\phi]$$

Keeping only the first two terms,

$$E_x = -\frac{2\lambda}{a\pi\epsilon_0}\left[\frac{\rho}{a}\cos(\phi) + \left(\frac{\rho}{a}\right)^5\cos(5\phi)\right]$$

$$E_y = -\frac{2\lambda}{a\pi\epsilon_0}\left[\frac{\rho}{a}\cos(3\phi) + \left(\frac{\rho}{a}\right)^5\cos(7\phi)\right]$$

## 2.21 Poisson Integral Solution and Cauchy's Theorem

**Use Cauchy's theorem to derive the Poisson integral solution. Cauchy's theorem states that if $F(z)$ is analytic in a region $R$ bounded by a closed curve $C$, then**

$$\frac{1}{2\pi i}\oint_C \frac{F(z')}{z'-z}dz' = \Big\{F(z), \quad if\ z\ is\ inside\ R; 0, \quad if\ z\ is\ outside\ R$$

**Hint: You may wish to add an integral that vanishes (associated with the image point) to the integral for the point inside the circle.**

---

Poisson's integral solution is given back in problem 2.12,

$$\Phi = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'$$

Cauchy's theorem says

$$\Phi(z) = \frac{1}{2\pi i}\oint_C \frac{\Phi(z')}{z'-z}dz'$$

Using the hint, we subtract Cauchy's theorem from an image charge, $z' = b^2/z^*$, which should be zero,

$$\Phi(z) = \frac{1}{2\pi i}\oint_C \frac{\Phi(z)'}{z'-z}dz' - \frac{1}{2\pi i}\oint_C \frac{\Phi(z')}{z'-\frac{b^2}{z^*}}dz'$$

Since $z'z'^* = b^2$,

$$= \frac{1}{2\pi i}\oint_C \frac{1}{z'}\frac{z'z'^*-zz^*}{|z'-z|^2}\Phi(z')\ dz'$$

Making the substitution $z = \rho e^{i\phi}$ and $z' = be^{i\phi'}$,

$$\Phi = \int_0^{2\pi} \Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'$$

## 2.22 Oppositely Charged Conducting Hemispherical Shells

### 2.22.1 Interior Potential

**For the example of oppositely charged conducting hemispherical shells separated by a tiny gap, show that the interior potential $(r < a)$ on the z axis is**

$$\Phi_{in}(z) = V\frac{a}{z}\left[1 - \frac{(a^2 - z^2)}{a\sqrt{a^2 + z^2}}\right]$$

**Find the first few terms of the expansion in powers of z and show that they agree with equation (2.22) with the appropriate substitutions.**

---

We'll start with Dirichlet's solution to the Poisson equation (1.45) with the associated spherical Green function (2.9),

$$\Phi(z) = \frac{1}{4\pi}\int \Phi \frac{a(a^2 - z^2)}{(z^2 + a^2 - 2az\ cos(\gamma))^{3/2}}\ d\Omega'$$

We see that this is similar to the solution (2.20) we derived for the potential along the z-axis back in section 2.2.

$$= \frac{Va(a^2 - z^2)}{2az}\left[\frac{2a}{a^2 - z^2} - \frac{2}{\sqrt{z^2 + a^2}}\right]$$

$$= V\frac{a}{z}\left[1 - \frac{a^2 - z^2}{a\sqrt{a^2 + z^2}}\right]$$

If we want to show that this is equivalent to equation (2.22), we need to use the binomial approximation,

$$= V\frac{a}{z}\left[1 - \frac{a^2\left(1 - \frac{z^2}{a^2}\right)}{a^2\left(1 - \frac{z^2}{a^2}\right)^{1/2}}\right]$$

$$\approx V\frac{a}{z}\left[1 - \left(1 - \frac{z^2}{a^2}\right)\left(1 - \frac{z^2}{2a^2}\right)\right]$$

$$= V\frac{a}{z}\left(\frac{3z^2}{2a^2}\right) = \frac{3Vz}{2a}$$

Which agrees with the first term in equation (2.22).

### 2.22.2 Electric Field

**From the result of part a and (2.20), show that the radial electric field on the positive z axis is**

$$E_r(z) = \frac{Va^2}{(z^2+a^2)^{3/2}}\left(3+\frac{a^2}{z^2}\right)$$

**For $z > a$, and**

$$E_r(z) = -\frac{V}{a}\left[\frac{3+(a/z)^2}{(1+(z/a)^2)^{3/2}} - \frac{a^2}{z^2}\right]$$

**for $|z| < a$. Show that the second form is well behaved at the origin, with the value, $E_r(0) = -3V/2a$. Show that at $z = a$ (north pole inside) it has the value $-(\sqrt{2}-1)V/a$. Show that the radial field at the north pole outside has the value $\sqrt{2}V/a$.**

---

For $z > a$, we want to start with the potential on the z-axis (2.20) and find the electric field using equation (1.22),

$$E_> = -\frac{\partial\Phi}{\partial z} = V\frac{d}{dz}\left(\frac{z^2-a^2}{(z^4+a^2z^2)^{1/2}}\right)$$

$$= \frac{Va^2}{(z^2+a^2)^{3/2}}\left(3+\frac{a^2}{z^2}\right)$$

For $z < a$, we want to use the potential derived in part (a).

$$E_< = -V\frac{d}{dz}\left(\frac{a}{z} - \frac{a^2-z^2}{(z^4+a^2z^2)}\right)$$

$$= -\frac{V}{a}\left[\frac{3+\left(\frac{a}{z}\right)^2}{\left(1+\left(\frac{z}{a}\right)^2\right)^{3/2}} - \frac{a^2}{z^2}\right]$$

At the origin, we want to use $E_<$ and evaluate at z=0,

$$E_<(0) = -\frac{V}{a}\left[\frac{3}{\left(1+\left(\frac{z}{a}\right)^2\right)^{3/2}} + \frac{\frac{a^2}{z^2}}{\left(1+\left(\frac{z}{a}\right)^2\right)^{3/2}} - \frac{a^2}{z^2}\right]$$

$$= -\frac{z}{a}\left[3\left(1+\left(\frac{z}{a}\right)^2\right)^{-3/2} + \frac{a^2}{z^2}\left(1+\left(\frac{z}{a}\right)^2\right)^{-3/2} - \frac{a^2}{z^2}\right]$$

Using the binomial approximation,

$$-\frac{V}{a}\left[3\left(1-\frac{3}{2}\left(\frac{z}{a}\right)^2\right)+\frac{a^2}{z^2}\left(1-\frac{3}{2}\left(\frac{z}{a}\right)^2\right)-\frac{a^2}{z^2}\right]$$

$$=-\frac{3V}{2a}$$

At the north pole, we want to evaluate the electric field at $z = a$. Just inside,

$$E_<(a) = -\frac{V}{a}\left[\frac{3+1}{(1+1)^{3/2}} - 1\right]$$

$$= -(\sqrt{2}-1)\frac{V}{a}$$

Just outside,

$$E_>(a) = \frac{Va^2}{(2a^2)^{3/2}}4$$

$$= \frac{\sqrt{2}V}{a}$$

## 2.23 Separation of Variables: Hollow Cube

**A hollow cube has conducting walls defined by six planes $x = 0$, $y = 0$, $z = 0$, and $z = a$, $y = a$, and $z = a$. The walls $z = 0$ and $z = a$ are held at a constant potential $V$. The other four sides are at zero potential.**

---

### 2.23.1 Potential Inside

**Find the potential $\Phi(x, y, z)$ at any point inside the cube.**

The general potential is given by equation (2.28),

$$\Phi = e^{\pm i\alpha x}\ e^{\pm i\beta y}\ e^{\pm\sqrt{\alpha^2+\beta^2}\ z}$$

Using the condition that the potential must vanish for $x = 0$ and $y = 0$,

$$\begin{cases} X(x) = sin(\alpha x) \\ \\ Y(y) = sin(\beta y) \\ \\ Z(z) = e^{\gamma z} + e^{-\gamma z} \end{cases}$$

Using the condition that the potential must vanish for $x = a$ and $y = a$, the solution must be periodic,

$$\begin{cases} \alpha_n = \dfrac{n\pi}{a} \\ \\ \beta_m = \dfrac{m\pi}{a} \\ \\ \gamma_{nm} = \dfrac{\pi}{a}\sqrt{n^2 + m^2} \end{cases}$$

At this point, our potential is given by

$$\Phi = \sum_{n,m} sin(\alpha x)\ sin(\beta y) \left[A_{mn}e^{\gamma z} + B_{mn}e^{-\gamma z}\right]$$

We'll first apply the condition $\Phi = V$ at $z = 0$.

$$V = \sum_{n,m} sin\left(\frac{n\pi x}{a}\right)\ sin\left(\frac{m\pi y}{a}\right)[A_{mn} + B_{mn}]$$

Multiplying both sides by $sin\left(\dfrac{n'\pi x}{a}\right)\ sin\left(\dfrac{m'\pi y}{a}\right)$ and integrating,

$$V\int_0^a\int_0^a sin\left(\frac{n\pi x}{a}\right)\ sin\left(\frac{n\pi y}{a}\right)\ dxdy = [A_{mn} + B_{mn}]\int_0^a\int_0^a sin^2\left(\frac{n\pi x}{a}\right)\ sin^2\left(\frac{m\pi y}{a}\right)\ dxdy$$

$$A_{mn} + B_{mn} = \frac{4V}{a^2} \int_0^a \int_0^a sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{a}\right)\ dxdy$$

$$= \frac{16V}{nm\pi^2}, \quad for\ n,\ m\ odd$$

We can do the same thing for the condition $\Phi = V$ at $z = a$,

$$A_{mn}\ e^{\gamma a} + B_{mn}\ e^{\gamma a} = \frac{16V}{nm\pi^2}$$

Solving this system of equations,

$$A_{mn} = \frac{16V}{nm\pi^2} \frac{e^{\gamma a} - 1}{e^{2\gamma a} - 1} = \frac{16V}{nm\pi^2} \frac{exp\left(\frac{\gamma a}{2}\right) sinh\left(\frac{\gamma a}{2}\right)}{e^{\gamma a}\ sinh(\gamma a)}$$

$$A_{mn} = \frac{8V}{nm\pi^2} \frac{exp\left(-\frac{\pi}{2}\sqrt{n^2+m^2}\right)}{cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}$$

$$B_{mn} = \frac{8V}{nm\pi^2} \frac{exp\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}{cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}$$

Substituting this back, out potential is now

$$\Phi = \sum_{n,\ m\ odd} = \frac{8V}{nm\pi^2} sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{a}\right) \frac{exp\left(-\frac{\pi}{2}\sqrt{n^2+m^2} + \gamma z\right) + exp\left(\frac{\pi}{2}\sqrt{n^2+m^2} - \gamma z\right)}{cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}$$

$$\Phi = \sum_{n,m\ odd} \frac{16V}{nm\pi^2} sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{a}\right) dfraccosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\left(\frac{2z}{a} - 1\right)\right) cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)$$

### 2.23.2 Computation

Skipping

### 2.23.3 Surface-Charge Density

**Find the surface-charge density on the surface $z = a$.**

To find the surface-charge density, we turn to equation (1.26),

$$\sigma = \epsilon_0 \frac{\partial \Phi}{\partial z}\bigg|_{z=a}$$

$$= \epsilon_0 \sum_{n,m\ odd} \frac{16V}{nm\pi^2} sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{a}\right) \frac{sinh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\left(\frac{2z}{a}-1\right)\right)}{cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)} \left(\frac{\pi}{a}\sqrt{n^2+m^2}\right)\Bigg|_{z=a}$$

$$\sigma = \frac{16\epsilon_0 V}{\pi a} \sum_{n,m\ odd} \frac{\sqrt{n^2+m^2}}{nm} sin\left(\frac{n\pi x}{a}\right) sin\left(\frac{m\pi y}{a}\right) tanh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)$$

## 2.24 Completeness Relation

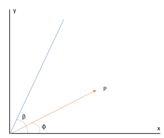

Figure 2.3:

**In the two-dimensional region shown in figure (2.3), the angular functions appropriate for Dirichlet boundary conditions at $\phi = 0$ and $\phi = \beta$ are $\Phi(\phi) = A_m sin(m\pi\phi/\beta)$. Show that the completeness relation for these functions is**

$$\delta(\phi - \phi') = \frac{2}{\beta}\sum_{m=1}^{\infty} sin(m\pi\phi/\beta) sin(m\pi\phi'/\beta) \quad for 0 < \phi,\ \phi' < \beta$$

---

We want our completeness relationship to be of the form

$$\delta(\phi - \phi') = N\sum_{m=1}^{\infty} sin\left(\frac{m\pi\phi}{\beta}\right) sin\left(\frac{m\pi\phi'}{\beta}\right)$$

To solve for $N$, we pick out the value $\phi = \phi'$, integrate and set equal to unity.

$$N\int_0^{\beta} sin^2\left(\frac{m\pi\phi}{\beta}\right)\ d\phi = 1$$

$$\frac{\beta}{2} - \frac{\beta}{4m\pi} sin(2m\pi) = 1$$

The second term drops out since $m$ takes integer values,

$$N = \frac{2}{\beta}$$

Now that we have our normalization constant, the completeness relation is given by

$$\delta(\phi - \phi') = \frac{2}{\beta}\sum_{m=1}^{\infty} sin\left(\frac{m\pi\phi}{\beta}\right) sin\left(\frac{m\pi\phi'}{\beta}\right)$$

## 2.25 Method of Images: Two Conducting Intersecting Planes

**Two conducting planes at zero potential meet along the z axis, making an angle $\beta$ between them, as in figure (2.3). A unit line charge parallel to the z axis is located between the planes at position $(\rho', \phi')$**

---

### 2.25.1 Green Function

**Show that $(4\pi\epsilon_0)$ times the potential in the space between the planes, that is, the Dirichlet Green function $G(\rho, \phi; \rho', \phi')$, is given by the infinite series**

$$G(\rho, \phi; \rho', \phi') = 4 \sum_{m=1}^{\infty} \frac{1}{m} \rho_<^{m\pi/\beta} \rho_>^{-m\pi/\beta} \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right)$$

---

We'll start with the general potential for this system (2.33),

$$\Phi = \sum_{m=1}^{\infty} \left(a_m \rho^{m\pi/\beta} + b_m \rho^{-m\pi/\beta}\right) \sin\left(\frac{m\pi\phi}{\beta}\right)$$

We break this up into two cases, one near the origin and other far away. The difference between these two states is a little arbitrary, but in the words of Justice Potter Stewart, "I know it when I see it."

$$\begin{cases} \Phi_< = \sum_{m=1}^{\infty} a_m \rho'^{m\pi/\beta} \sin\left(\frac{m\pi\phi}{\beta}\right) \\ \\ \Phi_> = \sum_{m=1}^{\infty} b_m \rho'^{-m\pi/\beta} \sin\left(\frac{m\pi\phi}{\beta}\right) \end{cases}$$

To solve for the constants, we first need to use the condition that the potential is continuous at the boundary (1.27), which we will set to the position of the line charge. That is, at $\rho' = \rho$, we expect $\Phi_> = \Phi_<$.

$$a_m \rho^{m\pi/\beta} = b_m \rho^{-m\pi/\beta}$$

We also use the condition that the electric field is discontinuous at the boundary (1.26),

$$\frac{b_m m\pi}{\beta} \rho^{-(m\pi/\beta)-1} + \frac{a_m m\pi}{\beta} \rho^{(m\pi/\beta)-1} = \frac{\delta(\phi - \phi')}{\epsilon_0 \rho \sin\left(\frac{m\pi\phi}{\beta}\right)}$$

From problem 2.24, we know that

$$\delta(\phi' - \phi) = \frac{2}{\beta} \sum_{m=1}^{\infty} \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right)$$

Solving for the constants,

$$\begin{cases} a_m = \dfrac{\rho'^{-m\pi/\beta}}{m\pi\epsilon_0} \sin\left(\dfrac{m\pi\phi'}{\beta}\right) \\ b_m = \dfrac{\rho'^{m\pi/\beta}}{m\pi\epsilon_0} \sin\left(\dfrac{m\pi\phi'}{\beta}\right) \end{cases}$$

Substituting them back,

$$\Phi_< = \sum_{m=1}^{\infty} \frac{1}{m\pi\epsilon_0} \rho^{m\pi/\beta} \rho'^{-m\pi/\beta} \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right)$$

$$\Phi_> = \sum_{m=1}^{\infty} \frac{1}{m\pi\epsilon_0} \rho'^{m\pi/\beta} \rho^{-m\pi/\beta} \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right)$$

$$\Phi = \sum_{m=1}^{\infty} \frac{\rho_<^{m\pi/\beta} \rho_>^{-m\pi/\beta}}{m\pi\epsilon_0} \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right)$$

Since $G = 4\pi\epsilon_0\Phi$,

$$G(\rho, \phi; \rho', \phi') = 4 \sum_{m=1}^{\infty} \frac{1}{m} \rho_<^{m\pi/\beta} \rho_>^{-m\pi/\beta} \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right)$$

### 2.25.2 Green Function Closed Form

**By means of complex-variable techniques or other means, show that the series can be summed to give a closed form,**

$$G(\rho, \phi; \rho', \phi') = \ln\left\{\frac{\rho^{2\pi/\beta} + \rho'^{2\pi/\beta} - 2(\rho\rho')^{\pi/\beta}\cos[\pi(\phi + \phi')/\beta]}{\rho^{2\pi/\beta} + \rho'^{2\pi/\beta} - 2(\rho\rho')^{\pi/\beta}\cos[\pi(\phi - \phi')/\beta]}\right\}$$

---

We want to make the substituion,

$$z = \rho_<^{m\pi/\beta} \rho_>^{-m\pi/\beta} \exp\left(\frac{im\phi}{\beta}\right)$$

We can write the Green function as

$$G = 4 \sum_{m=1}^{\infty} \frac{1}{m}\ Im(zz')$$

Using the identity,

$$2\ Im(zz') = Re(-zz' + z^* z'^*)$$

$$G = Re\left(-2\sum_{m=1}^{\infty}\frac{1}{m}z^m + 2\sum_{m=1}^{\infty}\frac{1}{m}z'^m\right)$$

$$= 2\ Re[\ln(1-z) - \ln(1-z')]$$

$$= \ln\left[\frac{\rho^{2\pi/\beta} + \rho'^{2\pi/\beta} - 2(\rho\rho')^{\pi/\beta}\cos\left(\dfrac{\pi(\phi+\phi')}{\beta}\right)}{\rho^{2\pi/\beta} + \rho'^{2\pi/\beta} - 2(\rho\rho')^{\pi/\beta}\cos\left(\dfrac{\pi(\phi-\phi')}{\beta}\right)}\right]$$

### 2.25.3 Expected Values

**Verify that you obtain the familiar results when $\beta = \pi$ and $\beta = \pi/2$.**

---

If $\beta = \pi$, we expect the normal method of images for a flat conductor.

$$G = \ln\left[\frac{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi+\phi')}{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi')}\right]$$

$$= \ln(|\vec{x} - \vec{x}''|) - \ln(|\vec{x} - \vec{x}'|)$$

If $\beta = \dfrac{\pi}{2}$,

$$G = \ln\left[\frac{\rho^4 + \rho'^4 - 2(\rho\rho')^2\cos[2(\phi+\phi')]}{\rho^4 + \rho'^4 - 2(\rho\rho')^2\cos[2(\phi-\phi')]}\right]$$

## 2.26 Separation of Variables: Intersecting Conductors

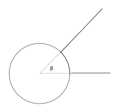

Figure 2.4:

**The two-dimensional region, $\rho \geq a$, $0 \leq \phi \leq \beta$, is bounded by conducting surfaces at $\phi = 0$, $\rho = a$, and $\phi = \beta$ held at zero potential, as indicated in figure (2.4). At large $\rho$ the potential is determined by some configuration of charges and/or conductors at fixed potential.**

### 2.26.1 Potential

**Write down a solution for the potential $\Phi(\rho, \phi)$ that satisfies the boundary conditions for finite $\rho$.**

We'll start with the potential for two intersecting planes (2.33),

$$\Phi(\rho, \phi) = \sum_{m=1}^{\infty} \left(A_m \rho^{m\pi/\beta} + B_m \rho^{-m\pi/\beta}\right) \sin\left(\frac{m\pi\phi}{\beta}\right)$$

Since we're looking at finite $\rho$, we can't delete any terms. Instead, let's look at the condition $\Phi(a, 0) = 0$,

$$A_m a^{m\pi/\beta} + B_m a^{-m\pi/\beta} = 0$$

$$B_m = -A_m a^{2m\pi/\beta}$$

$$\Phi(\rho, \phi) = \sum_{m=1}^{\infty} A_m \left(\rho^{m\pi/\beta} - \frac{a^{2m\pi/\beta}}{\rho^{m\pi/\beta}}\right) \sin\left(\frac{m\pi\phi}{\beta}\right)$$

For symmetry sake, we can rewrite this noting that the $A_m$ here is not the same $A_m$ used before since we can absorb a factor of $a^{2m\pi/\beta}$. Also note that we can't get any more specific than this without knowing if there are charges or potentials in this system.

$$\Phi(\rho, \phi) = \sum_{m=1}^{\infty} A_m \left[\left(\frac{\rho}{a}\right)^{m\pi/\beta} - \left(\frac{a}{\rho}\right)^{m\pi/\beta}\right] \sin\left(\frac{m\pi\phi}{\beta}\right)$$

### 2.26.2 Electric Field and Surface Charge Density

**Keeping only the lowest nonvanishing terms, calculate the electric field components $E_\rho$ and $E_\phi$ and also the surface-charge densities $\sigma(\rho, 0)$, $\sigma(\rho, \beta)$, and $\sigma(a, \phi)$ on the three boundary surfaces.**

The components of the electric field can be found using equation (1.22). The radial component,

$$E_\rho = -\frac{\partial \Phi}{\partial \rho} = -\sum_{m=1}^{\infty} \left[ \frac{m\pi}{\rho\beta} \left(\frac{\rho}{a}\right)^{m\pi/\beta} + \frac{m\pi}{\rho\beta} \left(\frac{a}{\rho}\right)^{m\pi/\beta} \right] \sin\left(\frac{m\pi\phi}{\beta}\right)$$

Keeping the lowest non-vanishing term, $m = 1$,

$$\approx -\frac{A_1\pi}{\rho\beta} \left[ \left(\frac{\rho}{a}\right)^{\pi/\beta} + \left(\frac{a}{\rho}\right)^{\pi/\beta} \right] \sin\left(\frac{\pi\phi}{\beta}\right)$$

Looking at the angular component,

$$E_\phi = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi} = -\sum_{m=1}^{\infty} \frac{A_m}{\rho} \left[ \left(\frac{\rho}{a}\right)^{m\pi/\beta} - \left(\frac{a}{\rho}\right)^{m\pi/\beta} \right] \frac{m\pi}{\beta} \cos\left(\frac{m\pi\phi}{\beta}\right)$$

Again, the lowest non-vanishing term is the $m = 1$ term,

$$\approx -\frac{A_1\pi}{\rho\beta} \left[ \left(\frac{\rho}{a}\right)^{\pi/\beta} - \left(\frac{a}{\rho}\right)^{\pi/\beta} \right] \cos\left(\frac{\pi\phi}{\beta}\right)$$

We can find the surface charge density (1.26) using the above electric fields,

$$\sigma(\rho, 0) = \epsilon_0 E_\phi(\rho, 0) = -\frac{A_1\pi\epsilon_0}{\rho\beta} \left[ \left(\frac{\rho}{a}\right)^{\pi/\beta} - \left(\frac{a}{\rho}\right)^{\pi/\beta} \right]$$

$$\sigma(\rho, \beta) = -\epsilon_0 E_\phi(\rho, \beta) = \frac{A_1\pi\epsilon_0}{\rho\beta} \left[ \left(\frac{\rho}{a}\right)^{\pi/\beta} - \left(\frac{a}{\rho}\right)^{\pi/\beta} \right]$$

$$\sigma(a, \phi) = \epsilon_0 E_\rho(a, \phi) = -\frac{2A_1\pi\epsilon_0}{a\beta} \sin\left(\frac{\pi\phi}{\beta}\right)$$

### 2.26.3 Plane Conductor with Half-Cylinder

**Consider $\beta = \pi$ (a plane conductor with a half-cylinder of radius $a$ on it). Show that far from the half-cylinder the lowest order terms of part b give a uniform electric field normal to the plane. For fixed electric field strength far from the plane, show that the total charge on the half-cylinder (actually charge per unit length in the z direction) is twice as large as would reside on a strip of width $2a$ in its absence. Show that the extra portion is drawn from regions of the plane nearby, so that the total charge on a strip of width large compared to $a$ is the same whether the half-cylinder is there or not.**

Using the above fields,

$$E_\rho = -\frac{A_1}{\rho}\left[\frac{\rho}{a} + \frac{a}{\rho}\right]\sin(\phi)$$

$$E_\phi = -\frac{A_1}{\rho}\left[\frac{\rho}{a} - \frac{a}{\rho}\right]\cos(\phi)$$

Rewriting and looking at the case $\rho >> a$,

$$E_\rho = -A_1\left[1 + \left(\frac{a}{\rho}\right)^2\right]\sin(\phi) = -A_1\sin(\phi)$$

$$E_\phi = -A_1\left[1 - \left(\frac{a}{\rho}\right)^2\right]\cos(\phi) = -A_1\cos(\phi)$$

In terms of Cartesian coordinates,

$$E_x = E_\rho\cos(\phi) - E_\phi\sin(\phi) = 0$$

$$E_y = E_\rho\sin(\phi) + E_\phi\cos(\phi) = -A_1$$

For $\rho >> a$, the electric field only consists of a y-component, which is normal to the plane.

$$Q = 2\sigma a\int_0^{pi}\sin(\phi)\ d\phi$$

$$= -4A_1\epsilon_0 a$$

A strip of width $2a$ will give a total charge of $-2A_1\epsilon_0 a$

$$Q = -\epsilon_0 A_1\int_a^L\left(1 - \frac{a^2}{\rho^2}\right)\ d\rho$$

$$= -\epsilon_0 A_1\left(L - a + \frac{a^2}{L} - L\right)$$

For large $L$,

$$\approx -\epsilon_0 A_1 a$$

## 2.27 Two-Dimensional Wedge

**Consider the two-dimensional wedge-shaped region of Problem 2.26, with $\beta = 2\pi$. This corresponds to a semi-infinite thin sheet of conductor on the positive x axis from x=a to infinity with a conducting cylinder of radius $a$ fastened to its edge.**

---

### 2.27.1 Surface-Charge Density

**What are the surface-charge densities on the cylinder and on the top and bottom of the sheet, using the lowest order solution.**

---

We'll use the electric field we determined in the previous problem, setting $\beta = \pi$,

$$E_\rho = -\frac{A_1}{2\rho}\left[\left(\frac{\rho}{a}\right)^{1/2} + \left(\frac{a}{\rho}\right)^{1/2}\right]\sin\left(\frac{\phi}{2}\right)$$

$$E_\phi = -\frac{A_1}{2\rho}\left[\left(\frac{\rho}{a}\right)^{1/2} - \left(\frac{a}{\rho}\right)^{1/2}\right]\cos\left(\frac{\phi}{2}\right)$$

The surface charge densities are thus given by

$$\sigma(a,\phi) = \epsilon_0 E_\rho = -\frac{A_1\epsilon_0}{a}\sin\left(\frac{\phi}{2}\right)$$

$$\sigma(\rho,0) = \epsilon_0 E_\phi = -\frac{A_1\epsilon_0}{2\rho}\left[\left(\frac{\rho}{a}\right)^{1/2} - \left(\frac{a}{\rho}\right)^{1/2}\right]$$

The bottom of the plane will be given by $-\sigma(\rho,0)$.

### 2.27.2 Total Charge

**Calculate the total charge on the cylinder and compare with the total deficiency of charge on the sheet near the cylinder, that is, the total difference in charge for $a$ finite compared with $a = 0$, assuming that the charge density far from the cylinder is the same.**

---

The total charge on the cylinder,

$$Q = -\frac{A_1\epsilon_0}{a}\int_0^{2\pi} a\sin\left(\frac{\pi}{2}\right)\, d\phi$$

$$= -4A_1\epsilon_0$$

The total charge on the sheet given large $L$,

$$Q = -\frac{A_1\epsilon_0}{2}2\pi\int_a^L \left(\frac{1}{a\rho^{1/2}} - \frac{a}{\rho^{3/2}}\right)\, d\rho$$

$$= -\frac{2A_1\epsilon_0\pi L^{1/2}}{a}$$

# Chapter 3

# Boundary-Value Problems in Electrostatics II

## 3.1 Separation of Variables: Concentric Spheres

**Two concentric spheres have radii $a$, $b$ ($b > a$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential $V$. The other hemispheres are at zero potential.**

**Determine the potential in the region $a \le r \le b$ as a series in Legendre polynomials. Include terms at least up to $l = 4$. Check your solution against known results in the limiting cases $b \to \infty$, and $a \to 0$.**

---

We can use the azimuthally symmetric solution here (3.19). Since we are looking at the region between the spheres, we can't get rid of any terms.

$$\Phi(r,\ \theta) = \sum_{l=0}^{\infty}[A_l r^l + B_l r^{-(l+1)}]\ P_l(\cos(\theta))$$

Let's start by looking at the $r = a$ boundary,

$$V(\theta) = \sum_{l=0}^{\infty}[A_l a^l + B_l a^{-(l+1)}]\ P_l(\cos(\theta))$$

To solve for the coefficients, we use the orthonormality of the Legendre polynomials (3.6). We multiply both sides by $P_{l'}(\cos(\theta))$ and integrate.

$$\int_0^{\pi} V(\theta)\ P_{l'}(\cos(\theta))\sin(\theta)\ d\theta = \int_0^{\pi}\sum_{l=0}^{\infty}[A_l a^l + B_l a^{-(l+1)}]\ P_l(\cos(\theta))\ P_{l'}(\cos(\theta))\ \sin(\theta)\ d\theta$$

Making the substitution $\cos(\theta) = x$,

$$\int_{-1}^{1} V(x)\ P_{l'}(x)\ dx = \sum_{l=0}^{\infty}[A_l a^l + B_l a^{-(l+1)}] \int_{-1}^{1} P_l(x) P_{l'}(x)\ dx$$

$$= \sum_{l=0}^{\infty}[A_l a^l + B_l a^{-(l+1)}] \frac{2}{2l+1}\delta_{ll'}$$

$$A_l a^l + B_l a^{-(l+1)} = \frac{2l+1}{2} V \int_0^1 P_l(x)\ dx$$

Note that we only integrate from 0 to 1 since only the top half of the hemisphere has a potential. Similarly, if we look at the boundary $r = b$,

$$A_l b^l + B_l b^{-(l+1)} = \frac{2l+1}{2} V(-1)^l \int_0^1 P_l(x)\ dx$$

Solving for the coefficients,

$$\begin{cases} A_l = \dfrac{a^{l+1} - (-1)^l b^{l+1}}{a^{2l+1} - b^{2l+1}} \left(\dfrac{2l+1}{2}\right) V \int_0^1 P_l(x)\ dx \\ \\ B_l = \dfrac{(-1)^l b^{l+1} a^{2l+1} - a^{l+1} b^{2l+1}}{a^{2l+1} - b^{2l+1}} \left(\dfrac{2l+1}{2}\right) V \int_0^1 P_l(x)\ dx \end{cases}$$

Plugging these back into our general potential,

$$\Phi(r,\phi) = \sum_{l=0}^{\infty} V \left(\frac{2l+1}{2}\right) \left(\frac{a^{l+1} - (-1)^l b^{l+1}}{a^{2l+1} - b^{2l+1}} r^l - \frac{a^{l+1} b^{2l+1} - (-1)^l a^{2l+1} b^{l+1}}{a^{2l+1} - b^{2l+1}} r^{-(l+1)}\right) P_l(\cos(\theta)) \int_0^1 P_l(x)\ dx$$

If we want to write out the explicit form, we can actually use a trick to solve the integral. The $l = 0$ case is fairly straightforward. For other $l$, we want to use one of the recursion relations for Legendre polynomials (3.7),

$$\int_0^1 P_l(x)\ dx = \frac{1}{2l+1} \int_0^1 \left(\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx}\right)\ dx$$

$$= \frac{1}{2l+1}[P_{l-1}(0) - P_{l+1}(0)]$$

We recognize that this is 0 when $l$ is even. Now, keeping up to the $l = 4$ term,

$$\Phi(r,\phi) = \frac{V}{2}\Big[1 + \frac{3}{2}\left(\frac{(a^2+b^2)r}{a^3-b^3} - \frac{(a^2b^3+a^3b^2)r^{-2}}{a^3-b^3}\right)\cos(\theta)$$
$$-\frac{7}{16}\left(\frac{(a^4+b^4)r^3}{a^7-b^7} - \frac{(a^4b^7+a^7b^4)r^{-4}}{a^7-b^7}\right)(5\cos^3(\theta) - 3\cos(\theta)) + ...\Big]$$

As $b \to \infty$,

$$\Phi \approx \frac{V}{2}\left[1 + \frac{3}{2}\left(\frac{a}{r}\right)^2 \cos(\theta) - \frac{7}{16}\left(\frac{a}{r}\right)^4 (5\cos^3(\theta) - 3\cos(\theta)) + ...\right]$$

If we let $a \to 0$,

$$V(\theta) = \sum_{l=0}^{\infty} B_l a^{-(l+1)} P_l(\cos(\theta))$$

$$\Phi \approx \frac{V}{2}$$

## 3.2 Sphere with a Cap

**A spherical surface of radius $R$ has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$.**

---

### 3.2.1 Potential

**Show that the potential inside the spherical surface can be expressed as**

$$\Phi = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}[P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))]\frac{r^l}{R^{l+1}}P_l(\cos(\theta))$$

**where, for $l = 0$, $P_{l-1}(\cos(\alpha)) = -1$. What is the potential outside?**

---

We have azimuthal symmetry, so we can use the general potential (3.19). We can suppress terms that explode both inside and outside,

$$\Phi_{in} = \sum_{l=0}^{\infty} A_l r^l \; P_l(\cos(\theta))$$

$$\Phi_{out} = A_0 + \sum_{l=0}^{\infty} B_l r^{-(l+1)} \; P_l(\cos(\theta))$$

Let's look at the potential as $r$ gets large. In this case, we can treat the sphere as a point particle, so the potential (1.23) does not depend on $\theta$.

$$\Phi_{out} = \frac{Q_{tot}}{4\pi\epsilon_0 r}$$

$$A_0 + \frac{B_0}{r} = \frac{1}{4\pi\epsilon_0 r}2\pi R^2\frac{Q}{4\pi R^2}\int_{\alpha}^{\pi}\sin(\theta)\; d\theta$$

$$= \frac{1}{4\pi\epsilon_0 r}\frac{Q}{2}(1+\cos(\alpha))$$

$$A_0 + \frac{B_0}{r} = \frac{Q(1+\cos(\alpha))}{8\pi\epsilon_0 r}$$

Matching solutions,

$$\begin{cases} A_0 = 0 \\ \\ B_0 = \dfrac{Q(1+cos(\alpha))}{8\pi\epsilon_0} \end{cases}$$

Using the condition that the potential must be continuous at the boundary (1.27),

$$A_0 + \sum_{l=1}^{\infty} A_l R^l \ P_l(\cos(\theta)) = \frac{Q(1+\cos(\alpha))}{8\pi\epsilon_0 R} + \sum_{l=1}^{\infty} B_l R^{-(l+1)} \ P_l(\cos(\theta))$$

$$\begin{cases} A_0 = \dfrac{Q(1+cos(\alpha))}{8\pi\epsilon_0 R} \\ \\ B_l = A_l R^{2l+1} \end{cases}$$

Using the condition that the electric field is discontinuous at the boundary (1.26),

$$\sigma = \epsilon_0 \left[ \frac{Q(1+\cos(\alpha))}{8\pi\epsilon_0 R^2} + \sum_{l=1}^{\infty} A_l \ P_l(\cos(\theta)) R^{l-1}(2l+1) \right]$$

Want to find $l \geq 1$, so we ignore the first term since that corresponds to $l = 0$. We use the orthonormality of Legendre polynomials (3.6). We multiply both sides by $P_{l'}(\cos(\alpha))$ and integrate,

$$\int_0^{\pi} \sigma \sin(\theta) \ P_{l'}(\cos(\theta)) \ d\theta = sum_{l=1}^{\infty} \epsilon_0 A_l R^{l-1}(2l+1) \int_0^{\pi} \sin(\theta) P_l(\cos(\theta)) \ P_{l'}(\cos(\theta)) \ d\theta$$

$$= \sum_{l=1}^{\infty} \epsilon_0 A_l R^{l-1} 2\delta_{ll'}$$

$$A_l = \frac{1}{2\pi\epsilon_0 R^{l-1}} \frac{Q}{4\pi R^2} \int_{-1}^{\cos(\alpha)} P_l(x) \ dx$$

Using the recurrence relations (3.7),

$$A_l = \frac{Q[P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))]}{8\pi\epsilon_0 R^{l+1}(2l+1)}$$

Thus, the general potentials are

$$\Phi_{in} = \frac{Q}{8\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{2l+1} [P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))] \frac{r^l}{R^{l+1}} P_l(\cos(\theta))$$

$$\Phi_{out} = \frac{Q}{8\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{2l+1} [P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))] \frac{R^l}{r^{l+1}} P_l(cos(\theta))$$

### 3.2.2 Electric Field

**Find the magnitude and the direction of the electric field at the origin.**

---

We use equation (1.22),

$$E_{in} = -\nabla\Phi_{in}$$

$$= -\frac{\partial\Phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}$$

$$= -\frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{l}{2l+1}[P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))]\frac{r^{l-1}}{R^{l+1}}P_l(\cos(\theta))\hat{r}$$
$$-\frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}[P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha))]\frac{r^{l-1}}{R^{l+1}}\frac{dP_l(\cos(\theta))}{d\theta}\hat{\theta}$$

If we're looking specifically at the origin, only the $l = 1$ term survives,

$$E_{in}(0) = -\frac{Q}{8\pi\epsilon_0}[P_2(\cos(\alpha)) - P_0(\cos(\alpha))]\left(\frac{\cos(\theta)}{3R^2}\hat{r} - \frac{\sin(\theta)}{3R^2}\hat{\theta}\right)$$

$$= -\frac{Q\sin^2(\alpha)}{16\pi\epsilon_0 R^2}(\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta})$$

### 3.2.3 Limiting Cases

**Discuss the limiting forms of the potential (part a) and electric field (part b) as the spherical cap becomes (1) very small, and (2) so large that the area with a charge on it becomes a very small cap at the south pole.**

---

As the cap becomes small, $\alpha \to 0$, $P_{l+1}(\cos(\alpha)) - P_{l-1}(\cos(\alpha)) = 0$. Only the $l = 0$ term survives,

$$\Phi_{in} = \frac{Q}{4\pi\epsilon_0 R}$$

$$\Phi_{out} = \frac{Q}{4\pi\epsilon_0 r}$$

Alternatively, if the cap becomes large, $\alpha \to \pi$, the potential goes to 0.

## 3.3 Flat, Conducting, Circular Disc

**A thin, flat, conducting, circular disc of radius $R$ is located in the x-y plane with its center at the origin, and is maintained at a fixed potential $V$. With the information that the charge density on a disc at fixed potential is proportional to $(R^2 - \rho^2)^{-1/2}$, where $\rho$ is the distance out from the center of the disc,**

---

### 3.3.1 Potential outside

**show that for $r > R$ the potential is**

$$\Phi(r, \theta, \phi) = \frac{2V}{\pi}\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l} P_{2l}(\cos(\theta))$$

---

We want to use the general solution to Poisson's equation using Dirichlet boundary conditions (1.45) and expand the solution using the conditions given in the problem. We can neglect the second term since we are looking at a point outside of the disc (where there is no fixed potential).

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G(\vec{x}, \vec{x}')\ d^3x'$$

The Green function for a sphere is simply the expansion of $1/R$ (3.29),

$$G(\vec{x}, \vec{x}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta', \phi')Y_{lm}(\theta, \phi)$$

Because we have azimuthal symmetry, we set $m = 0$, and this simplifies to

$$= \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos(\theta'))P_l(\cos(\theta))$$

For the charge density, we are told the surface charge density is proportional to $(R^2 - \rho^2)^{-1/2}$, so the charge density takes the form,

$$\rho = a\ \delta(\cos(\theta))\Theta(R - r)(R^2 - r^2)^{-1/2}\frac{1}{r}$$

Note that we have to add a factor of $1/r$ (see problem 1.3). We want to know what the constant $a$ should be such that this disk gives a potential $V$. Using equation (1.23),

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x})}{r}d^3x$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_{-1}^{1}\int_0^{\infty} a\ \delta(cos(\theta))\Theta(R - r)\frac{1}{\sqrt{R^2 - r^2}}\ drd(\cos(\theta))d\phi$$

$$V = \frac{a\pi}{4\epsilon_0}$$

The charge density is given by

$$\rho(\vec{x}) = \frac{4\epsilon_0 V}{\pi} \frac{\delta(\cos(\theta))\Theta(R-r)}{r\sqrt{R^2-r^2}}$$

Substituting this into the general potential,

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\int_0^{2\pi}\int_{-1}^{1}\int_0^{\infty}\frac{4\epsilon_0 V}{\pi}\frac{\delta(\cos(\theta'))\Theta(R-r')}{r'\sqrt{R^2-r'^2}}\frac{r'^{2l}}{r^{2l+1}}P_l(\cos(\theta'))P_l(\cos(\theta))r'^2\ dr'd(\cos(\theta'))d\phi'$$

$$= \frac{2V}{\pi}\sum_{l=0}^{\infty}\int_0^R \frac{1}{r^{l+1}}\frac{r'^{l+1}}{\sqrt{R^2-r'^2}}P_l(0)P_l(\cos(\theta))\ dr$$

We want to make the substitution $u = \sqrt{R^2-r'^2}$. We also note that only even $l$ terms are non-zero.

$$= \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{1}{r^{2l+1}}P_{2l}(\cos(\theta))\int_0^R P_{2l}(R^2-u^2)^l\ du$$

Doing this integral out explicitly for the first couple $l$ terms,

$$\Phi(r,\theta,\phi) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{1}{r^{2l+1}}\frac{(-1)^l}{2l+1}R^{2l+1}P_{2l}(\cos(\theta))$$

$$= \frac{2V}{\pi}\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l}P_{2l}(\cos(\theta))$$

### 3.3.2 Potential inside

**find the potential for $r < R$**

---

This seems a little more difficult than the previous part, and looking at solutions online, I don't think I completely follow/know which one to follow. I think I'll come back to this one later.

### 3.3.3 Capacitance

**What is the capacitance of the disc?**

---

We know the capacitance is given by $C = Q/V$, so we need to find the total charge on the disc.

$$Q = \int_0^{2\pi}\int_{-1}^{1}\int_0^{\infty}\frac{4\epsilon_0 V}{\pi}\frac{\delta(\cos(\theta))\Theta(R-r)}{r\sqrt{R^2-r^2}}r^2\ drd(\cos(\theta))d\phi$$

$$= 8\epsilon_0 V R$$

$$C = 8\epsilon_0 R$$

## 3.4 Orange Slices Potential

**The surface of a hollow conducting sphere of inner radius $a$ is divided into an *even number* of equal segments by a set of planes; their common line of intersection is the z axis and they are distributed uniformly in the angle $\phi$. (The segments are like the skin on wedges of an apple, or the earth's surface between successive meridians of longitude.) The segments are kept at fixed potentials $\pm V$, alternately.**

---

### 3.4.1 Potential Inside

**Set up a series representation for the potential inside the sphere for the general case of $2n$ segments, and carry the calculation of the coefficients in the series far enough to determine exactly which coefficients are different from zero. For the nonvanishing terms, exhibit the coefficients as an integral over $\cos(\theta)$.**

---

We start with the series solution (3.30). We want the solution inside, so we suppress $r^{-(l+1)}$ terms,

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} r^l Y_{lm}(\theta,\phi)$$

On the surface, the potential is given by

$$V(\phi) = \begin{cases} V, & \dfrac{2\pi i}{n} < \phi < \dfrac{\pi(2i+1)}{n}; \\ \\ -V, & \dfrac{\pi(2i+1)}{n} < \phi < \dfrac{\pi(2i+2)}{n} \end{cases}$$

At the boundary $r = a$,

$$V(\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} a^l Y_{lm}(\theta,\phi)$$

Multiplying both sides by $Y^*_{l'm'}(\theta,\phi)$ and integrating,

$$\int_0^{2\pi}\int_0^{\pi} V(\phi) Y_{l'm'} \sin(\theta)\ d\theta d\phi = \int_0^{2\pi}\int_0^{\pi} A_{lm} a^l Y_{lm} Y^*_{l'm'} \sin(\theta)\ d\theta d\phi$$

Using the orthonormality of the spherical harmonics (3.27) on the right side,

$$= A_{lm} a^l \delta_{ll'}\delta_{mm'}$$

Writing the spherical harmonics using equation (3.25),

$$A_{lm} = \frac{1}{a^l}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_0^{2\pi}\int_{-1}^{1} V(\phi) P_l^m(\cos(\theta)) e^{-im\phi}\ d(\cos(\theta)) d\phi$$

$$= \frac{V}{a^l}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\sum_{j=0}^{n-1}\int_{-1}^{1} P_l^m(\cos(\theta))\ d(\cos(\theta))\left[\int_{(2\pi j/n)}^{\pi(2j+1)/n} e^{-im\phi}\ d\phi - \int_{\pi(2j+1)/n}^{\pi(2j+2)/n} e^{-im\phi}\ d\phi\right]$$

To evaluate the last integral, we look at the $m = 0$ case,

$$\sum_{j=0}^{n-1}\left(\frac{(2j+1)\pi}{n} - \frac{2j\pi}{n} - \frac{(2j+2)\pi}{n} + \frac{(2j+1)\pi}{n}\right) = 0$$

In general,

$$\sum_{j=0}^{n-1}\left(\int_{2j\pi/n}^{\pi(2j+1)/n} e^{-im\phi}\ d\phi - \int_{(2j+1)\pi/n}^{(2j+2)\pi/n} e^{-im\phi}\ d\phi\right)$$

$$= \frac{i}{m}\sum_{j=0}^{n-1}\left[\exp\left(-\frac{im\pi(2j+1)}{n}\right) - \exp\left(-\frac{2im\pi j}{n}\right) - \exp\left(-\frac{im\pi(2j+2)}{n}\right) + \exp\left(-\frac{im\pi(2j+1)}{n}\right)\right]$$

$$= \frac{i}{m}\sum_{j=0}^{n-1}\exp\left(-\frac{2im\pi j}{n}\right)\left[2\exp\left(-\frac{im\pi}{n}\right) - 1 - \exp\left(-\frac{2im\pi}{n}\right)\right]$$

$$= \frac{i}{m}\sum_{j=0}^{n-1}\exp\left(-\frac{2im\pi j}{n}\right)\left[\exp\left(-\frac{im\pi}{n}\right) - 1\right]^2$$

This is 0 when $\frac{m}{2n}$ is an integer. Otherwise, we get a non-zero value.

### 3.4.2 Two Hemispheres

**For the special case of $n = 1$ (two hemispheres) determine explicitly the potential up to and including all terms with $l = 3$. By a coordinate transformation, verify that this reduces to**

$$\Phi(r,\theta) = V\left[\frac{3}{2}\frac{r}{a}P_1(\cos(\theta)) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos(\theta)) + \frac{11}{16}\left(\frac{r}{a}\right)^5 P_5(\cos(\theta))...\right]$$

---

Using the results from the previous section,

$$\Phi(r,\theta,\phi) = -V\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\frac{i}{m}\left[\exp\left(-\frac{im\pi}{n}\right) - 1\right]^2\sum_{j=0}^{n-1}\exp\left(-\frac{2imj\pi}{n}\right)\left(\frac{r}{a}\right)^l Y_{lm}(\theta,\phi)\int_{-1}^{1} P_l^m(\cos(\theta))\ d(\cos(\theta))$$

Setting $n = 1$, we see that only odd terms in $m$ survive.

$$= -\frac{4iV}{\sum}^{\infty}_{l=0}\sum_{m\ odd}^{l}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\left(\frac{r}{a}\right)^l Y_{lm}(\theta,\phi)\int_{-1}^{1} P_l^m(\cos(\theta))\ d(\cos(\theta))$$

$$= \sum_{l=0}^{\infty} \sum_{m\ odd}^{l} A_{lm} \left(\frac{r}{a}\right)^l Y_{lm}(\theta, \phi)$$

$$A_{lm} = -\frac{4iV}{m} \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \int_{-1}^{1} P_l^m(\cos(\theta))\ d(\cos(\theta))$$

Evaluating explicitly up to $l = 3$,

$$\Phi = \frac{3}{2} V \left(\frac{r}{a}\right) \sin(\theta)\sin(\phi) + \left(\frac{r}{a}\right)^3 V \left(\frac{35}{64} \sin^3(\theta)\sin^3(\phi) + \frac{21}{64} \sin(\theta)(5\cos^2(\theta) - 1)\sin(\phi)\right)$$

Making the substitution, $\cos(\theta') = \sin(\theta)\sin(\phi)$,

$$\Phi = \frac{3}{2} V \left(\frac{r}{a}\right) \cos(\theta') - \frac{7}{8} \left(\frac{r}{a}\right)^3 V \left[\frac{5}{2} \cos^3(\theta') - \frac{3}{2} \cos(\theta')\right]$$

## 3.5 Hollow Sphere Potential

**A hollow sphere of inner radius $a$ has the potential specified on its surface to be $\Phi = V(\theta, \phi)$. Prove the equivalence of the two forms of solution for the potential inside the sphere:**

---

### 3.5.1 Green Function

$$\Phi(\vec{x}) = \frac{a(a^2 - r^2)}{4\pi} \int \frac{V(\theta', \phi')}{(r^2 + a^2 - 2ar\cos(\gamma))^{3/2}} d\Omega'$$

**where** $\cos(\gamma) = \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi - \phi')$

---

To get to this solution, we need to use equation (1.45), which means we need the Green's function. Because we have a hollow sphere, we can just use the Green's function for a sphere (2.9). Since there is no charge distribution, our potential is given by

$$\Phi(\vec{x}) = -\frac{1}{4\pi} \oint_S \Phi(\vec{x}') \frac{\partial G}{\partial n'}\ da'$$

Our Green's function,

$$G(\vec{x}), \vec{x}') = \frac{1}{(x^2 + x'^2 - 2xx'\cos(\gamma))^{1/2}} - \frac{1}{\left(\dfrac{x^2 x'^2}{a^2} + a^2 - 2xx'\cos(\gamma)\right)^{1/2}}$$

$$-\frac{\partial G}{\partial x'} = \frac{2x' - 2x\cos(\gamma)}{2(x^2 + x'^2 - 2xx'\cos(\gamma))} - \frac{\dfrac{2x^2 x'}{a^2} - 2x\cos(\gamma)}{2\left(\dfrac{x^2 x'^2}{a^2} + a^2 - 2xx'\cos(\gamma)\right)^{3/2}}$$

Setting $x' = a$,

$$= \frac{a^2 - x^2}{a(x^2 + a^2 - 2ax\cos(\gamma))^{3/2}}$$

$$\Phi = \frac{1}{4\pi} \int \frac{V(\theta', \phi')\ (a^2 - r^2)}{a(r^2 + a^2 - 2ar\cos(\gamma))^{3/2}} a^2\ d\Omega'$$

$$= \frac{a(a^2 - r^2)}{4\pi} \int \frac{V(\theta', \phi')}{(r^2 + a^2 - 2ar\cos(\gamma))^{3/2}}\ d\Omega'$$

### 3.5.2 Spherical Harmonics Expansion

$$\Phi(\vec{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} \left(\frac{r}{a}\right)^l Y_{lm}(\theta, \phi)$$

**where** $A_{lm} = \int Y^*_{lm}(\theta', \phi') V(\theta', \phi')\ d\Omega'$

---

We start with equation (3.30). Since we are inside the sphere, we suppress $r^{-(l+1)}$ terms. At the boundary $r = a$,

$$V(\theta', \phi') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} a^l Y_{lm}(\theta, \phi)$$

Multiplying both sides by $Y_{l'm'}(\theta', \phi')$ and using the orthonormality of spherical harmonics (3.27),

$$\int V(\theta, \phi) Y_{l'm'}(\theta', \phi') \sin(\theta')\ d\Omega' = A_{lm} a^l \int \sin(\theta) Y_{lm}(\theta', \phi') Y^*_{l'm'}(\theta', \phi')\ d\Omega'$$

$$= A_{lm} a^l$$

$$A_{lm} = \frac{1}{a^l} \int Y^*_{lm}(\theta', \phi') V(\theta', \phi')\ d\Omega'$$

Our potential using this method,

$$\Phi(\vec{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} \left(\frac{r}{a}\right)^l Y_{lm}(\theta, \phi)$$

where we have a different $A_{lm}$ than before,

$$A_{lm} = \int Y^*_{lm}(\theta', \phi') V(\theta', \phi')\ d\Omega'$$

## 3.6 Spherical Harmonics Expansion

**Two point charges $q$ and $-q$ are located on the z-axis at $z = +a$ and $z = -a$, respectively.**

---

### 3.6.1 Potential

**Find the electrostatic potential as an expansion in spherical harmonics and powers of $r$ for both $r > a$ and $r < a$.**

---

Because we have two point charges, we can use equation (1.23) to write the potential,

$$\Phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{x} - a\hat{k}|} - \frac{1}{|\vec{x} + a\hat{k}|}\right)$$

The expansion of $1/R$ in spherical harmonics is given by (3.29),

$$\frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \phi')Y_{lm}(\theta, \phi)$$

Remember that primed variables refer to locations for the source. We say that the positive charge has $\phi' = 0$ and $\theta' = 0$ while the negative charge has $\phi' = 0$ and $\theta' = \pi$. Plugging these in,

$$\Phi = \frac{q}{\epsilon_0}\left[\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta, \phi)[Y_{lm}^*(0, 0) - Y_{lm}^*(\pi, 0)]\right]$$

If $r < a$, $r_< = r$ while $r_> = a$. If $r > a$, $r_< = a$ and $r_> = r$.

### 3.6.2 Dipole

**Keeping the product $qa = p/2$ constant, take the limit of $a \to 0$ and find the potential for $r \neq 0$. This is by definition a dipole along the z axis and its potential.**

---

As $a \to 0$, we will be in the $r > a$ region.

$$\Phi = \frac{q}{\epsilon_0}\left[\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{a^l}{r^{l+1}} Y_{lm}(\theta, \phi)[Y_{lm}^*(0, 0) - Y_{lm}^*(\pi, 0)]\right]$$

We have azimuthal symmetry, meaning we can set $m = 0$,

$$= \frac{q}{4\pi\epsilon_0}\left[\sum_{l=0}^{\infty} \frac{a^l}{r^{l+1}} P_l(\cos(\theta))[P_l(1) - P_l(-1)]\right]$$

By looking at the first couple terms in $l$, we see that only odd terms in $l$ survive,

$$= \frac{q}{4\pi\epsilon_0} \sum_{l\ odd} \frac{2a^l P_l(\cos(\theta))}{r^{l+1}}$$

$$= \frac{p}{4\pi\epsilon_0} \sum_{l\ odd} \frac{a^{l-1}}{r^{l+1}} P_l(\cos(\theta))$$

Keeping the leading term (l=1),

$$\Phi \approx \frac{p}{4\pi\epsilon_0 r^2} \cos(\theta)$$

### 3.6.3 Superposition

**Suppose now that the dipole of part b is surrounded by a *grounded* spherical shell of radius $b$ concentric with the origin. By linear superposition find the potential everywhere inside the shell.**

---

We have the potential found in the previous section in addition to the potential due to a grounded sphere. We can use the azimuthal expansion (3.19). Since we are looking for the potential inside the sphere, the potential just from the grounded sphere is

$$\Phi = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta))$$

At the boundary, $r = b$, the total potential is

$$0 = \frac{p\cos(\theta)}{4\pi\epsilon_0 b^2} + \sum_{l=0}^{\infty} A_l b^l P_l(\cos(\theta))$$

Looking at the above equation, we see that only the $l = 1$ term survives since we need to match terms, and the only way to get each term to disappear is if we only keep the l=1 term (so as to cancel with the potential due to the dipole).

$$0 = \frac{p\cos(\theta)}{4\pi\epsilon_0 b^2} + A_1 b\cos(\theta)$$

$$A_1 = -\frac{p}{4\pi\epsilon_0 b^3}$$

Thus, the total potential is

$$\Phi = \frac{p\cos(\theta)}{4\pi\epsilon_0 r^2} - \frac{pr\cos(\theta)}{4\pi\epsilon_0 b^3} = \frac{p\cos(\theta)}{4\pi\epsilon_0 b^2}\left[\frac{b^2}{r^2} - \frac{r}{b}\right]$$

## 3.7 Green Function: Spherical Expansion in Legendre Polynomials

**Three point charges $(q, -2q, q)$ are located in a straight line with separation $a$ and with the middle charge $(-2q$ at the origin of a grounded conducting spherical shell of radius $b$**

---

### 3.7.1 Potential without the Grounded Sphere

**Write down the potential of the three charges in the absence of the grounded sphere. Find the limiting form of the potential as $a \to 0$, but the product $qa^2 = Q$ remains finite. Write this latter answer in spherical coordinates.**

---

We have three point charges.

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{x} - a\hat{k}|} + \frac{1}{|\vec{x} + a\hat{k}|} - \frac{2}{|\vec{x}|}\right]$$

Since we have azimuthal symmetry, we can use the expansion (3.20),

$$= \frac{q}{4\pi\epsilon_0}\left[\sum_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}} + \frac{(-1)^l R_<^l}{r_>^{l+1}}\right) P_l(\cos(\theta)) - \frac{2}{r}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\sum_{l\ even} \frac{2r_<^l}{r_>^{l+1}} P_l(\cos(\theta)) - \frac{2}{r}\right]$$

As $a \to 0$,

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\sum_{l\ even} \frac{2a^l}{r^{l+1}} P_l(\cos(\theta)) - \frac{2}{r}\right]$$

$$= \frac{2Q}{4\pi\epsilon_0}\sum_{even}\frac{a^{l-2}}{r^{l+1}} P_l(\cos(\theta))$$

Keeping the lowest non-vanishing term,

$$= \frac{2Q}{4\pi\epsilon_0}\left[\frac{a^0}{r^3} P_2(\cos(\theta)) + ...\right] = \frac{Q(3\cos^2(\theta) - 1)}{4\pi\epsilon_0 r^3}$$

### 3.7.2 Potential with the Grounded Sphere

**The presence of the grounded sphere of radius $b$ alters the potential for $r < b$. The added potential can be viewed as caused by the surface-charge density induced on the inner surface at $r = b$ or by image charges located at $r > b$. Use linear superposition to satisfy the boundary conditions and find the potential everywhere inside the sphere for $r < a$ and $r > a$. Show that in the limit $a \to 0$,**

$$\Phi(r, \theta, \phi) \to \frac{Q}{2\pi\epsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right) P_2(\cos(\theta))$$

---

We start with the general potential in spherical with azimuthal symmetry (3.19). Since we are looking inside the sphere, we suppress $r^{-(l+1)}$ terms,

$$\Phi = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta))$$

Looking at the boundary $r = b$,

$$0 \frac{q}{4\pi\epsilon_0}\left[\sum_{l\ even} \frac{2a^l}{b^{l+1}} P_l(\cos(\theta)) - \frac{2}{r}\right] + A_l b^l P_l(\cos(\theta))$$

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\sum_{l\ even} \frac{2r_<^l}{r_>^{l+1}} P_l(\cos(\theta)) - \frac{2}{r}\right] - \sum_{l\ even} \frac{qa^l r^l P_l(\cos(\theta))}{2\pi\epsilon_0 b^{2l+1}}$$

As $a \to 0$, only the $l = 2$ term survives,

$$\Phi = \frac{Q(3\cos^2(\theta) - 1)}{4\pi\epsilon_0 r^3} - \frac{2qa^2r^2\frac{1}{2}(3\cos^2(\theta) - 1)}{4\pi\epsilon_0 b^5}$$

$$= \frac{Q(3\cos^2(\theta) - 1)}{4\pi\epsilon_0 r^3}\left[1 = \frac{r^5}{b^5}\right]$$

$$= \frac{Q}{2\pi\epsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right) P_2(\cos(\theta))$$

## 3.8 Grounded Sphere with a Uniformly Charged Wire

**There is a puzzling aspect of the solution**

$$\Phi(\vec{x}) = \frac{Q}{4\pi\epsilon_0 b}\left\{\ln\left(\frac{b}{r}\right) + \sum_{j=1}^{\infty}\frac{4j+1}{2j(2j+1)}\left[1-\left(\frac{r}{b}\right)^{2j}\right]P_{2j}(\cos(\theta))\right\}$$

**for the potential inside a grounded sphere with a uniformly charged wire along a diameter. Very close to the wire (i.e., for $\rho = r\sin(\theta) << b$), the potential should be that of a uniformly charged wire, namely, $\Phi = (Q/4\pi\epsilon_0 b)\ln(b/\rho) + \Phi_0$. The solution does not explicitly have this behaviour.**

---

### 3.8.1 Solution Expansion

**Show by use of the Legendre differential equation (3.2) and some integration by parts, that $\ln(\csc(\theta))$ has the appropriate expansion in spherical harmonics to permit the solution to be written in the alternative form,**

$$\Phi(\vec{x}) = \frac{Q}{4\pi\epsilon_0 b}\left\{\ln\left(\frac{2b}{r\sin(\theta)}\right) - 1 - \sum_{j=1}^{\infty}\frac{4j+1}{2j(2j+1)}\left(\frac{r}{b}\right)^{2j}P_{2j}(\cos(\theta))\right\}$$

**in which the expected behaviour near the wire is manifest. Give an interpretation of the constant term $\Phi_0 = -Q/4\pi\epsilon_0 b$. Note that in this form, for any $r/b < 1$ the Legendre polynomial series is rapidly convergent at all angles.**

---

We start with $\ln(\csc(\theta))$,

$$\ln(\csc(\theta)) = \ln\left(\frac{1}{\sin(\theta)}\right) = -\frac{1}{2}\ln(1-\cos^2(\theta))$$

We then want to expand this in terms of the Legendre polynomials. We do this in much the same way as a Fourier series,

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x)$$

Multiplying both sides by $P_{l'}(x)$ and using the orthonormality of the Legendre polynomials,

$$A_l = \frac{2l+1}{2}\int_{-1}^{1} f(x)P_l(x)\ dx$$

Looking at the $l=0$ term and setting $x = \cos(\theta)$

$$A_0 = \frac{1}{2}\int_{-1}^{1} -\frac{1}{2}\ln(1-x^2)P_0(x)\ dx$$

$$= 1 - \ln(2)$$

For general $l$, we can rearrange equation (3.2)

$$P = -\frac{1}{l(l+1)}\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]$$

$$A_l = \frac{2l+1}{4l(l+1)}\int_{-1}^{1}\ln(1-x^2)\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]\, dx$$

Integrating by parts with $u = \ln(1-x^2)$ and $dv = \frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]$,

$$= \frac{2l+1}{4l(l+1)}\left[\ln(1-x^2)(1-x^2)\frac{dP}{dx}\Big|_{-1}^{1} + \int_{-1}^{1}\frac{2x(1-x^2)}{1-x^2}\frac{dP}{dx}dx\right]$$

$$= \frac{2l+1}{2l(l+1)}\int_{-1}^{1}x\frac{dP}{dx}dx$$

Integrating again by parts with $u = x$ and $dv = \frac{dP}{dx}$,

$$= \frac{2l+1}{2l(l+1)}\left[xP_l(x)\Big|_{-1}^{1} - \int_{-1}^{1}P_l(x)\, dx\right]$$

$$= \begin{cases} \frac{2l+1}{l(l+1)}, & l\ even \\ 0, & l\ odd \end{cases}$$

$$\ln(\csc(\theta)) = 1 - \ln(2) + \sum_{j=1}^{\infty}\frac{4j+1}{2j(2j+1)}P_{2j}(\cos(\theta))$$

Substituting into the potential given in the problem,

$$\Phi(\vec{x}) = \frac{Q}{4\pi\epsilon_0 b}\left[\ln\left(\frac{b}{r}\right) + \ln\left(\frac{1}{\cos(\theta)}\right) - 1 + \ln(2) - \sum_{j=1}^{\infty}\frac{4j+1}{2j(2j+1)}\left(\frac{r}{b}\right)^{2j}P_{2j}(\cos(\theta))\right]$$

$$\Phi(\vec{x}) = \frac{Q}{4\pi\epsilon_0 b}\left[\ln\left(\frac{2b}{r\sin(\theta)}\right) - 1 - \sum_{j=1}^{\infty}\frac{4j+1}{2j(2j+1)}\left(\frac{r}{b}\right)^{2j}P_{2j}(\cos(\theta))\right]$$

### 3.8.2 Surface-charge Density

**Show by use of the expansion (3.20) that**

$$\frac{1}{2}\left(\frac{1}{\sin(\theta/2)}+\frac{1}{\cos(\theta/2)}\right)=2\sum_{j=0}^{\infty}P_{2j}(\cos(\theta))$$

**and that therefore the charge density on the inner surface of the sphere,**

$$\sigma(\theta)=-\frac{Q}{4\pi b^2}\left[1+\sum_{j=1}^{\infty}\frac{4j+1}{2j+1}P_{2j}(\cos(\theta))\right]$$

**can be expressed alternatively as**

$$\sigma(\theta)=-\frac{Q}{4\pi b^2}\left\{\frac{1}{2}\left(\frac{1}{\sin(\theta/2)}+\frac{1}{\cos(\theta/2)}\right)-\sum_{j=0}^{\infty}\frac{1}{2j+1}P_{2j}(\cos(\theta))\right\}$$

**The (integrable) singular behaviour at $\theta=0$ and $\theta=\pi$ is now exhibited explicitly. The series provides corrections in $\ln(1/\theta)$ as $\theta\to 0$.**

---

Equation (3.20) in spherical coordinates,

$$\frac{1}{\sqrt{r^2+r'^2-2rr'\cos(\theta)}}=\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos(\gamma))$$

Looking at what we want to show, this indicates that we want to set $r=r'$, which means $\phi=\phi'$, and this simplifies to

$$\frac{1}{\sqrt{2-2\cos(\theta)}}=\sum_{l=0}^{\infty}P_l(\cos(\theta))$$

$$\frac{1}{2}\sqrt{\frac{2}{1+\cos(\theta)}}=\sum_{l=0}^{\infty}P_l(\cos(\theta))(-1)^l$$

Similarly,

$$\frac{1}{2}\sqrt{\frac{2}{1-\cos(\theta)}}=\sum_{l=0}^{\infty}P_l(\cos(\theta))$$

Using half-angle formulas,

$$\sin\left(\frac{\theta}{2}\right)=\sqrt{\frac{1-\cos(\theta)}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos(\theta)}{2}}$$

$$\frac{1}{\sin\left(\frac{\theta}{2}\right)} + \frac{1}{\cos\left(\frac{\theta}{2}\right)} = \sqrt{\frac{2}{1-\cos(\theta)}} + \sqrt{\frac{2}{1+\cos(\theta)}}$$

$$= 2\sum_{l=0}^{\infty} P_l(\cos(\theta)) + (-1)^l P_l(\cos(\theta))$$

Only even terms in l survive,

$$\frac{1}{2}\left(\frac{1}{\sin(\theta/2)} + \frac{1}{\cos(\theta/2)}\right) = 2\sum_{j=0}^{\infty} P_{2j}(\cos(\theta))$$

Things are a little easier to see if we start with Jackson's solution,

$$\sigma = -\frac{Q}{4\pi b^2}\left[\frac{1}{2}\left(\frac{1}{\sin(\theta/2)} + \frac{1}{\cos(\theta/2)}\right) - \sum_{j=0}^{\infty}\frac{1}{2j+1}P_{2j}(\cos(\theta))\right]$$

$$= -\frac{Q}{4\pi b^2}\sum_{j=0}^{\infty}\left[2P_{2j}(\cos(\theta)) - \frac{1}{2j+1}P_{2j}(\cos(\theta))\right]$$

$$= -\frac{Q}{4\pi b^2}\sum_{j=0}^{\infty}\frac{4j+1}{2j+1}P_{2j}(\cos(\theta))$$

$$= -\frac{Q}{4\pi b^2}\left[1 + \sum_{j=1}^{\infty}\frac{4j+1}{2j+1}P_{2j}(\cos(\theta))\right]$$

## 3.9 Separation of Variables: Cylindrical Coordinates

**A hollow right circular cylinder of radius $b$ has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end faces is zero, while the potential on the cylindrical surface is given as $V(\phi, z)$. Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential anywhere inside the cylinder.**

---

We start with the general potential in cylindrical coordinates (3.49). We want to break up the $\phi$ component into sin and cos and the $z$ component as sinh.

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh(k_{mn}z)(A_{mn} \sin(m\phi) + B_{mn} \cos(m\phi))$$

Let's look at the boundary $z = L$,

$$0 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh(k_{mn}L)(A_{mn} \sin(m\phi) + B_{mn} \cos(m\phi))$$

We can solve for $k_{mn}$ by setting the sinh term equal to zero. By writing $\sinh(x) = [\exp(x) - \exp(-x)]/2$ and assuming that $k_{mn}$ is imaginary, we find

$$k_{mn} = \frac{in\pi}{L}$$

Using the modified Bessel function (3.41), we can write

$$\Phi = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} i^{m+1} I_m \left(\frac{n\pi\rho}{L}\right) \sin\left(\frac{n\pi z}{L}\right) [A_{mn} \sin(m\phi) + B_{mn} \cos(m\phi)]$$

Let's look at the boundary $\rho = b$,

$$V(\phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} i^{m+1} I_m \left(\frac{n\pi b}{L}\right) \sin\left(\frac{n\pi z}{L}\right) [A_{mn} \sin(m\phi) + B_{mn} \cos(m\phi)]$$

To solve for $A_{mn}$, we multiply both sides by $\sin(n'\pi z'/L)$ and $\sin(m'\phi')$ and integrate. Using the orthonormality of sin,

$$A_{mn} = \frac{2}{i^{m+1} L\pi} I_m^{-1} \left(\frac{n\pi b}{L}\right) \int_0^L \int_0^{2\pi} V(\phi', z') \sin(m\phi') \sin\left(\frac{n\pi z'}{L}\right) \, d\phi' dz'$$

We can do the same thing for $B_{mn}$,

$$A_{mn} = \frac{2}{i^{m+1} L\pi} I_m^{-1} \left(\frac{n\pi b}{L}\right) \int_0^L \int_0^{2\pi} V(\phi', z') \cos(m\phi') \sin\left(\frac{n\pi z'}{L}\right) \, d\phi' dz'$$

Substituting these back into our potential,

$$\Phi(\rho, \phi, z) = \frac{2}{L\pi} \sum_{m=0}^{\infty} \frac{I_m\left(\frac{n\pi\rho}{L}\right)}{I_m\left(\frac{n\pi b}{L}\right)} \sin\left(\frac{n\pi z}{L}\right) \left[A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)\right]$$

where here,

$$A_{mn} = \int_0^L \int_0^{2\pi} V(\phi', z') \sin(m\phi') \sin\left(\frac{n\pi z'}{L}\right)\, d\phi' dz'$$

$$B_{mn} = \int_0^L \int_0^{2\pi} V(\phi', z') \cos(m\phi') \sin\left(\frac{n\pi z'}{L}\right)\, d\phi' dz'$$

## 3.10 Separation of Variables: Cylinder

**For the cylinder in Problem 3.9 the cylindrical surface is made of two equal half-cylinders, one at potential $V$ and the other at potential $-V$, so that**

$$V(\phi, z) = \begin{cases} V \quad for \ -\pi/2 < \phi < \pi/2 \\ \\ -V \quad for \ \pi/2 < \phi < 3\pi/2 \end{cases}$$

---

### 3.10.1 Potential

**Find the potential inside the cylinder**

---

We can start with the solution we found in the previous problem.

$$A_{mn} = \int_0^L \int_{-\pi/2}^{\pi/2} V \sin(m\phi') \sin\left(\frac{n\pi z'}{L}\right) \, d\phi' dz' - \int_0^L \int_{\pi/2}^{3\pi/2} V \sin(m\phi') \sin\left(\frac{n\pi z'}{L}\right) \, d\phi' dz' = 0$$

$$B_{mn} = \int_0^L \int_{-\pi/2}^{\pi/2} V \cos(m\phi') \sin\left(\frac{n\pi z'}{L}\right) \, d\phi' dz' - \int_0^L \int_{\pi/2}^{3\pi/2} V \cos(m\phi') \sin\left(\frac{n\pi z'}{L}\right) \, d\phi' dz'$$

$$= \frac{8LV}{nm\pi}(-1)^{(m-1)/2} \ for \ n \ and \ m \ odd$$

The complete potential is

$$\Phi = \sum_{m \ odd} \sum_{n \ odd} \frac{(-1)^{(m-1)/2} 16V}{nm\pi^2} \sin\left(\frac{n\pi z}{L}\right) \frac{I_m\left(\frac{n\pi\rho}{L}\right)}{I_m\left(\frac{n\pi b}{L}\right)} \cos(m\phi)$$

### 3.10.2 Approximation

**Assuming $L >> b$, consider the potential at $z = L/2$ as a function of $\rho$ and $\phi$ and compare it with two-dimensional Problem 2.13**

---

At $z = L/2$,

$$\Phi = \sum_{m \ odd} \sum_{n \ odd} \frac{16V}{nm\pi^2} \frac{\rho^m}{b^m} (-1)^{(m-1)/2} \cos(m\phi)(-1)^{(n-1)/2}$$

The for $x << 1$,

$$I_\nu(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu$$

$$\Phi \to \frac{2V}{\pi}\tan^{-1}\left(\frac{2b\rho\cos(\phi)}{b^2-\rho^2}\right)$$

## 3.11 Bessel Functions

**A modified Bessel-Fourier series on the interval $0 \leq \rho \leq a$ for an arbitrary function $f(\rho)$ can be based on the "homogeneous" boundary conditions:**

$$At\ \rho = 0, \quad \rho J_\nu(k\rho)\frac{dJ_\nu(k'\rho)}{d\rho} = 0$$

$$At\ \rho = a, \quad \frac{d}{d\rho}\ln[J_\nu(k\rho)] = -\frac{\lambda}{a}\ (\lambda\ real)$$

**The first condition restricts $\nu$. The second condition yields eigenvalues $k = y_{\nu n}/a$, where $y_{\nu n}$ is the nth positive root of $x\frac{dJ_\nu(x)}{dx} + \lambda J_\nu(x) = 0$.**

---

### 3.11.1 Orthogonality

**Show that the Bessel functions of different eigenvalues are orthogonal in the usual way.**

---

I believe we prove orthogonality in much the same way we proved orthogonality of the Bessel functions (3.39). We start with the differential equation (3.31),

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dJ_\nu(k\rho)}{d\rho}\right] + \left(k^2 - \frac{\nu^2}{\rho^2}\right)J_\nu(k\rho) = 0$$

We multiply both sides by $\rho J_\nu(k'\rho)$ and integrate from 0 to a.

$$\int_0^a \frac{d}{d\rho}\left[\rho\frac{dJ_\nu(k\rho)}{d\rho}\right]J_\nu(k'\rho)\ d\rho + \int_0^a \rho J_\nu(k'\rho)J_\nu(k\rho)\left(k^2 - \frac{\nu^2}{\rho^2}\right)\ d\rho = 0$$

Looking at the first term, we can integrate by parts with $u = J_\nu(k'\rho)$ and $dv = \frac{d}{d\rho}\left[\rho\frac{dJ_\nu(k\rho)}{d\rho}\right]$. That first term then becomes,

$$\rho J_\nu(k'\rho)\frac{dJ_\nu(k\rho)}{d\rho}\bigg|_0^a - \int_0^a \rho\frac{dJ_\nu(k'\rho)}{d\rho}\frac{dJ_\nu(k\rho)}{d\rho}\ d\rho$$

Again, looking at the first term. We evaluate at $\rho = 0$ and we can use the first boundary condition given by Jackson to kill it. The boundary condition $\rho = a$,

$$\frac{d}{d\rho}\ln[J_\nu(k\rho)] = -\frac{\lambda}{a}$$

$$\frac{1}{J_\nu(ka)}\frac{dJ_\nu(k\rho)}{d\rho}\bigg|_{\rho=a} = -\frac{\rho}{a}$$

$$\left.\frac{dJ_\nu(ka)}{d\rho}\right|_{\rho=a} = -\frac{\lambda J_\nu(ka)}{a}$$

The first term of our integration of parts then becomes

$$-\lambda a J_\nu(k'a)J_\nu(ka)$$

We are left with

$$-\lambda a J_\nu(k'a)J_\nu(ka) - \int_0^a \rho \frac{dJ_\nu(k'\rho)}{d\rho}\frac{dJ_\nu(k\rho)}{d\rho}\ d\rho + \int_0^a \rho J_\nu(k'\rho)J_\nu(k\rho)\left(k^2 - \frac{\nu^2}{\rho^2}\right)\ d\rho = 0$$

We can convince ourselves that if we do the same thing with $k$ and $k'$ reversed, and then subtract the two from each other, we would be left with,

$$(k^2 - k'^2)\int_0^a \rho J_\nu(k'\rho)J_\nu(k\rho)\ d\rho = 0$$

Which shows orthogonality.

### 3.11.2 Normalization

**Find the normalization integral and show that an arbitrary function $f(\rho)$ can be expanded on the interval in the modified Bessel-Fourier series**

$$f(\rho) = \sum_{n=1}^{\infty} A_n J_\nu\left(\frac{y_{\nu n}\rho}{a}\right)$$

**with the coefficients $A_n$ given by**

$$A_n = \frac{2}{a^2}\left[\left(1 - \frac{\nu^2}{y_{\nu n}^2}\right)J_\nu^2(y_{\nu n}) + \left(\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}}\right)^2\right]^{-1}\int_0^a f(\rho)\rho J_\nu\left(\frac{y_{\nu n}\rho}{a}\right)\ d\rho$$

**The dependence on $\lambda$ is implicit in this form, but the square bracket has alternative forms:**

$$\left[\left(1 - \frac{\nu^2}{y_{\nu n}^2}\right)J_\nu^2(y_{\nu n}) + \left(\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}}\right)^2\right] = \left(1 + \frac{\lambda^2 - \nu^2}{y_{\nu n}^2}\right)J_\nu^2(y_{\nu n})$$

$$= \left(1 + \frac{y_{\nu n}^2 - \nu^2}{\lambda^2}\right)\left[\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}}\right]^2$$

$$= [J_\nu^2(y_{\nu n}) - J_{\nu-1}(y_{\nu n})J_{\nu+1}(y_{\nu n})]$$

**For $\lambda \to \infty$ we recover the result of**

$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)$$

$$A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} \int_0^a \rho f(\rho) J_\nu \left( \frac{x_{\nu n}\rho}{a} \right) \, d\rho$$

---

We start with the differential equation (3.31), multiply by $\rho^2 \dfrac{dJ_\nu(k\rho)}{d\rho}$ and integrate from 0 to a.

$$\int_0^a \rho \frac{dJ_\nu(k\rho)}{d\rho} \frac{d}{d\rho} \left[ \rho \frac{dJ_\nu(k\rho)}{d\rho} \right] \, d\rho + k^2 \int_0^a \rho^2 J_\nu(k\rho) \frac{dJ_\nu(k\rho)}{d\rho} \, d\rho - \nu^2 \int_0^a J_\nu(k\rho) \frac{dJ_\nu(k\rho)}{d\rho} \, d\rho = 0$$

The first and third terms can be integrated using $\int f(x)f'(x)\,dx = f^2(x)/2$ returning $\dfrac{1}{2}\left(\rho \dfrac{dJ_\nu}{d\rho}\right)^2 \Bigg|_0^a$ and $\nu^2 J_\nu^2/2$ respectively.

We use integration by parts on the second term with $u = \rho^2 J_\nu$ and $dv = \dfrac{dJ_\nu}{d\rho}$,

$$\int_0^a \rho^2 J_\nu J_\nu' \, d\rho = \rho^2 J_\nu J_\nu \Big|_0^a - \int_0^a 2\rho J_\nu J_\nu \, d\rho - \int_0^a \rho^2 J_\nu J_\nu' \, d\rho$$

From this, we see that

$$\int_0^a \rho^2 J_\nu J_\nu' \, d\rho = \frac{1}{2} a^2 J_\nu^2 - \int_0^a \rho J_\nu J_\nu \, d\rho$$

Combining all of these terms and making the substitution $k = y_{\nu n}/a$,

$$\int_0^a \rho J_\nu^2(y_{\nu n}) \, d\rho = \frac{a^2}{2} \left[ \left(1 - \frac{\nu^2}{y_{\nu n}^2}\right) J_\nu^2(y_{\nu n}) + \left( \frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}} \right)^2 \right]$$

Which for now, we term $N$.

To show that we can expand a function $f(\rho)$ using the Bessel functions, we start with

$$f(\rho) = \sum_{n=1}^{\infty} A_n J_\nu \left( \frac{y_{\nu n}\rho}{a} \right)$$

We multiply both sides by $\rho J_{k'\rho}$ and integrate. Using the orthonormality condition we just found,

$$\int_0^a f(\rho) \rho J_\nu \left( \frac{y_{\nu n}\rho}{a} \right) \, d\rho = A_n N$$

$$A_n = \frac{2}{a^2} \left[ \left(1 - \frac{\nu^2}{y_{\nu n}^2}\right) J_\nu^2(y_{\nu n}) + \left( \frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}} \right)^2 \right]^{-1} \int_0^a f(\rho) \rho J_\nu \left( \frac{y_{\nu n}\rho}{a} \right) \, d\rho$$

## 3.12 3.12

## 3.13 Green's Function Potential

**Solve for the potential in problem 3.1, using the appropriate Green function obtained in the text, and verify that the answer obtained in this way agrees with the direct solution from the differential equation.**

---

From the text, the Green function is given by

$$G(\vec{x},\vec{x}') = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$

Using equation (1.45) to solve for the potential, and since we are in charge-free space,

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G}{\partial n'}da'$$

We'll start by looking at the $r' = a$ boundary where $r_< = r'$ and $r_> = r$,

$$-\frac{\partial G}{\partial r'} = -4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left(lr'^{l-1} + (l+1)\frac{a^{2l+1}}{r'^{l+2}}\right)\left(\frac{1}{r^{l-1}} - \frac{r^l}{b^{2l+1}}\right)$$

Setting $r' = a$,

$$= -\frac{4\pi}{a^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left[\left(\frac{a}{r}\right)^{l+1} - \frac{a^{l+1}r^l}{b^{2l+1}}\right]$$

Similarly, for $r' = b$,

$$\frac{\partial G}{\partial r'} = -\frac{4\pi}{b^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left[\left(\frac{r}{b}\right)^{l} - \frac{a^{2l+1}}{b^l r^{l+1}}\right]$$

Since we have azimuthal symmetry, we set $m = 0$,

$$\Phi = \sum_{l=0}^{\infty}\left[\int_0^1 \frac{VP_l(\cos(\theta'))P_l(\cos(\theta))}{1-\left(\frac{a}{b}\right)^{2l+1}}\frac{2l+1}{4\pi}\left[\left(\frac{a}{r}\right)^{l+1} - \left(\frac{a}{b}\right)^{l+1}\left(\frac{r}{b}\right)^l\right]d(\cos(\theta'))\right.$$

$$\left. + \int_{-1}^0 \frac{VP_l(\cos(\theta'))P_l(\cos(\theta))}{1-\left(\frac{a}{b}\right)^{2l+1}}\frac{2l+1}{4\pi}\left[\left(\frac{r}{b}\right)^{l} - \left(\frac{a}{b}\right)^{l}\left(\frac{a}{r}\right)^{l+1}\right]d(\cos(\theta'))\right]$$

## 3.14 Line Charge with Varying Charge Density

**A line charge of length $2d$ with a total charge $Q$ has a linear charge density varying as $(d^2 - z^2)$, where $z$ is the distance from the midpoint. A grounded, conducting, spherical shell of inner radius $b > d$ is centered at the midpoint of the line charge.**

---

### 3.14.1 Potential Inside

**Find the potential everywhere inside the spherical shell as an expansion in Legendre polynomials.**

---

This problem is similar to an example worked out by Jackson, so we'll follow his example. We use equation (1.45) to find the potential. We only keep the first term since the potential vanishes at the boundaries (infinity and the shell)

$$\Phi = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G_D(\vec{x},\vec{x}')\ d^3x'$$

The general form of the potential is

$$\rho(\vec{x}) = a\frac{d^2 - r^2}{r^2}[\delta(\cos(\theta) - 1) + \delta(\cos(\theta) + 1)]$$

$$Q = \int_0^{2\pi}\int_{-1}^{1}\int_0^d \rho(\vec{x})r^2\ dr d\cos(\theta) d\phi$$

Solving for a,

$$a = \frac{3Q}{8\pi d^3}$$

$$\rho(\vec{x}) = \frac{3Q}{8\pi d^3}\frac{d^2 - r^2}{r^2}[\delta(\cos(\theta) - 1) + \delta(\cos(\theta) - 1)]$$

From the text,

$$G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{r_<^l r_>^l}{b^{2l+1}}\right)P_l(\cos(\theta'))P_l(\cos(\theta))$$

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{3Q}{8\pi d^3}2\pi\int_0^d (d^2 - r'^2)\left(\frac{r_<^l}{r_>^{l+1}} - \frac{r_>^l r_<^l}{b^{2l+1}}\right)[P_l(1) - P_l(-1)]P_l(\cos(\theta))\ dr'$$

$$= \frac{3Q}{8\pi\epsilon_0 d^3}\sum_{l\ even} P_l(\cos(\theta))\int_0^d (d^2 - r'^2)\left(\frac{r_<^l}{r_>^{l+1}} - \frac{r_>^l r_<^l}{b^{2l+1}}\right)\ dr'$$

If we are in the region $r > d$, $r_< = r'$ and $r_> = r$. Solving the integral,

$$\int_0^d (d^2 - r'^2)\left(\frac{r'^l}{r^{l+1}} - \frac{r'^l r^l}{b^{2l+1}}\right) dr'$$

$$= \frac{d^{l+3}}{r^{l+1}}\left(\frac{1}{l+1} - \frac{1}{l+3}\right) + \frac{r^l d^3}{b^{2l+1}}\left(-1 + \frac{d^l}{l+3}\right)$$

$$\Phi = \frac{3Q}{4\pi\epsilon_0}\sum_{l\ even} P_l(\cos(\theta))\frac{d^l}{(l+1)(l+3)}\left[\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right]$$

For symmetry,

$$= \frac{3Q}{4\pi\epsilon_0}\sum_{l\ even} P_l(\cos(\theta))\frac{d^l}{b^{l+1}(l+1)(l+3)}\left[\left(\frac{b}{r}\right)^{l+1} - \left(\frac{r}{b}\right)^l\right]$$

If we are in the region $r < d$, we need to break the integral into two parts: one where we are approaching the observation point and the other after we've passed that point.

$$\int_0^r (d^2 - r'^2)\left(\frac{r'^l}{r^{l+1}} - \frac{r^l r'^l}{b^{2l+1}}\right) dr' + \int_r^d (d^2 - r'^2)\left(\frac{r^l}{r'^{l+1}} - \frac{r^l r'^l}{b^{2l+1}}\right) dr'$$

$$= d^2\left(\frac{1}{l} + \frac{1}{l+1}\right) - r^2\left(\frac{1}{l+3} + \frac{1}{l-2}\right) + \frac{r^l d^{l+3}}{b^{2l+1}}\left(\frac{1}{l+3} - \frac{1}{l+1}\right) - \frac{r^l d^{2-l}}{l} + \frac{r^l}{(l-2)d^{l-2}}$$

$$\Phi = \frac{3Q}{8\pi\epsilon_0 d^3}\sum_{l\ even} P_l(\cos(\theta))\left[d^2\left(\frac{2l+1}{l(l+1)}\right) - r^2\left(\frac{2l+1}{(l+3)(l-2)}\right) - \frac{r^l d^{l+3}}{b^{2l+1}}\left(\frac{2}{(l+3)(l+1)}\right) + \frac{r^l}{d^{l-2}}\left(\frac{2}{l(l-2)}\right)\right]$$

### 3.14.2 Surface-Charge Density

**Calculate the surface-charge density induced on the shell**

---

We use equation (1.26). We want to look at the $r > d$ since we are looking at the shell,

$$\sigma = \epsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=b}$$

$$\frac{\partial \Phi}{\partial r} = \frac{3Q}{4\pi\epsilon_0}\sum_{l\ even} P_l(\cos(\theta))\frac{d^l}{b^{l+1}(l+1)(l+3)}\left[(-l-1)\frac{b^{l+1}}{r^{l+2}} - l\frac{r^{l-1}}{b^l}\right]$$

At $r = b$,

$$\sigma = -\frac{3Q}{4\pi\epsilon_0}\sum_{l\ even} P_l(\cos(\theta))\frac{d^l(2l+1)}{b^{l+2}(l+1)(l+3)}$$

### 3.14.3 Sphere limit

**Discuss your answers to parts a and b in the limit that $d << b$.** As $d << b$, the $r > d$ potential dominates. We want to rewrite this as

$$\Phi = \frac{3Q}{4\pi\epsilon_0 b} \sum_{l\ even} P_l(\cos(\theta)) \frac{1}{(l+1)(l+3)} \left(\frac{d}{b}\right)^l \left[\left(\frac{b}{r}\right)^{l+1} - \left(\frac{r}{b}\right)^l\right]$$

Since $d << b$, the $d/b$ term kills the potential unless $l = 0$,

$$\Phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$$

For the surface-charge density, we want to write as

$$\sigma = -\frac{3Q}{4\pi\epsilon_0 b^2} \sum_{l\ even} P_l(\cos(\theta)) \frac{2l+1}{(l+1)(l+3)} \left(\frac{d}{b}\right)^l$$

Using the same logic as before,

$$\sigma = -\frac{Q}{4\pi b^2}$$

## 3.15 Circuits

**Consider the following "spherical cow" model of a battery connected to an external circuit. A sphere of radius $a$ and conductivity $\sigma$ is embedded in a uniform medium of conductivity $\sigma'$. Inside the sphere there is a uniform (chemical) force in the z direction acting on the charge carriers; its strength as an effective electric field entering Ohm's law is $F$. In the steady state, electric fields exist inside and outside the sphere and surface charge resides on its surface.**

---

### 3.15.1 Electric Field and Current Density

**Find the electric field (in addition to $F$ and current density everywhere in space. Determine the surface-charge density and show that the electric dipole moment of the sphere is $p = 4\pi\epsilon_0\sigma a^3 F/(\sigma + 2\sigma')$.**

---

We want to start by solving for any surface-charge density. Because the force is pointing solely in the z direction, we have azimuthal symmetry, so we can use equation (3.19).

$$\Phi_{int} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta))$$

$$\Phi_{ext} = \sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos(\theta))$$

As we will see later, it is good to set $\sigma = \sum C_l P_l(\cos(\theta))$. We use the condition that the normal derivative of the potential is discontinuous (1.26),

$$-\frac{\partial \Phi_{ext}}{\partial r} + \frac{\partial \Phi_{int}}{\partial r} = \frac{\sigma}{\epsilon_0}$$

$$B_l(l+1)a^{-l-2} + A_l l a^{l-1} = \frac{C_l}{\epsilon_0}$$

We also want to use the condition that the tangent direction is continuous (1.27),

$$A_l a^l = B_l a^{-l-1}$$

$$B_l = A_l a^{2l+1}$$

Solving for the coefficients,

$$\begin{cases} A_l = \dfrac{C_l}{(2l+1)\epsilon_0}\dfrac{1}{a^{l-1}} \\[2ex] B_l = \dfrac{C_l}{(2l+1)\epsilon_0} a^{l+2} \end{cases}$$

$$\Phi_{int} = \sum_{l=0}^{\infty} \frac{C_l}{(2l+1)\epsilon_0} \frac{r^l}{a^{l-1}} P_l(\cos(\theta))$$

$$\Phi_{ext} = \sum_{l=0}^{\infty} \frac{C_l}{(2l+1)\epsilon_0} \frac{a^{l+2}}{r^{l+1}} P_l(\cos(\theta))$$

Using the orthonormality of the Legendre polynomials,

$$C_l = \frac{2l+1}{2} \int_{-1}^{1} \sigma P_l(\cos(\theta)) \; d\cos(\theta)$$

From the problem, the electric field on the interior is due solely to the chemical force,

$$\vec{E}_{int} = -E_{int}\hat{z}$$

$$\Phi_{int} = -E_{int} r \cos(\theta)$$

Setting this equal to the general potential and matching powers of $\cos(\theta)$, we see that only the $l = 1$ term survives. This pops out

$$C_1 = -3\epsilon_0 E_{int}$$

Plugging this in, and using equation (1.22),

$$\vec{E}_{int} = E_{int}\hat{z}$$

$$\vec{E}_{out} = E_{int} \frac{a^3}{r^3} (-3\cos(\theta)\hat{r} + \hat{z})$$

$$\sigma = -3\epsilon_0 E_{int} \cos(\theta)$$

Ohm's law states $\vec{J} = \sigma \vec{E}$ where $\vec{J}$ is the current density. Here, $\sigma$ and $\sigma'$ refer to the conductivity,

$$\vec{J}_{int} = \sigma(\vec{E}_{int} + \vec{F}) = \sigma(E_{int} + F)\hat{z}$$

$$\vec{J}_{ext} = \sigma' \vec{E}_{ext} = \sigma' E_{int} \frac{a^3}{r^3} (-3\cos(\theta)\hat{r} + \hat{z})$$

The normal component of the current density must be continuous at the boundary,

$$\vec{J}_{int} \cdot \hat{r} = \vec{J}_{ext} \cdot \hat{r}$$

$$\sigma(E_{int} + F)\cos(\theta) = -\sigma E_{int}(-3\cos(\theta) + \cos(\theta)$$

$$E_{int} = -\frac{\sigma}{\sigma + 2\sigma'} F$$

Substituting this back in, we can show the electric field in terms of conductivity,

$$\vec{E}_{int} = -\frac{\sigma}{\sigma + 2\sigma'} F\hat{z}$$

$$\vec{E}_{ext} = \frac{\sigma}{\sigma + 2\sigma'} \frac{a^3}{r^3} F(3\cos(\theta)\hat{r} - \hat{z})$$

$$\vec{J}_{int} = \frac{2\sigma\sigma'}{\sigma + 2\sigma'} F\hat{z}$$

$$\vec{J}_{ext} = \frac{\sigma\sigma'}{\sigma + 2\sigma'} \frac{a^3}{r^3} F(3\cos(\theta)\hat{r} - \hat{z})$$

$$\sigma = 3\epsilon_0 \frac{\sigma}{\sigma + 2\sigma'} F\cos(\theta)$$

The electric field due to a dipole is

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{p}{r^3} (3\cos(\theta)\hat{r} - \hat{z})$$

Comparing to $\vec{E}_{ext}$,

$$p = \frac{4\pi\epsilon_0\sigma a^3}{\sigma + 2\sigma'}$$

### 3.15.2 Current and Power Dissipation

**Show that the total current flowing out through the upper hemisphere of the sphere is**

$$I = \frac{2\sigma\sigma'}{\sigma + 2\sigma'} \pi a^2 F$$

**Calculate the total power dissipation outside the sphere. Using the lumped circuit relations, $P = I^2 R_e = IV_e$, find the effective external resistance $R_e$ and voltage $V_e$.**

---

We calculate the total current,

$$I = \int_0^{2\pi} \vec{J}_{int} \cdot \hat{r} a^2 \; d\cos(\theta) d\phi$$

$$= a^2 \frac{2\sigma\sigma'}{\sigma + 2\sigma'} 2\pi F \int_0^1 \cos(\theta) \; d\cos(\theta)$$

$$I = \frac{2\pi a^2 F\sigma\sigma'}{\sigma + 2\sigma'}$$

We calculate the total power using $P = \dfrac{1}{\sigma}\int J^2\, dV$,

$$P_{int} = \frac{1}{\sigma}\frac{4\sigma^2\sigma'^2}{(\sigma+2\sigma')^2}F^2\int_0^{2\pi}\int_{-1}^{1}\int_0^a r^2\, dr$$

$$= \frac{16\sigma\sigma'^2F^2\pi^2a^3}{3(\sigma+2\sigma')^2}$$

$$P_{ext} = \frac{\sigma^2\sigma'^2}{(\sigma+2\sigma')^2}F^2\frac{2\pi a^6}{\sigma'}\int_{-1}^{1}\int_a^{\infty}\frac{3\cos^2(\theta)+1}{r^4}\, drd\cos(\theta)$$

$$= \frac{8F^2\pi a^3\sigma'\sigma^2}{3(\sigma+2\sigma')^2}$$

Adding these together to get the total power dissipated,

$$P = \frac{8\sigma'\sigma}{3(\sigma+2\sigma')}F^2\pi a^3$$

Using the equations provided by Jackson,

$$R_{ext} = \frac{2}{3\sigma'\pi a}$$

$$V_{ext} = \frac{4\sigma}{3(\sigma+2\sigma')}Fa$$

### 3.15.3 Power dissipated Inside

**Find the power dissipated within the sphere and deduce the effective internal resistance $R_i$ and voltage $V_i$.**

---

We do the same thing we did in the previous section,

$$P_{int} = \frac{16\sigma\sigma'^2}{3(\sigma+2\sigma')^2}F^2\pi a^3$$

$$R_{int} = \frac{4}{3\sigma\pi a}$$

$$V_{int} = \frac{8\sigma'}{3(\sigma+2\sigma')}Fa$$

### 3.15.4 Ohm's Law

**Define the total voltage through the relation, $V_t = (R_e + R_i)I$ and show that $V_t = 4aF/3$, as well as $V_e + V_i = V_t$. Show that $IV_t$ is the power supplied by the "chemical force".**

---

We'll calculate the total voltage using Ohm's law first.

$$V_t = \left(\frac{2}{3\sigma'\pi a} + \frac{4}{3\sigma\pi a}\right)\frac{2\pi a^2 F\sigma\sigma'}{\sigma + 2\sigma'}$$

$$V_t = \frac{4Fa}{3}$$

If we instead add the voltages, we get the same result,

$$V_t = \frac{4Fa\sigma}{3(\sigma + 2\sigma')} + \frac{8Fa\sigma'}{3(\sigma + 2\sigma')} = \frac{4Fa}{3}$$

We can calculate the total power dissipated and compare to the result from part b,

$$P_t = \frac{8a^3F^2\pi\sigma\sigma'}{3(\sigma + 2\sigma')}$$

## 3.16 Bessel Functions

### 3.16.1 Orthonormality

**Starting from the Bessel differential equation and appropriate limiting procedures, verify the generalization,**

$$\frac{1}{k}\delta(k-k') = \int_0^\infty \rho J_\nu(k\rho)J_\nu(k'\rho)\ d\rho$$

**or equivalently that**

$$\frac{1}{\rho}\delta(\rho-\rho') = \int_0^\infty kJ_\nu(k\rho)J_\nu(k\rho')\ dk$$

**where Re($\nu$)$> -1$.**

---

We start with the Bessel differential equation (3.31),

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho J'_\nu) + \left(k^2 - \frac{\nu^2}{\rho^2}\right)J_\nu = 0$$

We multiply by $\rho J'_\nu$ and integrate.

$$\int_0^\infty \rho J'_\nu \frac{d}{d\rho}(\rho J'_\nu)\ d\rho + k^2\int_0^\infty \rho^2 J_\nu J'_\nu\ d\rho - \nu^2\int_0^\infty J_\nu J'_\nu\ d\rho = 0$$

Using the work we did in problem 3.11,

$$\frac{1}{2}\left(\rho\frac{dJ_\nu}{d\rho}\right)\Bigg|_0^\infty + k^2\left(\frac{1}{2}\rho^2J_\nu^2\Big|_0^\infty - \int_0^\infty \rho J_\nu J_\nu\ d\rho\right) - \nu^2\frac{J_\nu^2}{2}\Bigg|_0^\infty = 0$$

Using the limiting case $\rho >> 1$,

$$J_\nu \to \sqrt{\frac{2}{\pi k\rho}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

Substituting this in,

$$\frac{\rho^2}{4}\left[\frac{1}{4}\frac{2}{\pi k\rho^3}\cos^2 + \frac{2k}{\pi k\rho^2}\cos\sin + \frac{2k^2}{\pi k\rho}\sin^2\right] + \frac{k^2\rho^2}{2}\frac{2}{\pi k\rho}\cos^2 = \int_0^\infty \rho J_\nu J_\nu\ d\rho$$

Evaluating the left side at $\rho = \infty$ gives $1/k/$

### 3.16.2 Inverse Distance Expansion

**Obtain the following expansion**

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{m=-\infty}^{m=\infty}\int_0^\infty e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')e^{-k(z_>-z_<)}\ dk$$

We'll start with the expansion in cylindrical coordinates, equation (3.50),

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{m=\infty}\int_0^\infty e^{im(\phi-\phi')}\cos[k(z-z')]I_m(k\rho_<)K_m(k\rho)\ dk$$

We want to use the equations for the modified Bessel functions (3.41) and (3.42). $I_m(k\rho_<)K_m(k\rho_>)$ becomes

$$\frac{\pi}{2}J_\nu(ik\rho_<)\left[\frac{i\sin(\nu\pi)-J_\nu(ik\rho_>)\cos(\nu\pi)-(-1)^m J_\nu(ik\rho_>)}{\sin(\nu\pi)}\right]$$

I'm pretty sure the imaginary parts cancel, but beyond that, I'm not sure how to simplify this more.

### 3.16.3 Expansions

**By appropriate limiting procedures, prove the following expansions:**

$$\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^\infty e^{-k|z|}J_0(k\rho)\ dk$$

$$J_0(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi)} = \sum_{m=-\infty}^{m=\infty}e^{im\phi}J_m(k\rho)J_m(k\rho')$$

$$e^{ik\rho\cos(\phi)} = \sum_{m=-\infty}^{m=\infty}\sum_{m=-\infty}^{\infty}i^m e^{im\phi}J_m(k\rho)$$

We can prove the first relation by setting $m=0$, which implies that $x'\to 0$. We can prove the second by letting $\rho^2\to\rho^2+\rho'^2-2\rho\rho'\cos(\phi)$, which implies that $z'=0$.

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## 3.22 Separation of Variables: Polar Coordinates

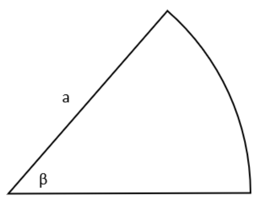


Figure 3.1:

**The geometry of a two-dimensional potential problem is defined in polar coordinates by the surfaces $\phi = 0$, $\phi = \beta$, and $\rho = a$, as indicated in figure (3.1). Using separation of variables in polar coordinates, show that the Green function can be written as**

$$G(\rho,\phi;\rho',\phi') = \sum_{m=1}^{\infty} \frac{4}{m}\rho_<^{m\pi/\beta}\left(\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right)\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

**Problem 2.25 may be of use.**

---

The Green function should follow equation (1.42), which in polar coordinates is given by

$$\nabla'^2 G(\vec{x},\vec{x}') = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')$$

From problem 2.24,

$$\delta(\phi-\phi') = \frac{2}{\beta}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

$$\nabla'^2 G = \frac{8\pi}{\rho\beta}\delta(\rho-\rho')\sum_{m=1}^{\infty}\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

We deduce that we can write the Green's function as a radial component and an angular component,

$$G = \sum_{m=1}^{\infty} g_m(\rho,\rho')\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

In polar coordinates, the laplacian of the Green function is given by

$$\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial G}{\partial\rho'}\right) + \frac{1}{\rho'^2}\frac{\partial^2 G}{\partial\phi'^2} = -\frac{8\pi}{\beta}\delta(\rho-\rho')\sum_{m=1}^{\infty}\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

$$\frac{1}{\rho'}\frac{d}{d\rho'}\left(\rho'\frac{dg_m}{d\rho}\right) - \frac{1}{\rho'^2}\frac{m^2\pi^2}{\beta^2} = -\frac{8\pi}{\rho\beta}\delta(\rho-\rho')$$

From problem 2.25, the general form of the radial component is

$$g_m = A_m(\rho')\rho^{m\pi/\beta} + B_m(\rho')\rho^{-m\pi/\beta}$$

For $\rho < \rho'$, we include the origin,

$$g_< = A_m \rho^{m\pi/\beta}$$

For $\rho > \rho'$, we look at the $\rho = a$ boundary,

$$A_m a^{m\pi/\beta} + B_m a^{-m\pi/\beta} = 0$$

$$A_m = -B_m a^{-2m\pi/\beta}$$

$$g_m = B_m \left( \frac{1}{\rho^{m\pi/\beta}} - \frac{\rho^{m\pi/\beta}}{a^{2m\pi/\beta}} \right)$$

Repeating this with $\rho'$ instead of $\rho$,

$$g_m = C_m \rho_<^{m\pi/\beta} \left( \frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}} \right)$$

Integrating the radial component of the laplacian, letting $\epsilon \to 0$, we get a discontinuity relation,

$$\int_{\rho'-\epsilon}^{\rho'+\epsilon} \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial g_m}{\partial \rho} \right) - \frac{m^2\pi^2}{\rho^2\beta^2} \right] \, d\rho = -\frac{8\pi}{\rho'\beta}$$

$$\int \left[ \frac{1}{\rho} \frac{\partial g_m}{\partial \rho} + \frac{\partial^2 g_m}{\partial \rho^2} - \frac{m^2\pi^2}{\rho^2\beta^2} \right] \, d\rho = -\frac{8\pi}{\rho'\beta}$$

Using integration by parts on the first term with $u = 1/\rho$ and $dv = \dfrac{\partial g_m}{\partial \rho}$, the first and third terms disappear since they are just polynomials in $\rho$,

$$\left. \frac{\partial g_m}{\partial \rho} \right|_{\rho'_+} - \left. \frac{\partial g_m}{\partial \rho} \right|_{\rho'_-} = -\frac{8\pi}{\rho'\beta}$$

$$g_{m,+} = C_m \rho'^{m\pi/\beta} \left( \frac{1}{\rho^{m\pi/\beta}} - \frac{\rho^{m\pi/\beta}}{a^{2m\pi/\beta}} \right)$$

$$\left. \frac{\partial g_{m,+}}{\partial \rho} \right|_{\rho=\rho'} = -\frac{C_m m\pi}{\beta} \rho'^{m\pi/\beta} \left( \frac{1}{\rho'^{(m\pi/\beta)+1}} - \frac{\rho'^{(m\pi/\beta)-1}}{a^{2m\pi/\beta}} \right)$$

$$g_{m,-} = C_m \rho^{m\pi/\beta} \left( \frac{1}{\rho'^{m\pi/\beta}} - \frac{\rho'^{m\pi/\beta}}{a^{2m\pi/\beta}} \right)$$

$$\left.\frac{\partial g_{m,-}}{\partial \rho}\right|_{\rho'=\rho} = \frac{C_m m\pi}{\beta}\rho'^{(m\pi/\beta)-1}\left(\frac{1}{\rho'^{m\pi/\beta}} - \frac{\rho'^{m\pi/\beta}}{a^{2m\pi/\beta}}\right)$$

$$C_m = \frac{4}{m}$$

Combining the radial and angular components,

$$G(\rho,\phi;\rho',\phi') = \sum_{m=1}^{\infty}\frac{4}{m}\rho_<^{m\pi/\beta}\left(\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right)\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

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## 3.26 Neumann Boundary Condition Green Function

**Consider the Green function appropriate for Neumann boundary conditions for the volume $V$ between the concentric spherical surfaces defined by $r = a$ and $r = b$, $a < b$. To be able to use equation (1.47) for the potential, impose the simple constraint $\frac{\partial G_N(\vec{x}, \vec{x}')}{\partial n'} = -4\pi/S$. Use an expansion in spherical harmonics of the form,**

$$G(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} g_l(r, r') P_l(\cos(\gamma))$$

**where $g_l(r, r') = r_<^l / r_>^{l+1} + f_l(r, r')$.**

---

### 3.26.1 Radial Green Function

**Show that for $l > 0$, the radial Green function has the symmetric form**

$$g_l(r, r') = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{(b^{2l+1} - a^{2l+1})} \left[ \frac{l+1}{l} (rr')^l + \frac{l}{l+1} \frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1} \left( \frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}} \right) \right]$$

---

Writing the Green function using the expansion provided by Jackson,

$$G = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos(\gamma)) + f_l P_l(\cos(\gamma))$$

We recognize the first term as the expansion of $1/R$ in Legendre polynomials (3.20). Comparing to the general form of the Green function (1.40), we see that $F(\vec{x}, \vec{x}') = f_l P_l(\cos(\gamma))$. The laplacian of $F$ (1.41) in spherical coordinates is the same as the azimuthally symmetric solution (3.19).

$$\nabla'^2 F = 0$$

$$f_l = A_l r'^l + B_l r'^{-(l+1)}$$

$$g_l = \frac{r_<^l}{r_>^{l+1}} + A_l r'^l + B_l \frac{1}{r'^{l+1}}$$

Looking at the condition from Neumann boundary conditions,

$$\frac{\partial G}{\partial n'} = -\frac{4\pi}{4\pi(a^2 + b^2)}$$

We can then multiply the right side by $P_0(\cos(\gamma))$ since that is just 1 to fix $l = 0$. We then look at the boundaries of each sphere,

$$\left.\frac{\partial g}{\partial(-r')}\right|_{r'=a} = -\frac{1}{a^2+b^2}$$

$$\left.\frac{\partial g}{\partial r'}\right|_{r'=b} = -\frac{1}{a^2+b^2}$$

Note that if $l \neq 0$, the right side of each equation is 0 since the condition is only valid on the surface of the sphere. For the $r' = a$ boundary,

$$g_l = \frac{r'^l}{r^{l+1}} + A_l r'^l + \frac{B_l}{r'^{l+1}}$$

For the $r' = b$ boundary,

$$g_l = \frac{r^l}{r'^{l+1}} + A_l r'^l + \frac{B_l}{r'^{l+1}}$$

Using the boundary conditions and solving for the constants,

$$\begin{cases} A_l = -\dfrac{1}{r^{l+1}} + \dfrac{(l+1)B_l}{la^{2l+1}} \\ \\ B_l = -r^l + \dfrac{lA_l b^{2l+1}}{l+1} \end{cases}$$

$$\begin{cases} A_l = \dfrac{r^l}{b^{2l+1}-a^{2l+1}}\left[\dfrac{l+1}{l} + \left(\dfrac{a}{r}\right)^{2l+1}\right] \\ \\ B_l = \dfrac{r^l}{b^{2l+1}-a^{2l+1}}\left[a^{2l+1} + \dfrac{l}{l+1}\left(\dfrac{ab}{r}\right)^{2l+1}\right] \end{cases}$$

$$g_l = \frac{r_<^l}{r_>^{l+1}} + \frac{r^l r'^l}{b^{2l+1}-a^{2l+1}}\left[\frac{l+1}{l} + \left(\frac{a}{r}\right)^{2l+1}\right] + \frac{r^l r'^{-(l+1)}}{b^{2l+1}-a^{2l+1}}\left[a^{2l+1} + \frac{l}{l+1}\left(\frac{ab}{r}\right)^{2l+1}\right]$$

$$= \frac{r_<^l}{r_>^{l+1}} + \frac{1}{b^{2l+1}-a^{2l+1}}\left[\frac{l+1}{l}(rr')^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1}\left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\right)\right]$$

### 3.26.2 l=0 Case

**Show that for $l = 0$,**

$$g_0(r, r') = \frac{1}{r_>} - \left(\frac{a^2}{a^2+b^2}\right)\frac{1}{r'} + f(r)$$

**where $f(r)$ is arbitrary. Show explicitly in equation (1.47) that answers for the potential $\Phi(\vec{x})$ are independent of $f(r)$.**

**The arbitrariness in the Neumann Green function can be removed by symmetrizing $g_0$ in $r$ and $r'$ with a suitable choice of $f(r)$.**

---

We start by going back to our definition of $g_l$ and setting $l = 0$,

$$g_0 = \frac{1}{r_>} + \frac{B_0}{r'}$$

The boundary conditions are

$$\left.\frac{\partial g_0}{\partial(-r')}\right|_{r'=a} = -\frac{1}{a^2+b^2}$$

$$\left.\frac{\partial g_0}{\partial r'}\right|_{r'=b} = -\frac{1}{a^2+b^2}$$

Solving for the constants,

$$-\frac{B_0}{a^2} = \frac{1}{a^2+b^2}$$

$$-\frac{1}{b^2} - \frac{B_0}{b^2} = -\frac{1}{a^2+b^2}$$

$$B_0 = -\frac{a^2}{a^2+b^2}$$

$$g_0(r,r') = \frac{1}{r_>} - \left(\frac{a^2}{a^2+b^2}\right)\frac{1}{r'} + f(r)$$

Note that we can insert $f(r)$ since our boundary condition involves taking a derivative according to $r'$, which kills that term.

We can show that $f(r)$ does not affect the potential by looking at equation (1.47),

$$\Phi(\vec{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G_N(\vec{x},\vec{x}')\ d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}G_N\ da'$$

We can convince ourselves that in order to show that $f(r)$ has no effect, we need to show,

$$\frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')f(r)\ d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}f(r)\ da' = 0$$

Using equation (1.22),

$$= \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')f(r)\ d^3x' - \frac{f(r)}{4\pi}\oint_S \vec{E}(\vec{x}')\cdot da'$$

$$= \frac{f(r)}{4\pi}\left(\frac{1}{\epsilon_0}\int_V \rho(\vec{x}')\ d^3x' + \oint_S \vec{E}(\vec{x}')\cdot da'\right)$$

Using the divergence theorem (1.17) on the second integral and then Gauss's Law (1.16), we can show that this is equal to 0.

## 3.27 Neumann Boundary Condition Green Function: Example

**Apply the Neumann Green function of Problem 3.26 to the situation in which the normal electric field is $E_r = -E_0 \cos(\theta)$ at the outer surface $(r = b)$ and is $E_r = 0$ on the inner surface $(r = a)$.**

---

### 3.27.1 Potential Inside

**Show that the electrostatic potential inside the volume $V$ is**

$$\Phi(\vec{x}) = E_0 \frac{r\cos(\theta)}{1-p^3}\left(1+\frac{a^3}{2r^3}\right)$$

**where $p = a/b$. Find the components of the electric field,**

$$E_r(r,\theta) = -E_0 \frac{\cos(\theta)}{1-p^3}\left(1-\frac{a^3}{r^3}\right)$$

$$E_\theta(r,\theta) = E_0 \frac{\sin(\theta)}{1-p^3}\left(1+\frac{a^3}{2r^3}\right)$$

---

The general solution to Poisson's equation with Neumann boundary conditions is given by equation (1.47). Since we are in charge-free space, this reduces to

$$\Phi = \frac{1}{4\pi}\oint_S \frac{\partial \Phi}{\partial n'} G_N \; da'$$

Using equation (1.22) and the Green's function given in problem 3.26,

$$= \frac{E_0 b^2}{4\pi}\sum_{l=0}^{\infty}\int g_l(r,r')P_l(\cos(\gamma))\cos(\theta') \; d\Omega'$$

Using the addition theorem (3.28), we can rewrite the Legendre polynomial in terms of the spherical harmonics. Further, we want to write the $\cos(\theta')$ in terms of spherical harmonics (you can look this up),

$$= \frac{E_0 b^2}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int g_l(r,r')\frac{4\pi}{2l+1}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\sqrt{\frac{4\pi}{3}}Y_{10}(\theta',\phi') \; d\Omega'$$

Using the orthonormality of the spherical harmonics (3.27) sets $l = 1$ and $m = 0$,

$$= \frac{E_0 b^2}{4\pi}\sqrt{\frac{4\pi}{3}}g_1(r,b)\frac{4\pi}{3}Y^*_{10}(\theta,\phi)$$

Using the $g_l$ that we found in problem 3.26a,

$$= \frac{E_0 b^2 \cos(\theta)}{3}\left(\frac{r}{b^2} + \frac{1}{b^3 - a^3}\left[2(rb) + \frac{1}{2}\frac{(ab)^3}{(rb)^2} + a^3\left(\frac{r}{b^2} + \frac{b}{r^2}\right)\right]\right)$$

$$\Phi(\vec{x}) = E_0 \frac{r\cos(\theta)}{1 - p^3}\left(1 + \frac{a^3}{2r^3}\right)$$

We can determine the components of the electric field using equation (1.22),

$$E_r = -\frac{\partial \Phi}{\partial r} = -E_0 \frac{\cos(\theta)}{1 - p^3}\left(1 - \frac{a^3}{r^3}\right)$$

$$E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = E_0 \frac{\sin(\theta)}{1 - p^3}\left(1 + \frac{a^3}{2r^3}\right)$$

### 3.27.2 Cartesian and Cylindrical Components of Electric Field

**Calculate the Cartesian and cylindrical components of the field.**

---

In Cartesian coordinates, the potential is written as

$$\Phi = \frac{E_0}{1 - p^3}\left(\sqrt{x^2 + y^2 + z^2}\arctan\left(\frac{y}{x}\right) + \frac{a^3}{2}\frac{\arctan(y/x)}{x^2 + y^2 + z^2}\right)$$

$$E_x = -\frac{\partial \Phi}{\partial x} = \frac{E_0}{1 - p^3}\left[x\arctan\left(\frac{y}{x}\right)\left(\frac{1}{(x^2 + y^2 + z^2)^{1/2}} - \frac{a^3}{(x^2 + y^2 + z^2)^2}\right) - \frac{y}{x^2 + y^2}\left(\sqrt{x^2 + y^2 + z^2} + \frac{a^3}{2(x^2 + y^2 + z^2)}\right)\right]$$

$$E_y = -\frac{\partial \Phi}{\partial y} = \frac{E_0}{1 - p^3}\left[y\arctan\left(\frac{y}{x}\right)\left(\frac{1}{(x^2 + y^2 + z^2)^{1/2}} - \frac{a^3}{(x^2 + y^2 + z^2)^2}\right) - \frac{x}{x^2 + y^2}\left(\sqrt{x^2 + y^2 + z^2} + \frac{a^3}{2(x^2 + y^2 + z^2)}\right)\right]$$

$$E_z = -\frac{\partial \Phi}{\partial z} = \frac{E_0}{1 - p^3}\left[\frac{z}{(x^2 + y^2 + z^2)^{1/2}} - \frac{a^3}{(x^2 + y^2 + z^2)^2}\arctan\left(\frac{y}{x}\right)\right]$$

In cylindrical coordinates, the potential is

$$\Phi = \frac{E_0}{1 - p^3}\left(z + \frac{a^3 z}{2(\rho^2 + z^2)^{3/2}}\right)$$

$$E_z = -\frac{\partial \Phi}{\partial z} = -\frac{E_0}{1 - p^3}\left[1 + \frac{a^3}{2(\rho^2 + z^2)^{3/2}} - \frac{3a^3 z}{2(\rho^2 + z^2)^{5/2}}\right]$$

$$E_\rho = -\frac{\partial \Phi}{\partial \rho} = \frac{E_0}{1 - p^3}\left(\frac{3a^3 z\rho}{2(\rho^2 + z^2)^{5/2}}\right)$$

# Chapter 4

# Multipoles, Electrostatics of Macroscopic Media, Dielectrics

## 4.1 Multipole Moments

**Calculate the moments $q_{lm}$ of the charge distributions shown as parts a and b. Try to obtain results for the nonvanishing moments valid for all $l$, but in each case find the first two sets of nonvanishing moments at the very least.**

---

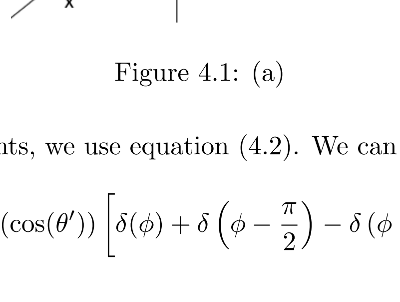


Figure 4.1: (a)

To find the multipole moments, we use equation (4.2). We can write the charge density as

$$\rho(\vec{x}') = \frac{q}{a^2}\delta(r'-a)\delta(\cos(\theta'))\left[\delta(\phi) + \delta\left(\phi - \frac{\pi}{2}\right) - \delta\left(\phi - \pi\right) - \delta\left(\phi - \frac{3\pi}{2}\right)\right]$$

$$q_{lm} = \int_0^\infty \int_{-1}^1 \int_0^{2\pi} Y^*_{lm}(\theta', \phi') r'^l \rho(\vec{x}') r'^2 \; d\phi' d\cos(\theta') dr'$$

Using the spherical harmonics (3.25) and delta functions, we get

$$= qa^l \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(0)[1 + (-1)^m i^m - (-1)^m - i^m]$$

We can convince ourselves that this goes to 0 if $m$ is even. If $m$ is odd,

$$q_{lm} = 2qa^l \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(0)[1 - i^m]$$

The first two non-vanishing terms are the dipole and octopole moments,

$$q_{11} = 2qa\sqrt{\frac{3}{4\pi}\frac{1}{2}}(-1)(1-) = -qa\sqrt{\frac{3}{2\pi}}(1-i)$$

$$q_{33} = 2qa^3\sqrt{\frac{7}{4\pi}\frac{1}{6!}}(-15)(1+i) = -5qa^3\sqrt{\frac{7}{80\pi}}(1+i)$$

$$q_{31} = 2qa^3\sqrt{\frac{7}{4\pi}\frac{2}{24}}\frac{3}{2}(1-i) = 3qa^3\sqrt{\frac{7}{48\pi}}(1-i)$$

We can use $q_{l,-m} = (-1)^m q_{lm}^*$ to find the other terms.

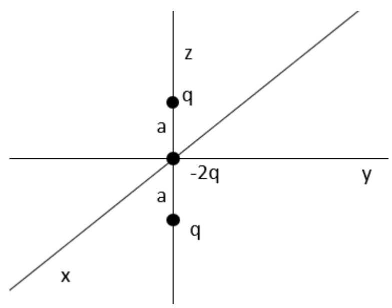


Figure 4.2: (b)

We can write the charge density as

$$\rho(\vec{x}') = \frac{q}{2\pi a^2}\delta(r-a)[\delta(\cos(\theta')-1) + \delta(\cos(\theta')+1)] - \frac{2q}{4\pi r'^2}\delta(r)$$

Looking at figure (4.2), we see that the system has azimuthal symmetry, so we can set $m = 0$.

$$q_{l0} = \sqrt{\frac{2l+1}{4\pi}} \int_0^\infty \int_{-1}^1 \int_0^{2\pi} P_l(\cos(\theta'))r'^l \rho(\vec{x}')r'^2 \; d\phi' d\cos(\theta')dr'$$

The non-origin charges give

$$qa^l \sqrt{\frac{2l+1}{4\pi}}[1+(-1)^l]$$

while the charge at the origin only gives a contribution when $l = 0$,

$$2q\sqrt{\frac{1}{4\pi}}$$

Further, we see that when $l = 0$, these two terms cancel. If $l$ is even and $m = 0$,

$$q_{lm} = \sqrt{\frac{2l+1}{\pi}} qa^l$$

The two lowest order non-vanishing terms are

$$q_{2,0} = \sqrt{\frac{5}{\pi}} qa^2$$

$$q_{4,0} = \sqrt{\frac{9}{4\pi}} qa^4$$

### 4.1.1 Multipole Expansion of Potential

**For the charge distribution of the second set b write down the multipole expansion for the potential.**

---

We want to start with the multipole expansion of the potential we derived in the text (4.1). Inserting the expression for the multipole moment we found earlier in the problem,

$$\Phi = \frac{1}{4\pi\epsilon_0} \sum_{l\ even} \frac{4\pi}{2l+1} \sqrt{\frac{2l+1}{\pi}} qa^l \sqrt{\frac{2l+1}{4\pi}} \frac{P_l(\cos(\theta))}{r^{l+1}}$$

$$= \frac{1}{2\pi\epsilon_0} \sum_{l\ even} qa^l \frac{P_l(\cos(\theta))}{r^{l+1}}$$

Keeping the lowest non-vanishing term ($l = 2$),

$$\Phi \approx \frac{q}{4\pi\epsilon_0} \frac{a^2}{r^3} (3\cos^2(\theta) - 1)$$

### 4.1.2 Potential from Point Charges

**Calculate directly from Coulomb's law the exact potential for b in the x-y plane.**

---

We want to use superposition and equation (1.23). The distance to a point in the x-y plane from either of the charges not at the origin is $\sqrt{a^2 + r^2}$.

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{2}{\sqrt{r^2 + a^2}} - \frac{2}{r}\right]$$

$$= \frac{q}{2\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2 + a^2}} - \frac{1}{r}\right]$$

## 4.2 Dipole Charge Density

**A point dipole with dipole moment $\vec{p}$ is located at the point $\vec{x}_0$. From the properties of the derivative of a Dirac delta function, show that for calculation of the potential $\Phi$ or the energy of a dipole in an external field, the dipole can be described by an effective charge density**

$$\rho_{eff}(\vec{x}) = -\vec{p} \cdot \nabla\delta(\vec{x} - \vec{x}_0)$$

---

We'll start with writing the potential due to a dipole (4.5) and setting it to the potential due to an arbitrary charge density (1.23),

$$\frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot (\vec{x} - \vec{x}_0)}{|\vec{x} - \vec{x}_0|^3} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3x'$$

On the left side, we can the derivative of $1/r$ (1.20),

$$\vec{p} \cdot \nabla \left( \frac{1}{|\vec{x} - \vec{x}_0|} \right) = \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3x'$$

Multiplying the left side by $\delta(\vec{x}' - \vec{x_0}$ and integrating,

$$\int \delta(\vec{x}' - \vec{x}_0) \vec{p} \cdot \nabla \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) \, d^3x'$$

Integrating by parts with $u = \delta(\vec{x}' - \vec{x}_0)$ and $dv = \nabla \left(1/|\vec{x} - \vec{x}'|\right)$,

$$= \vec{p}\delta(\vec{x}' - \vec{x}_0) \frac{1}{|\vec{x} - \vec{x}'|} - \int \frac{\vec{p} \cdot \nabla\delta(\vec{x}' - \vec{x}_0)}{|\vec{x} - \vec{x}'|} \, d^3x'$$

We can kill the first term using the delta function,

$$\int \frac{\vec{p} \cdot \nabla\delta(\vec{x}' - \vec{x}_0)}{|\vec{x} - \vec{x}'|} \, d^3x' = \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3x'$$

Matching terms,

$$\rho(\vec{x}) = -\vec{p} \cdot \nabla\delta(\vec{x} - \vec{x}_0)$$

## 4.3 Multipole Moments

**The $l$th term in the multipole expansion (4.1) of the potential is specified by the $(2l+1)$ multipole moments $q_{lm}$. On the other hand, the Cartesian multipole moments,**

$$Q^{(l)}_{\alpha\beta\gamma} = \int \rho(\vec{x}) x^\alpha y^\beta z^\gamma \; d^3x$$

**with $\alpha$, $\beta$, $\gamma$ non-negative integers subject to the constraint $\alpha+\beta+\gamma = l$, are $(l+1)(l+2)/2$ in number. Thus, for $l > 1$ there are more Cartesian multipole moments than seem necessary to describe the term in the potential whose radial dependence is $r^{-l-1}$.**

**Show that while the $q_{lm}$ transform under rotations as irreducible spherical tensors of rank $l$, the Cartesian multipole moments correspond to reducible spherical tensors of ranks $l$, $l-2$, $l-4$, ..., $l_{min}$, where $l_{min} = 0$ or 1 for $l$ even or odd, respectively. Check that the number of different tensorial components adds up to the total number of Cartesian tensors. Why are only the $q_{lm}$ needed in the expansion (4.1).**

---

I don't know what these words are.

## 4.4 Higher Order

### 4.4.1 Properties of Multipole Moments

**Prove the following theorem: For an arbitrary charge distribution $\rho(\vec{x})$ the values of the first nonvanishing multipole are independent of the origin of the coordinate axes, but the values of all higher multipole moments do in general depend on the choice of origin. (The different moments $q_{lm}$ for fixed $l$ depend, of course, on the orientation of the axes.)**

---

This problem is a little difficult to do in general since performing a translation on the origin $\vec{x}' \to \vec{x}' + \vec{R}$ is difficult to describe with the spherical harmonics. Instead, we'll look at the Cartesian moments which are given in the previous problem,

$$Q_{ij...l} = \int \rho(\vec{x}) x_i x_j ... x_l \ d^3x$$

Making the substitution, $\vec{x} \to \vec{x}' = \vec{x} - \vec{R}$,

$$Q'_{ij...l} = \int \rho(\vec{x}')(x_i - R_i)(x_j - R_j)...(x_l - R_l) \ d^3x'$$

Expanding the polynomials,

$$= \int \rho(\vec{x}') x_i x_j ... x_l \ d^3x' - R_i \int \rho(\vec{x}') x_j ... x_l \ d^3x' + ...$$

$$= Q_{ij...l} - R_i Q_{j...l} + ...$$

We can convince ourselves that the shifted multipole moments consists of the original multipole moments plus a bunch of lower order terms.

### 4.4.2 Up to Quadrupole

**A charge distribution has multipole moments $q$, $\vec{p}$, $Q_{ij}$,... with respect to one set of coordinate axes, and moments $q'$, $p'$, $Q'_{ij}$... with respect to another set whose axes are parallel to the first, but whose origin is located at the point $\vec{R} = (X, Y, Z)$ relative to the first. Determine explicitly the connections between the monopole, dipole, and quadrupole moments in the two coordinate frames.**

---

What this is saying is $\vec{x}' = \vec{x} - \vec{R}$. We want to make this substitution for monopole, dipole (4.3), and quadrupole (4.4) moments. We can convince ourselves that the monopole term remains the same since we integrate over all space,

$$q' = \int \rho(\vec{x}') \ d^3x' = q$$

For a dipole,

$$\vec{p}' = \int \vec{x}' \rho(\vec{x}')\ d^3x' = \int (\vec{x} - \vec{R})\rho(\vec{x}')\ d^3x'$$

$$= \int \vec{x}\rho(\vec{x})\ d^3x - \vec{R}\int \rho(\vec{x}')\ d^3x'$$

$$\vec{p}' = \vec{p} - q\vec{R}$$

For a quadrupole,

$$Q'_{ij} = \int (3x'_i x'_j - r'^2\delta_{ij})\rho(\vec{x}')\ d^3x'$$

$$= \int [3(x_i - R_i)(x_j - R_j) - (\vec{x} - \vec{R})^2\delta_{ij}]\rho(\vec{x}')\ d^3x'$$

Expanding the portion in the brackets,

$$= 3(x_i x_j - R_i x_j - R_j x_i + R_i R_j) - (r^2 - 2\vec{x}\cdot\vec{R} + R^2)\delta_{ij}$$

$$Q'_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(\vec{x})\ d^3x - 3R_i\int x_j\rho(\vec{x})\ d^3x - 3R_j\int x_i\rho(\vec{x})\ d^3x + 3R_iR_j\int \rho(\vec{x})\ d^3x - \int \delta_{ij}(R^2 - 2\vec{x}\cdot\vec{R})\rho(\vec{x})\ d^3x$$

$$= Q_{ij} - 3R_i p_j - 3R_j p_i + 3R_i R_j q - \delta_{ij}(R^2 q - 2\vec{p}\cdot\vec{R})$$

We see that shifted multipole moments are the original multipole moments plus a bunch of lower order terms.

### 4.4.3 Killing Lower Order Terms

**If $q \neq 0$, can $\vec{R}$ be found so that $\vec{p}' = 0$? If $q \neq 0$, $\vec{p} \neq 0$, or at least $\vec{p} \neq 0$, can $\vec{R}$ be found so that $Q'_{ij} = 0$?**

---

Going back to the previous forms, if we set $\vec{R} = \frac{1}{q}\vec{p}$, we can kill $\vec{p}'$. This is not true for higher order terms. $Q_{ij}$ has 5 terms $(2l+1)$, but $\vec{R}$ has only three degrees of freedom. In general, we cannot kill the quadrupole moments by choice of origin.

## 4.5 Force due to a Slowly-Varying Electric Field

**A localized charge density $\rho(x, y, z)$ is placed in an external electrostatic field described by a potential $\Phi^0(x, y, z)$. The external potential varies slowly in space over the region where the charge density is different from zero.**

---

### 4.5.1 Total Force

**From first principles calculate the total force acting on the charge distribution as an expansion in multipole moments times derivatives of the electric field, up to and including the quadrupole moments. Show that the force is**

$$\vec{F} = q\vec{E}^0(0) + \{\nabla[\vec{p}\cdot\vec{E}^0(\vec{x})]\}_0 + \left\{\nabla\left[\frac{1}{6}\sum_{j,k} Q_{jk}\frac{\partial E_j^0}{\partial x_k}(\vec{x})\right]\right\}_0$$

**Compare this to the expansion,**

$$W = q\Phi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\sum_i\sum_j Q_{ij}\frac{\partial E_j}{\partial x_i}(0) + ...$$

**Note that this expansion is a number, it is not a function of $\vec{x}$ that can be differentiated! What is its connection to $\vec{F}$?**

---

We start by using equation (1.4) to write the force. Let's look at a single component,

$$F_i = \int \rho E_i^0(\vec{x})\ d^3x$$

Expanding the electric field using a Taylor expansion,

$$E_i^0(\vec{x}) = \left[E_i^0(\vec{x}') + \sum_j \frac{\partial E_i(\vec{x}')}{\partial x_j'}x_j + \frac{1}{2}\sum_{jk}\frac{\partial}{\partial x_j'}\frac{\partial}{\partial x_k'}E_i^0(\vec{x}')x_j x_k\right]$$

Using Faraday's law (1.21), we can show that $\dfrac{\partial E_j}{\partial x_i} = \dfrac{\partial E_i}{\partial x_j}$. Write out the terms of $\nabla\times\vec{E} = 0$ to prove this to yourself. Inserting this into our force equation,

$$F_i = \int \rho E_i^0\ d^3x + \sum_j \frac{\partial E_j}{\partial x_i'}\int \rho x_j\ d^3x + \frac{1}{2}\sum_{jk}\frac{\partial}{\partial x_i'}\frac{\partial}{\partial x_k'}E_j^0\left(\frac{1}{3}Q_{jk}\right)$$

Using the definition of the dipole (4.3) and quadrupole (4.4),

$$\vec{F} = q\vec{E}^{(0)}(0) + \left\{\nabla[\vec{p}\cdot\vec{E}^{(0)}(\vec{x})]\right\}_0 + \left\{\nabla\left[\frac{1}{6}\sum_{jk} Q_{jk}\frac{\partial E_j^{(0)}}{\partial x_k}(\vec{x})\right]\right\}_0 + ...$$

To compare this to energy, we first want to use equation (1.22) on the first term. This means we can write the force as

$$F = -\nabla \left[ q\Phi - \vec{p} \cdot \vec{E}(0) - \frac{1}{6} \sum_{jk} Q_{jk} \frac{\partial E_j(0)}{\partial x_k} \right]$$

Comparing to the energy, we get the familiar $W = -\int \vec{F} \cdot d\vec{l}$.

### 4.5.2 Torque

**Repeat the calculation of part a for the total torque. For simplicity, evaluate only one Cartesian component of the torque, say $N_1$. Show that this component is**

$$N_1 = [\vec{p} \times \vec{E}^{(0)}(0)]_1 + \frac{1}{3} \left[ \frac{\partial}{\partial x_3} \left( \sum_j Q_{2j} E_j^{(0)} \right) - \frac{\partial}{\partial x_2} \left( \sum_j Q_{3j} E_j^{(0)} \right) \right]_0$$

---

The torque is given by

$$\vec{N} = \int \vec{x} \times (\rho(\vec{x}) \vec{E}^{(0)}(\vec{x}))\ d^3x$$

$$N_1 = \int \rho (x_2 E_3^{(0)} - x_3 E_2^{(0)})\ d^3x$$

Using the first two terms of the expansion from the previous part,

$$= \int \rho \left[ x_2 \left( E_3^{(0)} + \sum_j \frac{\partial E_3^{(0)}}{\partial x_j} x_j \right) - x_3 \left( E_2 + \sum_j \frac{\partial E_2}{\partial x_j} x_j \right) \right]\ d^3x$$

$$= \int \rho (x_2 E_3^{(0)} - x_3 E_2^{(0)})\ d^3x + \sum_j \frac{\partial E_j^0}{\partial x_3} \int \rho x_2 x_j\ d^3x - \sum_j \frac{\partial E_j^0}{\partial x_2} \int \rho x_3 x_j\ d^3x$$

We translate these into dipole (4.3) and quadrupole (4.4) moments,

$$= [\vec{p} \times \vec{E}^{(0)}(0)]_1 + \frac{1}{3} \left[ \frac{\partial}{\partial x_3} \left( \sum_j Q_{2j} E_j^{(0)} \right) - \frac{\partial}{\partial x_2} \left( \sum_j Q_{3j} E_j^{0} \right) \right]_0$$

## 4.6 Nucleus in Electric Field

**A nucleus with quadrupole moment $Q$ finds itself in a cylindrically symmetric electric field with a gradient $(\partial E_z/\partial z)_0$ along the z axis at the position of the nucleus.**

---

### 4.6.1 Energy

**Show that the energy of quadrupole interaction is**

$$W = -\frac{e}{4} Q \left(\frac{\partial E_z}{\partial z}\right)_0$$

---

Jackson tells us that the quadrupole energy term is

$$W = -\frac{1}{6} \sum_i \sum_j Q_{ij} \frac{\partial E_j}{\partial x_i}(0)$$

Furthermore, the only nonvanishing quadrupole moment is $Q_{33}$, and the total quadrupole moment is defined to be $Q = Q_{33}/e$ (see pg. 151). Since the quadrupole tensor is traceless and symmetric, this implies that $Q_{11} + Q_{22} + Q_{33} = 0$, or $Q_{11} = Q_{22} = -Q_{33}/2$. Further, we look at the electric field. From Gauss's Law (1.18), we know that $\nabla \cdot \vec{E} = 0$ since there is no charge density.

$$\frac{\partial E}{\partial x} + \frac{\partial E}{\partial y} + \frac{\partial E}{\partial z} = 0$$

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial y} = -\frac{1}{2}\frac{\partial E}{\partial z}$$

Substituting all of this in, ignoring off-diagonal terms since they are 0,

$$W = -\frac{1}{6}\left(Q_{11}\frac{\partial E}{\partial x} + Q_{22}\frac{\partial E}{\partial y} + Q_{33}\frac{\partial E}{\partial z}\right)$$

$$= -\frac{1}{6}\left(\frac{1}{4} + \frac{1}{4} + 1\right) Q_{33}\left(\frac{\partial E}{\partial z}\right)_0$$

$$W = -\frac{1}{4} Q_{33}\left(\frac{\partial E}{\partial z}\right)_0$$

### 4.6.2 Calculator

**If it is known that $Q = 2 \times 10^{-28}\ m^2$ and that $W/h$ is $10MHz$, where $h$ is Planck's constant, calculate $(\partial E_z/\partial z)_0$ in units of $e/4\pi\epsilon_0 a_0^3$, where $a_0 = 4\pi\epsilon_0\hbar^2/me^2$=$0.529 \times 10^{-10}m$ is the Bohr radius in hydrogen.**

---

In terms of the units that Jackson wants,

$$\frac{\partial E}{\partial z} = -\frac{4W}{eQ}\frac{e}{4\pi\epsilon_0 a_0^3}\frac{4\pi\epsilon_0 a_0^3}{e}$$

$$= -\left(\frac{16\pi\epsilon_0 a_0^3 h}{e^2 Q}\frac{W}{h}\right)\frac{e}{4\pi\epsilon_0 a_0^3}$$

Substituting in values,

$$\frac{\partial E}{\partial z} = 1.93 \times 10^{-50}\ \frac{e}{4\pi\epsilon_0 a_0^3}$$

### 4.6.3 Nuclear Charge Distributions

**Nuclear charge distributions can be approximated by a constant charge density throughout a spheroidal volume of semimajor axis $a$ and semiminor axis $b$. Calculate the quadrupole moments of such a nucleus, assuming that the total charge is $Ze$. Given that $Eu^{153}$ $(Z = 63)$ has a quadrupole moment $Q = 2.5 \times 10^{-28}m^2$ and a mean radius**

$$R = (a + b)/2 = 7 \times 10^{-15}m$$

**determine the fractional difference in radius $(a - b)/R$.**

---

We know from the text that $Q = Q_{33}/e$, so we only need to calculate

$$Q_{33} = \int (3z^2 - r^2)\rho(\vec{x})\ d^3x$$

In cylindrical coordinates, this becomes,

$$= \int \rho(\vec{x})[3z^2 - (r^2 + z^2)]r\ drdz$$

$$= 2\pi \int \rho(\vec{x})(2z^2 r - r^3)\ drdz$$

Since the charge density is constant,

$$\rho = \frac{Ze}{V} = \frac{3Ze}{4\pi ab^2}$$

For the radius, we need to use

$$\frac{R^2}{b^2} + \frac{z^2}{a^2} = 1$$

$$R = b\sqrt{1 - \frac{z^2}{a^2}}$$

Substituting all of this in,

$$Q_{33} = \frac{3Ze}{2ab^2} \int_{-a}^{a} \int_{0}^{b\sqrt{1-(z^2/a^2)}} (2z^2 r - r^3)\ drdz$$

$$= \frac{2Ze}{5}(a^2 - b^2)$$

$$Q = \frac{2Z}{5}(a^2 - b^2)$$

If we want to calculate $(a - b)/R$,

$$\frac{a-b}{R} = \frac{5Q}{4ZR^2} = 0.101$$

## 4.7 Multipole Expansion of Charge Distribution

**A localized distribution of charge has a charge density**

$$\rho(\vec{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2(\theta)$$

---

### 4.7.1 Multipole Expansion

**Make a multipole expansion of the potential due to this charge density and determine all the nonvanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.**

---

The potential is given by equation (4.1), and the multipole moments are given by (4.2).

$$\Phi = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$$

$$q_{lm} = \int Y^*_{lm}(\theta', \phi') r'^l \rho(\vec{x}')\ d^3x'$$

Looking at the charge density, we see that there is no $\phi$ component, so $m = 0$. Further, only the $l = 0$ and $l = 2$ terms survive due to orthonormality of the Legendre polynomials (3.6). To show this, we convert $\sin^2(\theta)$ into cos and then write our charge density in terms of the Legendre polynomials.

$$\cos^2(\theta) = \frac{2}{3} P_2(\cos(\theta)) + \frac{1}{3} P_0(\cos(\theta))$$

$$\rho = \frac{1}{64\pi} r^2 e^{-r} \left( \frac{2}{3} P_0(\cos(\theta) - \frac{2}{3} P_2(\cos(\theta)) \right)$$

$$q_l = \frac{1}{64\pi} \sqrt{\frac{2l+1}{4\pi}} 2\pi \int_0^{\infty} \int_{-1}^{1} P_l \left( \frac{2}{3} P_0 - \frac{2}{3} P_2 \right) r^{4+l} e^{-r}\ d\cos(\theta) dr$$

$$q_0 = \sqrt{\frac{1}{4\pi}}$$

$$q_2 = -6\sqrt{\frac{5}{4\pi}}$$

The potential in terms of the Legendre polynomials is

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{P_0(\cos(\theta))}{r} - 6\frac{P_2(\cos(\theta))}{r^3}\right]$$

If we want to write the potential very far away, we want to keep only the leading order term, which would be $\Phi = \frac{1}{4\pi\epsilon_0 r}$. If we calculate the total charge by this charge distribution,

$$Q = 2\pi\frac{1}{64\pi}\int_0^\infty\int_{-1}^1 r^4 e^{-r}(1-\cos^2(\theta))\ d\cos(\theta)dr = 1$$

We can approximate the charge as a point charge of magnitude unity since it falls off very quickly as we move away from the origin. Doing this, we can use equation (1.23) to get the potential.

### 4.7.2 Potential

**Determine the potential explicitly at any point in space, and show that near the origin, correct to $r^2$ inclusive,**

$$\Phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{r^2}{120}P_2(\cos(\theta))\right]$$

---

Using equation (1.23), we can calculate the potential,

$$\Phi = \frac{1}{4\pi\epsilon_0}\int\frac{1}{64\pi}\frac{r'^2 e^{-r'}(1-\cos^2(\theta'))}{|\vec{x}-\vec{x}'|}r'^2\ d^3x'$$

Converting the trig function to Legendre polynomials and using the orthonormality of the Legendre polynomials (3.6), we see that only the $l=0$ and $l=2$ terms survive,

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{32}\left[\frac{4}{3}\int_0^\infty r'^4 e^{-r'}\frac{1}{r_>}P_0(\cos(\theta))\ dr' - \frac{4}{15}\int_0^\infty r'^4 e^{-r'}\frac{r_<^2}{r_>^3}P_2(\cos(\theta))\ dr'\right]$$

Breaking the integral up,

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{8}\Big[\frac{P_0(\cos(\theta))}{3r}\int_0^r r'^4 e^{-r'}\ dr' + \frac{P_0(\cos(\theta))}{3}\int_r^\infty r'^3 e^{-r'}\ dr'$$
$$-\frac{P_2(\cos(\theta))}{15r^3}\int_0^r r'^6 e^{-r'}\ dr' - \frac{r^2 P_2(\cos(\theta))}{15}\int_r^\infty r' e^{-r'}\ dr'\Big]$$

$$= \frac{1}{32\pi\epsilon_0}\left[\frac{P_0(\cos(\theta))}{3}\left(\left(-r^2-6r-18-\frac{24}{r}\right)e^{-r}+\frac{24}{r}\right)\right.$$
$$\left.-\frac{P_2(\cos(\theta))}{15}\left(\left(-5r^2-30r-120-\frac{360}{r}-\frac{720}{r^2}-\frac{720}{r^3}\right)e^{-r}+\frac{720}{r^3}\right)\right]$$

Looking close to the origin,

$$e^{-r} \approx 1 - r + \frac{1}{2}r^2 - \frac{1}{6}r^3 + \frac{1}{24}r^4 - \frac{1}{120}r^5$$

Plugging this in, we get

$$\Phi \approx \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{r^2}{120}P_2(\cos(\theta))\right]$$

### 4.7.3 Nucleus Quadrupole

**If there exists at the origin a nucleus with a quadrupole moment $Q = 10^{-28}m^2$, determine the magnitude of the interaction energy, assuming that the unit of charge in $\rho(\vec{r})$ above is the electronic charge and the unit of length is the hydrogen Bohr radius $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 0.529 \times 10^{-10}m$. Express your answer as a frequency by dividing by Planck's constant $h$.**

**The charge density in this problem is that for the $m = \pm 1$ states of the $2p$ level in hydrogen, while the quadrupole interaction is of the same order as found in molecules.**

---

Converting the potential near the origin to units of electron charge and units of Bohr radius,

$$\Phi = -\frac{e}{4\pi\epsilon_0 a_0}\left[\frac{1}{4} - \frac{r^2}{120a_0^2}P_2(\cos(\theta))\right]$$

The interaction energy is given by

$$W = \int \rho\Phi \; d^3x$$

$$= -\frac{e}{4\pi\epsilon_0 a_0}\left[\frac{1}{4}q - \frac{1}{240a^2}\int \rho(3z^2 - r^2) \; d^3x\right]$$

Keeping the quadrupole term, which is the last term,

$$\frac{W}{h} = \frac{\alpha c Q}{480\pi a^3} \approx 1MHz$$

## 4.8 Boundary-Value Problem: Cylindrical Shell

**A very long, right circular, cylindrical shell of dielectric constant $\epsilon/\epsilon_0$ and inner and outer radii $a$ and $b$ respectively, is placed in a previously uniform electric field $E_0$ with its axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity.**

---

### 4.8.1 Potential and Electric Field

**Determine the potential and electric field in the three regions, neglecting end effects.**

---

Because the cylinder is very long, we can use the polar solution to Laplace's equation (2.32). In the three regions, the general potential is given by

$$\begin{cases} \Phi_1 = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \cos(n\phi), \quad \rho < a \\ \\ \Phi_2 = b_0 + c_0 \ln(\rho) + \sum_{n=1}^{\infty} \left( b^n \rho^n + \dfrac{c^n}{\rho^n} \right) \cos(n\phi), \quad a < \rho < b \\ \\ \Phi_3 = d_0 + \sum_{n=1}^{\infty} d_n \rho^{-n} \cos(n\phi) - E_0 \rho \cos(\phi), \quad \rho > b \end{cases}$$

From the potential outside of the cylinder, we see that only the $n = 1$ terms survive,

$$\begin{cases} \Phi_1 = a_1 \rho \cos(\phi) \\ \Phi_2 = b_1 \rho \cos(\phi) + c_1 \rho^{-1} \cos(\phi) \\ \Phi_3 = d_1 \rho^{-1} \cos(\phi) - E_0 \rho \cos(\phi) \end{cases}$$

From the tangential boundary conditions (4.13),

$$-\frac{1}{\rho}\frac{\partial \Phi_1}{\partial \phi}\bigg|_{\rho=a} = -\frac{1}{\rho}\frac{\partial \Phi_2}{\partial \phi}\bigg|_{\rho=a}$$

$$-\frac{1}{\rho}\frac{\partial \Phi_2}{\partial \phi}\bigg|_{\rho=b} = -\frac{1}{\rho}\frac{\partial \Phi_3}{\partial \phi}\bigg|_{\rho=b}$$

From the normal boundary conditions (4.12),

$$-\frac{\partial \Phi_1}{\partial \rho}\bigg|_{\rho=a} = -\frac{\epsilon}{\epsilon_0}\frac{\partial \Phi_2}{\partial \rho}\bigg|_{\rho=a}$$

$$-\frac{\epsilon}{\epsilon_0}\frac{\partial \Phi_2}{\partial \rho}\bigg|_{\rho=b} = -\frac{\partial \Phi_3}{\partial \rho}\bigg|_{\rho=b}$$

This gives the following conditions:

$$\begin{cases} a_1 = b_1 + \dfrac{c_1}{a^2} \\ \\ b_1 + \dfrac{c_1}{b^2} = \dfrac{d_1}{b^2} - E_0 \\ \\ a_1 = \dfrac{\epsilon}{\epsilon_0}\left(b_1 - \dfrac{c_1}{a^2}\right) \\ \\ \dfrac{\epsilon}{\epsilon_0}\left(b_1 - \dfrac{c_1}{b^2}\right) = -\dfrac{d_1}{b^2} - E_0 \end{cases}$$

Solving for the constants, where $\alpha = a^2(\epsilon_0 - \epsilon)^2 - b^2(\epsilon + \epsilon_0)^2$,

$$\begin{cases} a_1 = \dfrac{4E_0\epsilon_0\epsilon b^2}{\alpha} \\ \\ b_1 = \dfrac{2E_0\epsilon_0(\epsilon + \epsilon_0)b^2}{\alpha} \\ \\ c_1 = -\dfrac{2E_0\epsilon_0(\epsilon_0 - \epsilon)a^2b^2}{\alpha} \\ \\ d_1 = \dfrac{E_0(\epsilon_0^2 - \epsilon^2)b^2(b^2 - a^2)}{\alpha} \end{cases}$$

We can then substitute these into our formulas above to get the potentials,

$$\Phi_1 = \frac{4E_0\epsilon_0\epsilon b^2}{\alpha}\rho\cos(\phi)$$

$$\Phi_2 = \frac{2E_0\epsilon_0 b^2}{\alpha}\left(\rho + \frac{a^2}{\rho}\right)\cos(\phi)$$

$$\Phi_3 = E_0\left(\frac{b^2(\epsilon_0^2 - \epsilon^2)(b^2 - a^2)}{\alpha}\frac{1}{\rho} - E_0\rho\right)\cos(\phi)$$

To find the electric field, we use equation (1.22),

$$\vec{E} = -\frac{\partial \Phi}{\partial \rho}\hat{\rho} - \frac{1}{\rho}\frac{\partial \Phi}{\partial \phi}\hat{\phi}$$

For simplicity, I set

$$\beta = \frac{b^2(\epsilon_0^2 - \epsilon^2)(b^2 - a^2)}{\alpha}$$

$$\vec{E}_1 = -\left(\frac{4E_0\epsilon_0\epsilon b^2}{\alpha}\right)\left(\cos(\phi)\hat{\rho} - \sin(\phi)\hat{\phi}\right)$$

$$\vec{E}_2 = -\frac{2E_0\epsilon_0 b^2}{\alpha}\left[\left(1 - \frac{a^2}{\rho^2}\right)\cos(\phi)\hat{\rho} - \left(1 + \frac{a^2}{\rho^2}\right)\sin(\phi)\hat{\phi}\right]$$

$$\vec{E}_3 = -E_0\left[\left(-\frac{\beta}{\rho^2} - E_0\right)\cos(\phi)\hat{\rho} - \left(\frac{\beta}{\rho^2} - E_0\right)\sin(\phi)\hat{\phi}\right]$$

### 4.8.2 Sketching

**Sketch the lines of force for a typical case of $b \approx 2a$.**

---

The force is proportional to the electric field (1.4). I'm not going to sketch anything.

### 4.8.3 Limiting Forms

**Discuss the limiting forms of your solution appropriate for a solid dielectric cylinder in a uniform field, and a cylindrical cavity in a uniform dielectric.**

---

We can see the first case by letting $a \to 0$. The second case by substituting $\epsilon/\epsilon_0$ with $\epsilon_0/\epsilon$.

## 4.9 Method of Images: Dielectric Sphere

**A point charge $q$ is located in free space a distance $d$ from the center of a dielectric sphere of radius $a$ $(a < d)$ and dielectric constant $\epsilon/\epsilon_0$.**

---

### 4.9.1 Potential

**Find the potential at all points in space as an expansion in spherical harmonics.**

---

From our previous work with sphere and method of images (2.7), we want to place the original charge at $z = d$ and the image charge $q'$ at $z = a^2/d$. Outside the sphere, this gives a potential,

$$\Phi_{out} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{r} - d\hat{z}|} + \frac{q'}{\left|\vec{r} - \frac{a^2}{d}\hat{z}\right|}\right)$$

Because we have the azimuthal symmetry, we don't have to use spherical harmonics, we can expand $1/R$ using equation (3.20). Note that there is an implicit sum over $l$ in all of these equations,

$$\Phi_{out}(r > d) = \frac{1}{4\pi\epsilon_0}\left[q\frac{d^l}{r^{l+1}} + q'\frac{a^{2l}}{d^l r^{l+1}}\right] P_l(\cos(\theta))$$

$$\Phi_{out}(r < d) = \frac{1}{4\pi\epsilon_0}\left[q\frac{r^l}{d^{l+1}} + q'\frac{a^{2l}}{d^l r^{l+1}}\right] P_l(\cos(\theta))$$

Meanwhile, inside the sphere, the potential is similar to having a charge $q''$ at $z = d$,

$$\Phi_{in} = \frac{1}{4\pi\epsilon}\frac{q''}{|\vec{r} - d\hat{z}|} = \frac{q''}{4\pi\epsilon}\frac{r^l}{d^{l+1}} P_l(\cos(\theta))$$

Because we have no surface charge density, the normal boundary condition (4.12),

$$\epsilon\frac{\partial\Phi_{in}}{\partial r}\bigg|_{r=a} = \epsilon_0\frac{\partial\Phi_{out}}{\partial r}\bigg|_{r=a}$$

The tangential boundary condition (4.13),

$$\frac{\partial\Phi_{in}}{\partial\theta}\bigg|_{r=a} = \frac{\partial\Phi_{out}}{\partial\theta}\bigg|_{r=a}$$

We want to use $\Phi_{out}(r < d)$. This gives the system of equations,

$$lq'' = lq - (l+1)q'\frac{d}{a}$$

$$q'' = \frac{\epsilon}{\epsilon_0}\left(q + q'\frac{d}{a}\right)$$

Solving,

$$q' = \frac{qal}{d}\frac{\epsilon_0 - \epsilon}{\epsilon l + \epsilon_0 l + \epsilon_0}$$

$$q'' = q\epsilon\left(\frac{2l+1}{\epsilon l + \epsilon_0 l + \epsilon_0}\right)$$

Substituting these in, our potentials are

$$\Phi_{out}(r > d) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{d^l}{r^{l+1}}\left[1 + \frac{a^{2l+1}}{d^{2l+1}}\frac{\epsilon_0 - \epsilon}{\epsilon l + \epsilon_0 l + \epsilon_0}\right]P_l(\cos(\theta))$$

$$\Phi_{out}(r < d) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{d^{l+1}}\left[r^l + \frac{a^{2l+1}}{r^{l+1}}\frac{\epsilon_0 - \epsilon}{\epsilon l + \epsilon_0 l + \epsilon_0}\right]P_l(\cos(\theta))$$

$$\Phi_{in} = \frac{q}{4\pi}\sum_{l=0}^{\infty}\frac{2l+1}{\epsilon l + \epsilon_0 l + \epsilon_0}\frac{r^l}{d^{l+1}}P_l(\cos(\theta))$$

### 4.9.2 Electric Field

**Calculate the rectangular components of the electric field near the center of the sphere.**

---

We want to take the $l = 0$ and $l = 1$ terms. Higher order terms die since $r << d$,

$$\Phi_{in} = \frac{q}{4\pi\epsilon_0 d} + \frac{3qr\cos(\theta)}{4\pi(\epsilon + 2\epsilon_0)d^2}$$

Using equation (1.22),

$$\vec{E} = -\frac{3q}{4\pi(\epsilon + 2\epsilon_0)d^2}\hat{z}$$

## 4.10 Two Concentric Conducting Spheres

**Two concentric conducting spheres of inner and outer radii $a$ and $b$, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant $\epsilon/\epsilon_0$.**

---

### 4.10.1 Electric Field

**Find the electric field everywhere between the spheres.**

---

The electric field is given by

$$\vec{E} = \frac{B_0}{r^2}\hat{r}$$

When we factor in the dielectric (4.9),

$$\vec{D}_L = \epsilon_0 \frac{B_0}{r^2}\hat{r}$$

$$\vec{D}_R = \epsilon \frac{B_0}{r^2}\hat{r}$$

We know the total charge on the inside is $Q$,

$$Q = 2\pi \left( \int_{-1}^{1} (\vec{D}_R \cdot \hat{r}) r^2 \; dr d\cos(\theta) + \int_{0}^{1} (\vec{D}_R \cdot \hat{r}) \; dr d\cos(\theta) \right)$$

$$B_0 = \frac{Q}{2\pi(\epsilon + \epsilon_0)}$$

Substituting these back in,

$$\vec{E} = \frac{Q}{2\pi(\epsilon_0 + \epsilon) r^2}\hat{r}$$

$$\vec{D}_L = \frac{\epsilon_0 Q}{2\pi(\epsilon + \epsilon_0) r^2}\hat{r}$$

$$\vec{D}_R = \frac{\epsilon Q}{2\pi(\epsilon + \epsilon_0) r^2}\hat{r}$$

### 4.10.2 Surface-charge Distribution

**Calculate the surface-charge distribution on the inner sphere.**

---

Using equation (4.12),

$$\sigma = \vec{D} \cdot \hat{r}$$

$$\sigma_L = \vec{D}_L \cdot \hat{r}|_{r=a} = \frac{\epsilon_0 Q}{2\pi(\epsilon + \epsilon_0)a^2}$$

$$\sigma_R = \vec{D}_R \cdot \hat{r}|_{r=a} = \frac{\epsilon Q}{2\pi(\epsilon + \epsilon_0)a^2}$$

### 4.10.3 Polarization-Charge Density

**Calculate the polarization-charge density induced on the surface of the dielectric at $r = a$.**

---

Similarly, we have if the polarization is given by equation (4.8),

$$\vec{P} = (\epsilon - \epsilon_0)\vec{E}$$

$$\sigma_R = (\epsilon - \epsilon_0)\vec{E} \cdot \hat{r}|r = a$$

$$\sigma_R = \frac{(\epsilon - \epsilon_0)Q}{2\pi(\epsilon + \epsilon_0)r^2}$$

There is no polarization-charge density on the left side since there is no dielectric.