Jose Solutions

Benjamin D. Suh

August 27, 2019

Contents

Chapter 1

Fundamentals of Mechanics

1.1 Parabolic Motion

A gun is mounted on a hill of height h above a level plane. Neglecting air resistance, find the angle of elevation α for the greatest horizontal range at a given muzzle speed v. Find this range.

Since there is no horizontal force, the distance traveled is given by

$$d = vt\cos(\alpha)$$

In the vertical direction, we have gravity,

$$-h = vt\sin(\alpha) - \frac{1}{2}gt^2$$

Solving for t, and keeping the positive value,

$$t = \frac{v\sin(\alpha) + \sqrt{v^2\sin^2(\alpha) + 2gh}}{g}$$

Substituting into the distance,

$$d = \frac{v^2\sin(\alpha)\cos(\alpha) + v\cos(\alpha)\sqrt{v^2\sin^2(\alpha) + 2gh}}{g}$$

To maximize this value, we take the derivative according to α and set the result equal to 0. Note that we can divide by v^2/g to get rid of some constants,

$$\cos^2(\alpha) - \sin^2(\alpha) - \sin(\alpha)\sqrt{\sin^2(\alpha) = \frac{2gh}{v^2}} + \sin(\alpha)\cos^2(\alpha)\left(\sin^2(\alpha) + \frac{2gh}{v^2}\right)^{-1/2} = 0$$

Using the relation $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$ and multiplying through by the inverse root part,

$$\cos(2\alpha)\left(\sin^2(\alpha) + \frac{2gh}{v^2}\right)^{1/2} - \sin(\alpha)\left(\sin^2(\alpha) + \frac{2gh}{v^2}\right) + \sin(\alpha)\cos^2(\alpha) = 0$$

$$\cos(2\alpha)\left(\sin^2(\alpha) + \frac{2gh}{v^2}\right)^{1/2} = \sin(\alpha)\left(\frac{2gh}{v^2} - \cos(2\alpha)\right)$$

Using the relation $\sin^2(\alpha) = \frac{1}{2}(1 - \cos(2\alpha))$, we can solve for α,

$$\cos^2(2\alpha)\left[\sin^2(\alpha) + \frac{2gh}{v^2}\right] = \sin^2(\alpha)\left[\left(\frac{2gh}{v^2}\right)^2 - 2\frac{2gh}{v^2}\cos(2\alpha) + \cos^2(2\alpha)\right]$$

$$\cos^2(2\alpha) = \frac{2gh}{v^2}\sin^2(\alpha) - 2\cos(2\alpha)\sin^2(2\alpha)$$

$$2\cos^2(2\alpha) = \frac{2gh}{v^2} - \frac{2gh}{v^2}\cos(2\alpha) - 2\cos(2\alpha) + 2\cos^2(2\alpha)$$

$$\cos(2\alpha) = \frac{gh}{v^2 + gh}$$

To find the maximum range, we plug this back into our equation for the distance. It is easiest if we convert all of the trigonometric functions to $\cos(2\alpha)$. We use the relations, $\cos(\alpha) = \frac{1}{\sqrt{2}}\sqrt{1 + \cos(2\alpha)}$ and $\sin(\alpha) = \frac{1}{\sqrt{2}}\sqrt{1 - \cos(2\alpha)}$. We find that the maximum range is given by

$$R = \frac{v}{g}\sqrt{v^2 + 2gh}$$

1.2 Block on a Ramp

A mass m slides without friction on a plane tilted at an angle θ in a vertical uniform gravitational field g. The plane itself is on rollers and is free to move horizontally, also without friction; it has mass M. Find the acceleration A of the plane and the acceleration a of the mass m.

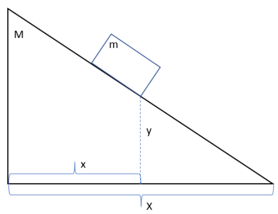

Figure 1.1:

We draw the block as in figure (1.1) with X being the position of the plane and x being the position of the block. From the diagram, we get a relation between the positions,

$$\tan(\theta) = \frac{y}{X - x}$$

$$X = x + y\cot(\theta)$$

Taking the derivative twice,

$$\ddot{X} = \ddot{x} + \ddot{y}\cot(\theta)$$

Since there are no external forces in the x-direction, we can use conservation of momentum. Since everything is initially at rest,

$$M\ddot{X} + m\ddot{x} = 0$$

From drawing free-body diagrams,

$$m\ddot{x} = N\sin(\theta)$$

$$m\ddot{y} = -mg + N\cos(\theta)$$

$$M\ddot{X} = -N\sin(\theta)$$

We can now solve for the necessary values,

$$m\ddot{y} = -mg - m\left(\frac{M}{m+M}\right)\cot^2(\theta)\ddot{y} = -mg$$

$$\ddot{y}\left(\frac{m + M + M\cot^2(\theta)}{m+M}\right) = -g$$

$$\ddot{y} = -g\frac{(m+M)\sin^2(\theta)}{m\sin^2(\theta) + M}$$

$$\ddot{x} = g\frac{M\sin(\theta)\cos(\theta)}{m\sin^2(\theta) + M}$$

$$\ddot{X} = -g\frac{m\sin(\theta)\cos(\theta)}{m\sin^2(\theta) + M}$$

1.3 Variations on a Rolling Cylinder

Imagine a hand pulling a circular cylindrical object (whose mass is distributed with cylindrical symmetry). The cylinder has a radius R, mass M, and moment of inertia I about its symmetry axis. The hand applies a force F by means of a weightless, flexible string. In all five cases find the acceleration A of the center of mass and the angular acceleration α of the cylindrical object; show explicitly that the work-energy theorem is satisfied.

1.3.1

Empty space, no gravity, the string passes through the center of mass of the cylinder.

In this case, we have no angular acceleration since there is no friction to cause the cylinder to roll. The only force is from the string,

$$A = \frac{F}{M}$$

The work done on this system is

$$W = Fx$$

$$= MA \cdot \frac{1}{2}At^2 = \frac{1}{2}M(At)^2 = \frac{1}{2}Mv^2$$

1.3.2

Empty space, no gravity, the string is wrapped around the cylinder. [Question: How can the hand, applying the same force as in Part (a) plus the extra rotational kinetic energy?]

This time, we have the same linear acceleration as before, but now we add an additional rotational component.

$$A = \frac{F}{M}$$

$$\alpha = \frac{FR}{I}$$

The work done in this system is given by

$$W = F(x + R\theta)$$

$$= MA \cdot \frac{1}{2}At^2 + R\frac{I\alpha}{R} \cdot \frac{1}{2}\alpha t^2$$

$$= \frac{1}{2}M(At)^2 + \frac{1}{2}I(\alpha t)^2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

I don't think the hint given in this problem is correct. I think what is trying to be asked is why does the wheel move the same distance for an equal amount of force. Normally what makes a wheel move is friction, but here we don't have that. Part of the translational energy from the previous part is turned into rotational energy, and that rotation becomes extra distance moved by the cylinder.

1.3.3

Uniform vertical gravitation sufficient, together with friction, to constrain the cylinder to roll without slipping on the surface. The string passes through the center of mass of the cylinder.

Now we introduce friction, which opposes the motion of the cylinder,

$$F - f = MA$$

$$fR = I\alpha$$

The condition for rolling without slipping is

$$\alpha R = A$$

We can solve for A and α,

$$A = \frac{FR^2}{MR^2 + I}$$

$$\alpha = \frac{FR}{MR^2 + I}$$

The work,

$$W = (F - f)x + fR\theta$$

$$= (F - f)\frac{1}{2}\frac{F - f}{M}t^2 + R\frac{I\alpha}{R}\frac{1}{2}I\alpha t^2$$

$$= \frac{1}{2}M\left(\frac{F - f}{M}t\right)^2 + \frac{1}{2}I(\alpha t)^2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

1.3.4

Same as part c, but with the string wrapped around the cylinder. [Question: In which direction is the frictional force? How does the hand manage to supply the necessary translation and kinetic energies different from part c?]

I think the answer here is similar to that for part b. The different force comes from applying it at a different point. If we had no friction, we would expect the wheel to rotate clockwise, so the frictional force must oppose that motion, and point counterclockwise.

$$F - f = MA$$

$$(F + f)R = I\alpha$$

Using the rolling without slipping condition,

$$A = \frac{2FR^2}{MR^2 + I}$$

$$\alpha = \frac{2FR}{MR^2 + I}$$

The work is given by

$$W = (F - f)x + (F - f)R\theta$$

$$= (F - f)\frac{1}{2}\left(\frac{F - f}{M}\right)t^2 + I\alpha\frac{1}{2}\alpha t^2 = \frac{1}{2}M\left(\frac{F - f}{M}t\right)^2 + \frac{1}{2}I(\alpha t)^2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

1.3.5

Same as part c, but with the string now wrapped around a shaft of radius $r < R$ within the cylinder (it's a kind of yo-yo). [Question: In which direction is the frictional force?]

This time, without friction, the wheel would rotate counter-clockwise, so the frictional force must go clockwise. Our equations are similar to before,

$$F - f = MA$$

$$fR - Fr = I\alpha$$

Using the rolling without slipping condition,

$$A = \frac{F(R^2 - rR)}{MR^2 + I}$$

$$\alpha = \frac{F(R-r)}{MR^2 + I}$$

The work is given by

$$W = (F - f)x + fR\theta - Fr\theta$$

$$= MA\frac{1}{2}At^2 + I\alpha\frac{1}{2}\alpha t^2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

1.4 Elastic Collision

A particle of mass m_1 makes an elastic (kinetic-energy conserving) collision with another particle of mass m_2. Before the collision m_1 has velocity $\vec{v}_1$ and m_2 is at rest relative to a certain inertial frame which we shall call the laboratory system. After the collision m_1 has velocity $\vec{u}_1$ making an angle θ with $\vec{v}_1$.

1.4.1 Lab Frame

Find the magnitude of $\vec{u}_1$

From conservation of momentum,

$$m_1\vec{v}_1 = m_1\vec{u}_1 + m_2\vec{u}_2$$

From conservation of energy,

$$m_1 v_1^2 = m_1 u_1^2 + m_2 u_2^2$$

Rearranging the conservation of momentum equation and squaring both sides. We do this since we know the dot product of $\vec{v}_1 \cdot \vec{u}_1$,

$$m_1\vec{v}_1 - m_1\vec{u}_1 = m_2\vec{u}_2$$

$$m_1^2 v_1^2 + m_1^2 u_1^2 - 2m_1^2 u_1 v_1 \cos(\theta) = m_2^2 u_2^2$$

Substituting this into the conservation of energy equation,

$$m_1 v_1^2 - m_1 u_1^2 - \frac{m_1^2}{m_2}(v_1^2 + u_1^2 - 2u_1 v_1 \cos(\theta)) = 0$$

$$u_1^2(m_1 m_2 + m_1^2) - 2m_1^2 u_1 v_1 \cos(\theta) + v_1^2(m_1^2 - m_1 m_2) = 0$$

$$u_1 = v_1 \frac{m_1 \cos(\theta) \pm \sqrt{m_2^2 - m_1^2 \sin^2(\theta)}}{m_1 + m_2}$$

We then think about the limiting case $m_2 = m_1$. Here, we expect that when the two balls collide, m_2 should continue moving in a straight line while m_1 stops moving. This is why you want to hit pool balls on the bottom since that gives the ball no spin and causes it to stop moving when it hits another ball (I think, I'm extremely poor at pool). Since we expect u_1 to vanish, this leads us to choose the negative,

$$u_1 = v_1 \frac{m_1 \cos(\theta) - \sqrt{m_2^2 - m_1^2 \sin^2(\theta)}}{m_1 + m_2}$$

1.4.2 CoM Frame

Relative to another inertial frame, called the center-of-mass system, the total linear momentum of the two-body system is zero. Find the velocity of the center-of-mass system relative to the laboratory system.

In the CoM frame,

$$v_1' = v_1 - V$$

$$v_2' = -V$$

From conservation of momentum,

$$m_1 v_1' + m_2 v_2' = 0$$

$$\frac{m_1}{m_1 + m_2} v_1 = V$$

1.4.3 Scattering Angle

Find the velocities $\vec{v}_1'$, $\vec{v}_2'$, $\vec{u}_1'$, $\vec{u}_2'$ of the two bodies before and after the collision in the center-of-mass system. Find the scattering angle θ' (the angle between $\vec{v}_1$' and $\vec{u}_1$' in terms of θ.

Solving for v_1' and v_2',

$$v_1' = \frac{m_2}{m_1 + m_2} v_1$$

$$v_2' = -\frac{m_1}{m_1 + m_2} v_1$$

From conservation of momentum,

$$m_1 u_1' = m_2 u_2' = 0$$

$$u_1' = u_1 - V$$

To find the scattering angle,

$$\cot(\theta') = \frac{u_{1x}'}{u_{1y}'} = \frac{u_{1x}}{u_{1y}} - \frac{V}{u_{1y}}$$

$$= \cot(\theta) - \frac{m_1}{M} \frac{v_1}{u_1 \sin(\theta)}$$

$$= \cot(\theta) - \frac{m_1}{\sin(\theta)\left(m_1 \cos(\theta) - \sqrt{m_2^2 - m_1^2 \sin^2(\theta)}\right)}$$

1.5 Center of Mass and Reduced Mass

Two masses m_1 and m_2 in a uniform gravitational field are connected by a spring of unstretched length h and spring constant k. The system is held by m_1 so that m_2 hangs down vertically, stretching the spring. At $t = 0$ both m_1 and m_2 are at rest, and m_1 is released, so that the system starts to fall. Set up a suitable coordinate system and describe the subsequent motion of m_1 and m_2.

In this system, we have two equations of motion. First, the center of mass falls as though it were just in the gravitational field,

$$\ddot{X} = -g$$

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

We also have the spring,

$$\mu \ddot{x} = -kx$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$x = x_2 - x_1$$

We recognize this as the simple harmonic oscillator,

$$x = A\cos(\omega t)$$

$$\omega^2 = k/\mu$$

Meanwhile, the center of mass gives

$$X = -\frac{1}{2}gt^2 + v_0 t + X_0$$

Our initial conditions are

$$\begin{cases} x_1(0) = 0 \\ x_2(0) = h + \dfrac{m_2 g}{k} \end{cases}$$

$$\dot{x}_1(0) = \dot{x}_2(0) = 0$$

From this,

$$A = h + \frac{m_2 g}{k}$$

$$v_0 = 0$$

$$\frac{m_1 x_1(0) + m_2 x_2(0)}{m_1 + m_2} = X_0$$

$$X_0 = \frac{m_2 \left(h + \frac{m_2 g}{k} \right)}{m_1 + m_2}$$

For convenience, let's define

$$\alpha = h + \frac{m_2 g}{k}$$

Solving for x_1 and x_2,

$$x_2 - x_1 = \alpha \cos(\omega t)$$

$$\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = -\frac{1}{2} g t^2 + \frac{m_2 \alpha}{m_1 + m_2}$$

$$x_1 = -\frac{1}{2} g t^2 + \frac{m_2 \alpha}{m_1 + m_2} (1 - \cos(\omega t))$$

$$x_2 = -\frac{1}{2} g t^2 + \frac{\alpha}{m_1 + m_2} (m_2 + m_1 \cos(\omega t))$$

1.6 Three Body Problem

The Earth and the Moon form a two-body system interacting through their mutual gravitational attraction. In addition, each body is attracted by the gravitational field of the Sun, which is an external force. Take the Sun as the origin and write down the equations of motion for the center of mass X and the relative position x of the Earth-Moon system. Expand the resulting expression in powers of x/X, the ratio of the magnitudes. Show that to lowest order in x/X the center of mass and relative position are uncoupled, but that in higher orders they are coupled because the Sun's gravitational force is not constant.

The equations of motion are

$$\ddot{\vec{X}} = \frac{m_e\ddot{\vec{r}}_e + m_m\ddot{\vec{r}}_m}{M}$$

$$\ddot{\vec{x}} = \ddot{\vec{r}}_m - \ddot{\vec{r}}_e$$

The equations of motion for the earth and moon are

$$\begin{cases} \ddot{\vec{r}}_e = -\dfrac{Gm_s}{r_e^3}\vec{r}_e + \dfrac{Gm_m}{x^3}\vec{x} \\ \\ \ddot{\vec{r}}_m = -\dfrac{Gm_s}{r_m^3}\vec{r}_m - \dfrac{Gm_e}{x^3}\vec{x} \end{cases}$$

Substituting these in,

$$\ddot{\vec{x}} = -Gm_s\left(\frac{\vec{r}_m}{r_m^3} - \frac{\vec{r}_e}{r_e^3}\right) - \frac{GM}{x^3}\vec{x}$$

$$\ddot{\vec{X}} = -\frac{Gm_s}{M}\left(\frac{m_e\vec{r}_e}{r_e^3} + \frac{m_m\vec{r}_m}{r_m^3}\right)$$

We now want to write $\vec{r}_e$ and $\vec{r}_m$ in terms of $\vec{x}$ and $\vec{X}$,

$$\vec{r}_e = \vec{X} - \frac{m_m}{M}\vec{x}$$

$$\vec{r}_m = \vec{X} + \frac{m_e}{M}\vec{x}$$

Now let's expand r_e^{-3},

$$r_e^{-3} = \left[\left(\vec{X} - \frac{m_m}{M}\vec{x}\right)\left(\vec{X} - \frac{m_m}{M}\vec{x}\right)\right]^{-3/2}$$

$$= \left[X^2 - 2\frac{m_m}{M}\vec{X}\cdot\vec{x} + \left(\frac{m_m}{M}\right)^2 x^2\right]^{-3/2}$$

Using the binomial approximation,

$$= \frac{1}{X^3}\left[1 - \frac{3}{2}\left(-\frac{2m_m}{M}\frac{\vec{X}\cdot\vec{x}}{X^2} + \left(\frac{m_m}{M}\right)^2\frac{x^2}{X^2}\right)\right]$$

Similarly,

$$r_m^{-3} = \frac{1}{X^3}\left[1 - \frac{3}{2}\left(-\frac{2m_e}{M}\frac{\vec{X}\cdot\vec{x}}{X^2} + \left(\frac{m_e}{M}\right)^2\frac{x^2}{X^2}\right)\right]$$

Substituting back in,

$$\ddot{\vec{x}} = -Gm_s\left[\vec{r}_m\left(\frac{1}{X^3} - \frac{3m_e}{M}\frac{\vec{X}\cdot\vec{x}}{X^2}\right) - \vec{r}_e\left(\frac{1}{X^3} + \frac{3m_m}{M}\frac{\vec{X}\cdot\vec{x}}{X^2}\right)\right] - \frac{GM\vec{x}}{x^3}$$

The last term is the term we want to keep,

$$\ddot{\vec{x}} \approx -\frac{GM}{x^3}\vec{x}$$

Similarly, we want to keep the first term in the expansion when solving for $\vec{X}$,

$$\vec{X} = -\frac{Gm_s}{M}\left(\frac{m_e\vec{r}_e}{X^3} + \frac{m_m\vec{r}_m}{X^3}\right)$$

$$= -\frac{Gm_s}{X^3}\vec{X}$$

1.7 Oscillating Particle

Show that a one-dimensional particle subject to the force $F = -kx^{2n+1}$, where n is an integer, will oscillate with a period proportional to A^{-n}, where A is the amplitude. Pay special attention to the case of $n \leq 0$.

The potential (1.10) corresponding to this force is

$$V = \frac{k}{2n+2} x^{2n+2}$$

From the text, the period is given by

$$P = \sqrt{2m} \int_{-A}^{A} \frac{dx}{\sqrt{E-V}}$$

We know that the energy is equal to the maximum value of the potential, i.e., $E = V(A)$. Setting $k/(2n+2) = \alpha$,

$$P = \sqrt{\frac{2m}{\alpha}} \int_{-A}^{A} \frac{dx}{\sqrt{A^{2n+2} - x^{2n+2}}}$$

We want to set $u = x/A$,

$$= \sqrt{\frac{2m}{A}} \frac{1}{\sqrt{A^{2n+2}}} \int_{-1}^{1} \frac{A\ dx}{\sqrt{1 - u^{2n+2}}}$$

$$= \sqrt{\frac{2m}{A}} \frac{1}{A^n} \int_{-1}^{1} \frac{dx}{\sqrt{1 - u^{2n+2}}}$$

The integral doesn't depend on A, so the period goes by A^{-n}.

This has the potential to fall apart when $n = -1$. In this case, the potential is given by $V = k \ln(x)$. The period is given by

$$P = \sqrt{\frac{2m}{k}} \int_{-A}^{A} \frac{dx}{\sqrt{\ln(A) - \ln(x)}}$$

$$= \sqrt{\frac{2m}{k}} \int_{-A}^{A} \frac{dx}{\sqrt{-\ln(x/A)}}$$

Setting $u = x/A$,

$$P = \sqrt{\frac{2m}{k}} A \int_{-1}^{1} \frac{du}{\sqrt{-\ln(u)}}$$

Once again the integral does not depend on A, so the period goes by A as expected.

1.8 Yo-Yo motion

A yo-yo consists of two disks of mass M and radius R connected by a shaft of mass m and radius r; a weightless string is wrapped around the shaft.

1.8.1 Fixed End

The free end of the string is held stationary in the Earth's gravitational field. Assuming that the string starts out vertical, find the motion of the yo-yo's center of mass.

The total energy of the system is given by

$$\frac{1}{2}(m+2M)v^2 + \frac{1}{2}I\omega^2 - (m+2M)gx = E$$

We define $\mu = m + 2M$. Since the total energy of the system should be constant, we can take the time derivative,

$$\mu va + I\alpha\omega - \mu gv = 0$$

Using

$$\begin{cases} v = \omega r \\ a = \alpha r \end{cases}$$

Substituting,

$$\mu va + Iavr^2 - \mu gv = 0$$

$$a = \frac{\mu g}{\mu + Ir^2}$$

1.8.2 Unfixed End

The free end is moved so as to keep the yo-yo's center of mass stationary. Describe the motion of the free end of the string and the rotation of the yo-yo

---

The end of the string must accelerate with the yo yo,

$$a = \frac{\mu(g+a)}{\mu + Ir^2}$$

$$a = \frac{\mu g}{1 - \mu}$$

1.8.3 No Gravity

The yo-yo is transported to empty space, where there is no gravitational field, and a force F is applied to the free end of the string. Describe the motion of the center of mass of the yo-yo, the yo-yo's rotation, and the motion of the free end of the string.

In this case, the only force on the yo-yo is from the force on the string,

$$\begin{cases} a = \dfrac{F}{\mu} \\ \\ \alpha = \dfrac{Fr}{I} \end{cases}$$

$$A = a + \alpha r = \frac{F}{\mu} + \frac{Fr^2}{I}$$

1.9 Terminal Velocity

A particle in a uniform gravitational field experiences an additional retarding force $\vec{F} = -\alpha\vec{v}$, where $\vec{v}$ is its velocity. Find the general solution to the equations of motion and show that the velocity has an asymptotic value (called the terminal velocity). Find the terminal velocity.

We start by writing the force as

$$\vec{F} = m\vec{g} - \alpha\vec{v}$$

The equations of motion can be found by solving equation (1.5), i.e.,

$$m\ddot{x} = mg - \alpha\dot{x}$$

We can turn this from a second-order differential equation to a first-order differential equation,

$$m\dot{v} = mg - \alpha v$$

$$m\frac{dv}{dt} = mg - \alpha v$$

$$\left[\frac{dv}{dt} + \frac{\alpha}{m}v = g\right]\exp\left(\frac{\alpha t}{m}\right)$$

$$\frac{d}{dt}\left[\exp\left(\frac{\alpha t}{m}\right)v\right] = g\exp\left(\frac{\alpha t}{m}\right)$$

$$v = \frac{mg}{\alpha} + c\exp\left(-\frac{\alpha t}{m}\right)$$

Setting the initial velocity to v_0,

$$v = \frac{mg}{\alpha} - \left(\frac{mg}{\alpha} - v_0\right)\exp\left(-\frac{\alpha t}{m}\right)$$

We see that as time goes to infinity, the exponential term dies, which means the velocity has some asymptotic behaviour. Further, we see that the velocity maxes out at $v = \frac{mg}{\alpha}$.

For the position, we integrate the above and set $x(0) = x_0$,

$$x = \frac{mg}{\alpha}t + \left(\frac{m}{\alpha}\right)\left(\frac{mg}{\alpha} - v_0\right)\left[\exp\left(-\frac{\alpha t}{m}\right) - 1\right] + x_0$$

1.10 Distance

Change the variable of integration in the given equation from s to any other parameter in order to show that the distance between two points on the trajectory (given equation) is indeed independent of the parameter.

$$l(s_0, s_1) = \int_{s_0}^{s_1} \left(\frac{dx_i}{ds} \frac{dx_i}{ds} \right)^{1/2} ds$$

We'll perform a change of variables by changing $s \to \alpha$. Further, we say that $\alpha(s_0) = \alpha_0$ and $\alpha(s_1) = \alpha_1$.

$$l = \int_{\alpha_0}^{\alpha_1} \left(\frac{dx_i}{d\alpha} \frac{d\alpha}{ds} \frac{dx_i}{d\alpha} \frac{d\alpha}{ds} \right)^{1/2} \frac{ds}{d\alpha} d\alpha$$

$$= \int_{\alpha_0}^{\alpha_1} \left(\frac{dx_i}{d\alpha} \frac{dx_i}{d\alpha} \right)^{1/2} \frac{d\alpha}{ds} \frac{ds}{d\alpha} \, d\alpha$$

$$= \int_{\alpha_0}^{\alpha_1} \left(\frac{dx_i}{d\alpha} \frac{dx_i}{d\alpha} \right)^{1/2} d\alpha$$

Which is the same form as the equation in the problem.

1.11 Curvature

1.11.1 Curvature

The concept of curvature and radius of curvature are defined by extending those concepts from circles to curves in general. The curvature κ is defined, as in

$$\lim_{t_1 \to t_2} \frac{|\vec{\tau}(t_1) - \vec{\tau}(t_2)|}{|l(t_1) - l(t_2)|} = \left| \frac{d\vec{\tau}}{dl} \right| = \kappa$$

as the rate (with respect to length along the curve) of rotation of the tangent vector. Show that what the above equation defines is in fact the rate of rotation of $\vec{\tau}$ (i.e., that it gives the rate of change of the angle $\vec{\tau}$ makes with a fixed direction). Show also that for a circle in the plane $\kappa = 1/R$, where R is the radius of the circle.

We start by taking the scalar product of $\vec{\tau}$ and a normal vector to the curve,

$$\frac{d}{dt}(\vec{\tau} \cdot \hat{N}) = \left| \frac{d\vec{\tau}}{dl} \right| \cos(\phi) = -\sin(\theta) \frac{d\theta}{dl}$$

Using $\cos(\phi) = \sin(\theta)$,

$$\kappa = \left| \frac{d\hat{\tau}}{dl} \right| = \frac{d\theta}{dl}$$

For a circle, $l = R\theta$.

1.11.2 Frenet Formulas

Derive the second of the Frenet formulas from the fact that $\hat{\tau}$, $\hat{n}$, and $\hat{B}$ are a set of orthogonal unit vectors and from the definition of θ.

The second Frenet formula is

$$\dot{\hat{n}} = -\kappa \dot{l} \vec{\tau} + \theta \dot{l} \hat{B}$$

We'll start by looking at the time derivative of $\hat{B}$.

$$\dot{\hat{B}} = \dot{\hat{\tau}} \times \hat{n} + \hat{\tau} \times \dot{\hat{n}}$$

The first term dies because of the first Frenet formula,

$$\dot{\hat{\tau}} = \kappa \dot{l} \hat{n}$$

We end up with

$$-\theta \dot{l} \hat{n} = \hat{\tau} \times \dot{\hat{n}}$$

If we look at the term $\theta\dot{l}\hat{B}$,

$$\theta\dot{l}\hat{B} = \theta\dot{l}(\hat{\tau} \times \hat{n})$$

$$= -\theta\dot{l}(\hat{\tau} \times (\hat{\tau} \times \dot{\hat{n}}))$$

Using the vector triple product (BACCAB),

$$= \theta\dot{l}[\hat{\tau}(\hat{\tau} \cdot \dot{\hat{n}} - \dot{\hat{n}}(\hat{\tau} \times \hat{\tau})]$$

$\hat{\tau}$ and $\dot{\hat{n}}$ are in the same direction,

$$\dot{\hat{n}} = -\kappa\dot{l}\hat{\tau} + \theta\dot{l}\hat{B}$$

1.12 Particle on an Ellipse

A particle is constrained to move at constant speed on the ellipse $a_{ij}x^i x^j = 1(i, j = 1, 2)$. Find the Cartesian components of its acceleration as a function of position on the ellipse.

The acceleration is given by

$$a = \frac{v^2}{R}$$

$$R = \frac{(1 + y'^2)^{3/2}}{y''}$$

The individual components are given by

$$a_x = a\sin(\theta) = ay'(1 + y'^2)^{-1/2}$$

$$a_y = a\cos(\theta) = a(1 + y'^2)^{-1/2}$$

If y is given by, we can find the derivatives,

$$y = \frac{B}{A}\left(A^2 - x^2\right)^{1/2}$$

$$y' = -\frac{B}{A}x\left(A^2 - x^2\right)^{-1/2}$$

$$y'' = -\frac{BA}{\left(A^2 - x^2\right)^{3/2}}$$

We can now solve for the acceleration components,

$$a_x = \frac{v^2 y' y''}{(1 + y'^2)^2}$$

$$= \frac{v^2 A^4 B^2 x}{[A^4 + x^2(B^2 - A^2)]^2}$$

$$a_y = \frac{a_x}{y'} = -\frac{v^2 A^5 B^2 (A^2 - x^2)^{1/2}}{[A^4 + x^2(B^2 - A^2)]^2}$$

1.13 Existence of Mass

Show that if

$$\mu_{12}\mu_{23}\mu_{31} = 1$$

is satisfied, there exists constants such that equations

$$\vec{v}_1(t) + \mu_{12}\vec{v}_2(t) = \vec{K}$$

$$\vec{v}_2(t) + \mu_{23}\vec{v}_3(t) = \vec{L}$$

$$\vec{v}_3(t) + \mu_{31}\vec{v}_1(t) = \vec{M}$$

can be put in the form of

$$m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{P}_{12}$$

$$m_2\vec{v}_2 + m_3\vec{v}_3 = \vec{P}_{23}$$

$$m_3\vec{v}_3 + m_1\vec{v}_1 = \vec{P}_{31}$$

We'll start with

$$\vec{v}_1 + \mu_{12}\vec{v}_2 = \vec{K}$$

We want to choose a μ_{12} such that we get

$$m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{P}_{12}$$

One such choice is

$$\mu_{12} = \frac{m_2}{m_1}$$

$$\vec{P}_{12} = m_1\vec{K}$$

We can do similar,

$$\mu_{23} = \frac{m_3}{m_2}$$

$$\mu_{31} = \frac{m_1}{m_3}$$

1.14 Non-inertial Frames

Consider equation

$$y_i = f_i(x, t)$$

$$x_i = g_i(y, t)$$

in two rather than three dimensions, and assume that the $\vec{x}$ and $\vec{y}$ coordinates are not both inertial, but rotating with respect to each other:

$$y_1 = x_1 \cos(\omega t) - x_2 \sin(\omega t)$$

$$y_2 = x_1 \sin(\omega t) + x_2 \cos(\omega t)$$

Show that in general even if the $\vec{x}$ acceleration vanishes, the $\vec{y}$ acceleration does not. Find $\ddot{\vec{y}}$ for $\ddot{\vec{x}} = 0$, but $\dot{\vec{x}} \neq 0$ and $\vec{x} \neq 0$. Give the physical significance of the terms you obtain.

We can write the given conditions in matrix notation,

$$|y\rangle = A|x\rangle$$

where

$$A = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

Following the prescription in the text,

$$|\ddot{y}\rangle = \ddot{A}|x\rangle + 2\dot{A}|\dot{x}\rangle + A|\ddot{x}\rangle$$

From this, we can see that $|\ddot{y}\rangle$ does not vanish if $|\ddot{x}\rangle$ does. We can find $|\ddot{y}\rangle$ by taking the necessary time derivatives. As we will see in a later chapter, the first term corresponds to centripetal acceleration, and the second corresponds to coriolis force.

$$\dot{A} = \begin{bmatrix} -\omega \sin(\omega t) & -\omega \cos(\omega t) \\ \omega \cos(\omega t) & -\omega \sin(\omega t) \end{bmatrix}$$

$$\ddot{A} = \begin{bmatrix} -\omega^2 \cos(\omega t) & \omega^2 \sin(\omega t) \\ -\omega^2 \sin(\omega t) & -\omega^2 \cos(\omega t) \end{bmatrix}$$

1.15 Particle in a Force Field

A particle of mass m moves in one dimension under the influence of the force

$$F = -kx + \frac{a}{x^3}$$

Find the equilibrium points, show that they are stable, and calculate the frequencies of oscillation about them. Show that the frequencies are independent of the energy.

The equilibrium points can be found by taking the first derivative of potential and setting that equal to 0. However, we're given force, which is already the negative derivative of the potential (1.10). All we need to do is set the given force equal to 0 and solve for x.

$$x = \pm \left(\frac{a}{k}\right)^{1/4}$$

To determine if these are stable, we take the second derivative of the potential (or the first derivative of the force and negatify it), plug in the equilibrium points and see if the result is positive.

$$-\frac{dF}{dx} = k + \frac{3a}{x^4}$$

Substituting in the equilibrium points, we see that this is positive as long as a and k have the same sign. Different signs would give imaginary solutions.

To find the period of oscillation, we substitute $x \to x + \epsilon$,

$$m\ddot{x} = -kx + \frac{a}{x^3}$$

$$m(\ddot{x} + \ddot{\epsilon}) - k(x + \epsilon) + \frac{a}{(x+\epsilon)^3}$$

Using the binomial approximation on the second term,

$$= -kx - \epsilon k + ax^{-3}\left(1 - 3\frac{\epsilon}{x}\right)$$

$$m\ddot{\epsilon} = -\epsilon k - 3\frac{\epsilon}{x^4}$$

Substituting in the equilibrium point,

$$m\ddot{\epsilon} = -4k\epsilon$$

We recognize this as the simple harmonic oscillator, so the frequency is

$$\omega = 2\sqrt{\frac{k}{m}}$$

The frequency does not depend on the position, so it does not depend on the energy.

1.16 Center of Mass

Consider a system of particles made up of K subsystems, each itself a system of particles. Let M_I be the mass and $\vec{X}_I$ the center of mass of the Ith subsystem. Show that the center of mass of the entire system is given by an equation similar to Ith subsystem. Show that the center of mass of the entire system is given by an equation similar to

$$\vec{X} = \frac{1}{M}\sum_i m_i \vec{x}_i$$

but with m_i and $\vec{x}_i$ replaced by M_I and $\vec{X}_I$ and the sum taken from $I = 1$ to $I = K$.

The center of mass of each subsystem is given by

$$\vec{X}_I = \frac{1}{M_I}\sum_i m_i \vec{x}_i$$

To get the total center of mass,

$$\vec{X} = \frac{1}{M}\sum_I \sum_i m_i \vec{x}_i$$

$$= \frac{1}{M}\sum_I M_I \vec{X}_I$$

1.17 Kinetic Energy, Center of Mass

Express the total kinetic energy of a system of N particles in terms of their center of mass and the relative positions of the particles [i.e., derive]

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\sum_i m_i \dot{y}_i^2$$

$$\vec{y}_i = \vec{x}_i - \vec{X}$$

Extend the result to a continuous distribution of particles with mass density $\rho(\vec{x})$.

The kinetic energy of a system of particles is given by

$$T = \frac{1}{2}\sum_i m_i \dot{x}^2 = \frac{1}{2}\sum_i m_i \dot{y}_i^2 + \frac{1}{2}\sum_i m_i \dot{\vec{y}}_i \cdot \vec{X} + \frac{1}{2}\sum_i m_i \dot{X}^2$$

The second term, we can kill by looking at the definition of center of mass,

$$\vec{X} = \frac{1}{M}\sum_i m_i \vec{x}_i = \frac{1}{M}\sum_i m_i(\vec{y}_i + \vec{X})$$

$$= \vec{X} + \frac{1}{M}\sum_i m_i \vec{y}_i$$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\sum_i m_i \dot{y}_i^2$$

To convert to continuous, we look at the second term and turn the sum into an integral,

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\int \rho \dot{y}^2 \ d^3x$$

1.18 Internal Forces

In deriving

$$\vec{N}_z = \dot{\vec{L}}_z$$

we assumed that the internal forces do not contribute to the total torque on a system of particles. Show explicitly that if for each i and j the internal force $\vec{F}_{ij}$ lies along the line connecting the ith and jth particles, then the internal forces indeed do not contribute to the total torque.

The definition of torque is

$$\vec{N} = \sum \vec{z}_i \times \vec{F}_i$$

The internal torque is given by

$$\vec{N}_{int} = \sum_{ij} \vec{z}_{ij} \times \vec{F}_{ij} = \frac{1}{2} \sum_{ij} \left(\vec{z}_i \times \vec{F}_{ij} + \vec{z}_j \times \vec{F}_{ji} \right)$$

Using Newton's third law,

$$= \frac{1}{2} \sum_{ij} (\vec{z}_i - \vec{z}_j) \times \vec{F}_{ij}$$

Since $\vec{z}_i - \vec{z}_j$ is in the same direction as $\vec{F}_{ij}$, it dies.

1.19 1.19

1.20 Stable Equilibrium of a Potential

A particle of mass m moves along the x axis under the influence of the potential

$$V(x) = V_0 x^2 e^{-ax^2}$$

where V_0 and $a > 0$ are constants. Find the equilibrium points of the motion.

We find the equilibrium points by taking the first derivative of potential and setting it equal to 0.

$$\frac{dV}{dx} = 2V_0 x e^{-ax^2} - 2V_0 a x^3 e^{-ax^2} = 0$$

$$x = \pm\sqrt{\frac{1}{a}}$$

1.21 1.21

1.22 1.22

1.23 1.23

1.24 1.24

1.25 1.25

1.26 Cross Product

Derive equation

$$\ddot{\vec{y}} = \vec{\omega} \times (\vec{\omega} \times \vec{y}) + 2\vec{\omega} \times \dot{\vec{y}}$$

We prove this by using

$$\vec{y} = (y_1, y_2, y_3)$$

$$\omega = (0, 0, \omega)$$

Substituting these into the above, we return equation 1.84.

$$\begin{cases} \ddot{y}_1 = \omega^2 y_1 - 2\omega \dot{y}_2 \\ \ddot{y}_2 = \omega^2 y_2 + 2\omega \dot{y}_1 \\ \ddot{y}_3 = 0 \end{cases}$$

1.27 1.27