c. Holding your fists near each other produces no perceptible repulsion. Use this fact to place an upper bound on the charge difference between an electron and a proton.

- $F = \dfrac{1}{4\pi\varepsilon_0} \cdot \dfrac{q\,q}{r^2}$

$r = 10^{-2}\ m$

$\varepsilon = 10^{-12}\ F/m$

Say you can notice 10^{-1} N

$10^{-1} = \dfrac{1}{10^{-11}} \cdot \dfrac{q^2}{10^{-4}}$ $\quad q = q_0 + \varepsilon$

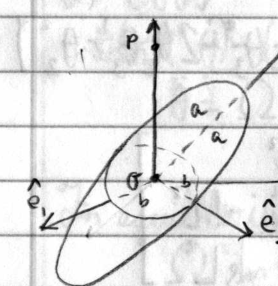
$10^{14} = 10^{-19}\,\varepsilon$

"I believe in my townspeople. You can knock on any door in our town say, 'I'm hungry,' and you will be fed. Our town is no exception; I've found the same ready charity everywhere. For the one who says, 'To heck with you – I got mine,' there are a hundred, a thousand, who will say, 'Sure, pal, sit down.'" – Robert Heinlein

2013-2014

1. A bacterium in the shape of a spheroid with principal axes b, b, a and uniform density is spinning in free space about its axis of symmetry $\hat{e}_3$ with angular velocity ω_r. The symmetry axis of the bacterium is inclined at angle θ with respect to an axis OP fixed in space, and precesses around it with angular velocity ω_p. The moments of inertia of a spheroid of uniform density ρ_0 are given by

$$I_{11} = I_{22} = \tfrac{4\pi}{15}\rho_0 a b^2 (a^2 + b^2)\ ;\ I_{33} = \tfrac{8\pi}{15}\rho_0 b^4 a$$



a. Determine ω_p

•

2.* Three identical pendulums of length L and mass m near the Earth's surface are connected by springs with constant k. The masses are constrained to move in a plane. Find the normal frequencies for small oscillations of the masses.

· $T = \frac{m}{2} L^2 (\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2)$

$V = -mgL(\cos\theta_1 + \cos\theta_2 + \cos\theta_3) + \frac{1}{2}k[(x_2 - x_1)^2 + (x_3 - x_2)^2]$

$x_1 = L\sin\theta_1 \approx L\theta_1$

$x_2 = L\sin\theta_2 \approx L\theta_2$

$x_3 = L\sin\theta_3 \approx L\theta_3$

$V = -mgL(\cos\theta_1 + \cos\theta_2 + \cos\theta_3) + \frac{k}{2}L^2(\theta_2^2 - 2\theta_1\theta_2 + \theta_1^2 + \theta_3^2 - 2\theta_2\theta_3 + \theta_2^2)$

$$T = \begin{bmatrix} mL^2 & & \\ & mL^2 & \\ & & mL^2 \end{bmatrix} \qquad V = \begin{bmatrix} kL^2 & -kL^2 & \\ -kL^2 & 2kL^2 & -kL^2 \\ & -kL^2 & kL^2 \end{bmatrix}$$

$$\det(V - \lambda T) = \det\begin{bmatrix} kL^2 - \lambda mL^2 & -kL^2 & \\ -kL^2 & 2kL^2 - \lambda mL^2 & -kL^2 \\ & -kL^2 & kL^2 - \lambda mL^2 \end{bmatrix}$$

$= (kL^2 - \lambda mL^2)[(2kL^2 - \lambda mL^2)(kL^2 - \lambda mL^2) - k^2L^4]$

$+ kL^2(-kL^2)(kL^2 - \lambda mL^2) = 0$

$(k - \lambda m)[(2k - \lambda m)(k - \lambda m) - 2k^2] = 0$

$(k - \lambda m)[\lambda^2 m^2 - 3k\lambda m] = 0$

$(k - \lambda m)\lambda m(\lambda m - 3k) = 0$

$\omega = 0, \sqrt{k/m}, \sqrt{3k/m}$

3.* A bead of mass m is constrained to slide on a circular wire of radius R oriented vertically near the surface of the Earth. The wire is rotating about the $\hat z$ axis at a frequency Ω.

a. Write the Lagrangian in terms of the variables θ and $\dot\theta$

· $\mathcal{L} = \frac{m}{2}(R^2\dot\theta^2 + R^2\Omega^2\sin^2\theta) + mgR\cos\theta$

b. Derive the equation of motion and determine the equilibrium angle θ_0 at which the mass can be stationary in θ.

· $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$

$\frac{d}{dt}(mR^2\dot\theta) - (mR^2\Omega^2\sin\theta\cos\theta - mgR\sin\theta) = 0$

$\ddot\theta = \Omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta$

In equilibrium, $\ddot\theta = \dot\theta = 0$

$\Omega^2\cos\theta = g/R$

$\theta_0 = \cos^{-1}(g/R\Omega^2)$

c. Calculate the frequency of small oscillations around θ_0

· $(\ddot\theta + \ddot\varepsilon) = \Omega^2\sin(\theta_0+\varepsilon)\cos(\theta_0+\varepsilon) - \frac{g}{R}\sin(\theta_0+\varepsilon)$

$= (\Omega^2\cos(\theta_0+\varepsilon) - g/R)\sin(\theta_0+\varepsilon)$

$\ddot\varepsilon = [\Omega^2(\cos\theta_0 - \varepsilon\sin\theta_0) - g/R](\sin\theta_0 + \varepsilon\cos\theta_0)$

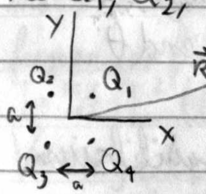
$= [-\varepsilon\Omega^2\sin\theta_0](\sin\theta_0 + \varepsilon\cos\theta_0)$

$= -\varepsilon\Omega^2\sin^2\theta_0$

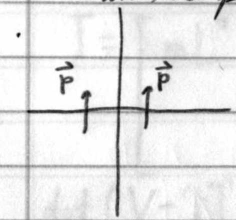
$= -\varepsilon\cdot\Omega^2\cdot(1 - \cos^2\theta_0) = -\varepsilon\cdot\Omega^2(1 - g^2/R^2\Omega^4)$

$\omega = \Omega\sqrt{1 - \frac{g^2}{R^2\Omega^4}}$

4*. Consider a distribution of four charges located at the corners of a square of side a that is centered at the origin. The charges are Q_1, Q_2, Q_3, and Q_4 as shown in the accompying figure



a. Calculate the electrostatic potential at the point P a distance R from the origin, with $R \gg a$, to lowest non-zero order in a/R with $Q_1 = Q_2 = +q$, $Q_3 = Q_4 = -q$. (Note that P does not necessarily lie in the plane of the square).



$\vec{p} = (0, qa, 0)$

$$\Phi = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\vec{p}\cdot\vec{r}}{r^3} \qquad \vec{r}_1 = (x - a/2, y, z)$$

where $x^2 + y^2 + z^2 = R^2$

$$\Phi = \frac{qa}{4\pi\varepsilon_0}\left[\frac{x - a/2}{((x-a/2)^2 + y^2 + z^2)^{3/2}} + \frac{x + a/2}{((x+a/2)^2 + y^2 + z^2)^{3/2}}\right]$$

$$= \frac{qa}{4\pi\varepsilon_0}\left[\frac{x - a/2}{(R^2 - ax + a^2/4)^{3/2}} + \frac{x + a/2}{(R^2 + ax + a^2/4)^{3/2}}\right]$$

$$= \frac{qa}{4\pi\varepsilon_0 R^3}\left[\frac{x - a/2}{(1 - ax/R^2 + a^2/4R^2)^{3/2}} + \frac{x + a/2}{(1 + ax/R^2 + a^2/4R^2)^{3/2}}\right]$$

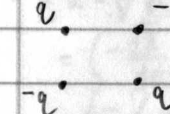
$$\simeq \frac{qa}{4\pi\varepsilon_0 R^3}\left[(x - a/2)\left(1 - 3/2\left(-ax/R^2 + a^2/4R^2\right)\right) + (x + a/2)\left(1 - 3/2\left(ax/R^2 + a^2/4R^2\right)\right)\right]$$

$$= \frac{qa}{4\pi\varepsilon_0 R^3}\left[\begin{array}{l} x - 3x/2\left(-\cancel{ax/R^2} + a^2/4R^2\right) - \cancel{a/2} + 3a/4\left(-ax/R^2 + \cancel{a^2/4R^2}\right) \\ + x - 3x/2\left(\cancel{ax/R^2} + a^2/4R^2\right) + \cancel{a/2} - 3a/4\left(ax/R^2 + \cancel{a^2/4R^2}\right)\end{array}\right]$$

$$= \frac{qa}{4\pi\varepsilon_0 R^3}\left[2x - 3x\left(\frac{a^2}{4R^2}\right) - \frac{3a^2x}{4R^2}\right]$$

$$= \frac{qa\,x}{2\pi\varepsilon_0 R^3}\left[1 - \frac{3}{4}\frac{a^2}{R^2}\right]$$

b. Calculate the electrostatic potential at the point P a distance R from the origin, with $R \gg a$, to lowest non-zero order in a/R, when $Q_1 = Q_3 = -q$, $Q_2 = Q_4 = +q$. (Again, P is not necessarily in the plane of the square).



$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5}\right]$$

$$Q_{ij} = \int (3x_i' x_j' - r^2\delta_{ij})\rho(\vec{x}')\, d^3x'$$

$$Q_{xy} = \int 3x'y'\rho(\vec{x}')\, d^3x'$$

$$= 3/2 \cdot q a^2$$

$$\Phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{2}\left(\frac{3qa^2\,xy \cdot 2}{2r^5}\right)\right] = \frac{3qa^2xy}{8\pi\varepsilon_0(x^2+y^2+z^2)^{5/2}}$$

5. A uniform dielectric sphere with dielectric constant ε and radius R has a surface charge density $\sigma = \sigma_0 \cos^2\theta$ placed on its surface, where θ is the polar angle of a point on the sphere measured from the north pole. Outside the sphere is vacuum with dielectric constant ε_0.

a. Find the electrostatic potential inside the sphere.

$$\Phi = [A_\ell r^\ell + B_\ell r^{-(\ell+1)}] P_\ell(\cos\theta)$$

$$V_{out} = V_{in} \Big|_{r=R}$$

$$\varepsilon \frac{\partial V_{in}}{\partial r} - \varepsilon_0 \frac{\partial V_{out}}{\partial r} = \sigma_0 \cos^2\theta \Big|_{r=R}$$

$$\Phi_{in} = A_2 r^2 P_2(\cos\theta) = A_2 r^2 \cdot \tfrac{1}{2}(3\cos^2\theta - 1)$$

$$\Phi_{out} = B_2 r^{-3} P_2(\cos\theta) = B_2 r^{-3} \cdot \tfrac{1}{2}(3\cos^2\theta - 1)$$

$$A_2 R^2 = B_2 R^{-3}$$

$$A_2 R^5 = B_2$$

$$\varepsilon A_2 R(3\cos^2\theta - 1) + \varepsilon_0 \tfrac{3}{2} B_2 R^{-4}(3\cos^2\theta - 1) = \sigma_0 \cos^2\theta$$

$$3\varepsilon A_2 R^5 \cos^2\theta - \varepsilon A_2 R^5 + \tfrac{9\varepsilon_0}{2} B_2 \cos^2\theta - \tfrac{3\varepsilon_0}{2} B_2 = \sigma_0 \cos^2\theta$$

$$6\varepsilon A_2 R^5 + 9\varepsilon_0 B_2 = 2\sigma_0$$

$$6\varepsilon B_2 + 9\varepsilon_0 B_2 = 2\sigma$$

$$B_2 = \frac{2\sigma}{6\varepsilon + 9\varepsilon_0}$$

$$A_2 = \frac{2\sigma}{R^5(6\varepsilon + 9\varepsilon_0)}$$

$$\Phi_{in} = \frac{\sigma r^2 (3\cos^2\theta - 1)}{R^5(6\varepsilon + 9\varepsilon_0)}$$

b. Find the electrostatic potential outside the sphere

$$\Phi = \frac{\sigma(3\cos^2\theta - 1)}{r^3(6\varepsilon + 9\varepsilon_0)}$$

c. Calculate the electric field inside the sphere.

$$E = -\nabla\Phi$$

$$= \left(-\frac{\partial\Phi}{\partial r}, -\frac{1}{r}\frac{\partial\Phi}{\partial\theta}, 0\right)$$

$$= \left(\frac{-2\sigma r(3\cos^2\theta - 1)}{R^5(6\varepsilon + 9\varepsilon_0)}, -\frac{2\sigma r \cos\theta \sin\theta}{R^5(6\varepsilon + 9\varepsilon_0)}, 0\right)$$

6*. Assume the Earth's atmosphere is an ideal gas of particles with molecular weight μ in static equilibrium in the Earth's gravitational field.

a. Assuming the atmosphere is in mechanical equilibrium, derive the relationship between an infinitesimal change in pressure dP and a change in height dZ.

$$A + PV = \mu N$$

$$E - TS + PV = \mu N$$

$$dE + VdP = (mg(z + dz))N$$

7. Consider a model for thin film growth in which atoms stack in individual columns. The energy of a column n atoms high is $E_c(n) = -n\varepsilon_0$. The physical height of such a column is na. Atoms in different columns do not interact in this model.

a. Using the grand canonical ensemble, find the average column height in terms of temperature T, chemical potential μ, ε_0, and a.

- $Z_G = [z \cdot \exp(\beta n \varepsilon_0)]^N$

$$\sum \exp(\beta n \varepsilon_0) = \frac{1}{1-\exp(\beta \varepsilon_0)}$$

$$Z_G = \left[\frac{1}{1-\exp(\beta(\varepsilon_0+\mu))}\right]^N$$

$$\langle h \rangle = a\langle n \rangle = a\left(-\frac{\partial \ln Z_G}{\partial(-\beta\mu)}\right) \qquad PV = -k_B T \ln Z_G$$

$$\ln Z_G = N \ln \frac{1}{1-\exp(\beta\varepsilon_0+\beta\mu)} = -N\ln(1-\exp(\beta\varepsilon_0+\beta\mu))$$

$$\langle h \rangle = a\left(\frac{1}{1-\exp(\beta(\varepsilon_0+\mu))} \cdot -\exp(\beta(\varepsilon_0+\mu)) \cdot (-1)\right)$$

$$= \frac{a\exp(\beta(\varepsilon_0+\mu))}{1-\exp(\beta(\varepsilon_0+\mu))}$$

b. The thin film is in equilibrium with a classical gas of the same type of atoms, with which it can exchange particles. The atoms have mass m. Find an expression for the average height of a column in terms of the gas pressure, and show that it diverges above some critical pressure P_0. (Hint: What is the pressure of an ideal gas in terms of its chemical potential and temperature)

$$P = z/\lambda^3\beta \qquad P_0 = \exp(-\beta\varepsilon_0)/\lambda^3\beta$$

$$z = P\lambda^3\beta \qquad \exp(-\beta\varepsilon_0) = 1/P_0\lambda^3\beta$$

$$\langle h \rangle = \frac{a \cdot P/P_0}{1-P/P_0} = \frac{aP}{P_0-P}$$

8. The universe is filled with blackbody radiation in the form of photons at $T = 3K$.

a. Write an expression for the photon number density in terms of T, some fundamental constants, and a dimensionless constant which you may express as an integral.

9*. A monochromatic plane wave described by $\vec{E} = A(\hat{x} + i\hat{y})\exp(i(kz - \omega t))$ is normally incident on a polaroid plate that allows through the $\hat{x}$ polarization component and absorbs the $\hat{y}$ component. A is a real constant.

a. What is the magnetic field $\vec{B}$ in this plane wave?

$|\vec{E}| = |\vec{B}|$

$\vec{E} \times \vec{B} \propto \hat{z}$

$$\begin{array}{ccc} i & j & k \end{array}$$
$$\begin{vmatrix} 1 & i & 0 \\ a & b & c \end{vmatrix} = (ic, -(c), b - ia) = \hat{z}$$

$\cancel{b = ia}$

$\vec{B} = A(-\hat{x} - j\hat{y})\exp(i(kz - \omega t))$

b. What is the intensity (average power / unit area) of the incident wave (in terms of A, k, and ω)?

$I = \frac{1}{2} \varepsilon v E_0^2$

$= \frac{1}{2} \cdot \varepsilon \cdot \frac{1}{\sqrt{\mu\varepsilon}} \cdot A^2 = \frac{1}{2} \varepsilon \cdot \frac{\omega}{k} \cdot A^2$

c. What fraction of the incident wave intensity is absorbed by the plate?

$\frac{1}{2}$ since x and y have equal magnitude

d. What is the pressure exerted on the plate?

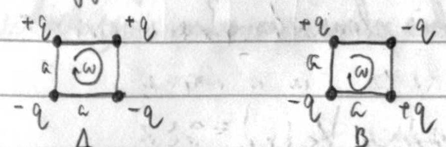
$F = IA\,\cancel{dt}$

$\cancel{= \frac{1}{2} \cdot \varepsilon \cdot \frac{\omega}{k} \cdot A^2 \cdot \cos^2 \omega t}\ dt$

$\cancel{= \frac{1}{4} \cdot \varepsilon \cdot \omega t}$

$= \frac{1}{2} \cdot \varepsilon \cdot \frac{\omega}{k} \cdot A^2 \cdot \text{Area}$

10. Consider two systems of four charges arranged in a square of side a, that rotates about an axis normal to the plane of the square with a rotational frequency ω. The charges are arranged in different ways, as shown in the figure below.



a. Find expressions for the time dependent electric dipole moments and Cartesian components of the electric quadrupole moments Q_{ij} for both systems A and B.

· B has no net dipole moment

For A,

$p_0 = q \cdot \sqrt{2}\, a = \sqrt{2}\, qa$

Initially: $\vec{p} = p_0(-x, y) + p_0(x, y)$

$\cancel{= 2\sqrt{2}\, qa \cos(\omega t)}$

$\vec{p} = 2\sqrt{2}\,(\hat{x} + i\hat{y})$

Also, no quadrupole moment

For the quadrupole moments of B, we can convince ourselves that only $Q_{xy} = Q_{yx} \neq 0$

$Q_{xy} = \int 3xy\, \rho(\vec{x})\, d^3x$

$\rho(\vec{x}) = -q\delta(x - a/2)\delta(y - a/2) + q\delta(x + a/2)\delta(y - a/2) + q\delta(x - a/2)\delta(y + a/2)$
$- q\delta(x + a/2)\delta(y + a/2)$

$Q_{xy} = Q_{yx} = 3q\left[-a^2/4 - a^2/4 - a^2/4 - a^2/4\right]$
$= 3qa^2$

b. What is the dominant frequency of radiation for system A? What is it for system B?

· A corresponds to $l = 1$, so ω
B corresponds to $l = 2$, so 2ω

c. In the long wavelength limit, which of the two systems radiates more power.

Dipole since it has a ~~smaller~~ inverse r dependance.

11*. A particle of charge q and mass m is confined to move in one dimension, in a harmonic potential $V(x) = \frac{1}{2} m\omega^2 x^2$ and is subject to a weak electric field potential $V_\varepsilon(x) = e\varepsilon x$.

a. The Hamiltonian of the system is $\mathcal{H} = p^2/2m + V + V_\varepsilon$. Determine the exact spectrum.

· $$-\frac{\hbar^2}{2m} \cdot \frac{d^2\psi}{dx^2} + \left(\frac{m\omega^2 x^2}{2} + e\varepsilon x\right)\psi = E\psi$$

$x' = x + e\varepsilon/m\omega^2$

$dx' = dx$

$$-\frac{\hbar^2}{2m} \cdot \frac{d^2\psi}{dx^2} + \left(\frac{m\omega^2}{2}\left(x' - e\varepsilon/m\omega^2\right)^2 + e\varepsilon\left(x' - e\varepsilon/m\omega^2\right)\right)\psi = E\psi$$

$$-\frac{\hbar^2}{2m} \cdot \frac{d^2\psi}{dx^2} + \left(\frac{m\omega^2}{2}\left(x'^2 - 2e\varepsilon x'/m\omega^2 + e^2\varepsilon^2/m^2\omega^4\right) + e\varepsilon\left(x' - e\varepsilon/m\omega^2\right)\right)\psi = E\psi$$

$$-\hbar^2/2m \cdot \frac{d^2\psi}{dx^2} + \left(\tfrac{1}{2} m\omega^2 x'^2 - e^2\varepsilon^2/2m\omega^2\right)\psi = E\psi$$

$$-\hbar^2/2m \cdot \frac{d^2\psi}{dx^2} + \left(\tfrac{1}{2} m\omega^2 x'^2\right)\psi = \left(E + e^2\varepsilon^2/2m\omega^2\right)\psi$$

$$E_n = (n + \tfrac{1}{2})\hbar\omega - \frac{e^2\varepsilon^2}{2m\omega^2}$$

b. Taking $\mathcal{H}_0 = p^2/2m + V$ and treating V_ε as a small perturbation, calculate the energies and eigenstates including the lowest order non-zero corrections due to V_ε. Compare the energies to the exact result from part (a) above. (Perturbative corrections to the eigenstates may be written in terms of the eigenstates of $\mathcal{H}_0$ as bra and ket vectors. The energies and their perturbative corrections should be computed explicitly in terms of q, ε, m, ω, and any relevant constants).

$\mathcal{H}_0|n\rangle = (n+\tfrac{1}{2})\hbar\omega|n\rangle$

$q\varepsilon x|n\rangle = q\varepsilon\sqrt{\tfrac{\hbar}{2m\omega}}\,(a+a^+)|n\rangle$

$= q\varepsilon\sqrt{\tfrac{\hbar}{2m\omega}}\left[\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right]$

To first order, there is no energy shift since $\langle n|q\varepsilon x|n\rangle = 0$

$$\sum_{n'\neq n}\frac{|\langle n'|q\varepsilon x|n\rangle|^2}{E_n - E_{n'}}$$

The only non-zero terms are $n'=n+1$ and $n'=n-1$

$$\frac{|\langle n+1|q\varepsilon x|n\rangle|^2}{E_n - E_{n+1}} = \frac{\left(q\varepsilon\sqrt{\tfrac{\hbar}{2m\omega}}\cdot\sqrt{n+1}\right)^2}{(n+\tfrac{1}{2})\hbar\omega - (n+\tfrac{3}{2})\hbar\omega}$$

$$\frac{|\langle n-1|q\varepsilon x|n\rangle|^2}{E_n - E_{n-1}} = \frac{\left(q\varepsilon\sqrt{\tfrac{\hbar}{2m\omega}}\cdot\sqrt{n}\right)^2}{(n+\tfrac{1}{2})\hbar\omega - (n-\tfrac{1}{2})\hbar\omega}$$

$$\Delta E^{(2)}_{(n)} = \frac{q^2\varepsilon^2\cdot\hbar}{2m\omega}\left(\frac{n+1}{-1} + \frac{n}{1}\right)\cdot\frac{1}{\hbar\omega}$$

$$= \frac{q^2\varepsilon^2}{2m\omega^2}\left(n-n-1\right) = \frac{-q^2\varepsilon^2}{2m\omega^2}$$

$$E = (n+\tfrac{1}{2})\hbar\omega - \frac{q^2\varepsilon^2}{2m\omega^2}$$

12*. Consider a particle of mass m moving in one dimension in a potential $V(x)$

a. Using an expansion in $\hbar$, derive the general expression for a wavefunction in the WKB approximation for regions of x where the particle is classically propagating. Your result should be valid to $O(1/\hbar)$, and can be written in terms of undetermined coefficients A_+ and A_- for left- and right-propagating states.

13. Consider a system of two particles, each with angular momentum 1, with Hamiltonian

$$\mathcal{H} = \varepsilon_1/\hbar^2 (\vec{L}_1 + \vec{L}_2)\cdot\vec{L}_2 + \varepsilon_2/\hbar^2 (L_{1,z}, L_{2,z})^2$$

where $\vec{L}_1$ is the angular momentum of particle 1 and $\vec{L}_2$ is the angular momentum of particle 2. Find the energy levels (both states and energies) and their degeneracies for states of the system with total angular momentum $2\hbar$.

$$\vec{L}^2 |l,m\rangle = l(l+1)\hbar^2 |l,m\rangle$$

$$L_z |l,m\rangle = m\hbar |l,m\rangle$$

$$\mathcal{H} = \varepsilon_1/\hbar^2 (\vec{L}_1\cdot\vec{L}_2 + \vec{L}_2^2) + \varepsilon_2/\hbar^2 (L_{1,z}^2 + 2L_{1,z}L_{2,z} + L_{2,z}^2)$$

$$= \varepsilon_1/2\hbar^2 (\vec{L}^2 - \vec{L}_1^2 + \vec{L}_2^2) + \varepsilon_2/\hbar^2 (L_{1,z}^2 + 2L_{1,z}L_{2,z} + L_{2,z}^2)$$

$$= \varepsilon_1/2\,(2\cdot 3 - 1\cdot 2 + 1\cdot 2) + \varepsilon_2 (m_1^2 + 2m_1 m_2 + m_2^2)$$

$$= \varepsilon_1/2\,(6) + \varepsilon_2 (m_1^2 + 2m_1 m_2 + m_2^2)$$

$$\mathcal{H}|1,1;1,1\rangle = 3\varepsilon_1 + 4\varepsilon_2$$

$$\mathcal{H}|1,0;1,1\rangle = \mathcal{H}|1,1;1,0\rangle = 3\varepsilon_1 + \varepsilon_2 = \mathcal{H}|1,-1;1,0\rangle = \mathcal{H}|1,0;1,-1\rangle$$

$$\mathcal{H}|1,0;1,0\rangle = 3\varepsilon_1$$

$$\mathcal{H}|1,-1;1,-1\rangle = 3\varepsilon_1$$

14. Consider N identical particles with Hamiltonian $\mathcal{H} = \sum_{i=1}^{N} h(\vec{r}_i)$ where $h(\vec{r})$ is a spin-independent one-particle Hamiltonian acting only on the spatial degrees of freedom of the particle, whose eigenstates satisfy $h|\phi_n\rangle = \varepsilon_n|\phi_n\rangle$.
The eigenenergies are ordered by their label, i.e. $\varepsilon_n < \varepsilon_{n+1}$ with $n = 1, 2, \ldots, \infty$. Assume the spectrum of h is non-degenerate.

a. What is the groundstate energy if this is a system of spin-0 particles?

15*. A relativistic particle of rest mass m traveling at speed $0.8c$ strikes and is absorbed by a second particle of equal rest mass m. The combined object then strikes and is absorbed by a third particle of rest mass m, producing a final particle of rest mass M traveling at speed V.

a. Find the numerical value of M/m

$$\gamma = (1-u^2/c^2)^{-1/2} = (1-u^2)^{-1/2}$$

$$P_1 = (\sqrt{m^2+\gamma_1^2 m^2 u_1^2}, \gamma_1 m u_1, 0, 0)$$

$$P_m = (m, 0, 0, 0)$$

$$P_2 = (\sqrt{m_2^2+\gamma_2^2 m_2^2 u_2^2}, \gamma_2 m_2 u_2, 0, 0)$$

$$P_1 + P_m = P_2 \qquad \gamma_1 m u_1 = \gamma_2 m_2 u_2$$

$$E_1 + E_m = E_2 \qquad \sqrt{m^2+\gamma_1^2 m^2 u^2} + m = \sqrt{m_2^2+\gamma_1^2 m^2 u^2}$$

$$m^2 + \cancel{\gamma_1^2 m^2 u_1^2} + 2m\sqrt{m^2+\gamma_1^2 m^2 u^2} = m_2^2 + \cancel{\gamma_1^2 m^2 u_1^2}$$

$$m_2^2 = m^2 + 2m^2\sqrt{1+\gamma_1^2 u^2}$$

$$= m^2 + 2m^2\sqrt{1+\frac{u_1^2}{1-u^2/c^2}}$$

$$= m^2 + 2m^2\sqrt{\frac{1-u^2+u^2}{1-u^2}}$$

$$m_2^2 = m^2 + \frac{2m^2}{\sqrt{1-u_1^2}}$$

$$m_3^2 = m_2^2 + \frac{2m_2^2}{\sqrt{1-u_2^2}}$$

$$= \left(m^2 + \frac{2m^2}{\sqrt{1-u^2}}\right) + \frac{2}{\sqrt{1-u_2^2}}\left(m^2 + \frac{2m^2}{\sqrt{1-u^2}}\right)$$

$$\left(\frac{M}{m}\right)^2 = \left(1+\frac{2}{\sqrt{1-u^2}}\right)\left(1+\frac{2}{\sqrt{1-u^2}}\right)$$

$$= 1 + \frac{4}{\sqrt{1-u^2}} + \frac{4}{1-u^2} = 1 + \frac{4}{.6} + \frac{4}{.36} =$$

$$\frac{M}{m} = 4.3$$

Find the numerical value of V/c

b. $$\gamma_2 u_2 = \gamma_1 u_1 \frac{m}{m_2}$$

$$\gamma_3 u_3 = \gamma_2 u_2 \cdot \frac{m_2}{M} = \gamma_1 u_1 \cdot \frac{m}{M} = \frac{u}{\sqrt{1-u^2}} \cdot \frac{1}{4.3} = .31$$

$$\frac{V}{\sqrt{1-V^2}} = a$$

$$V^2 = a - aV^2$$

$$V^2(1+a) = a$$

$$V^2 = \frac{a}{1+a}$$

$$V = .49$$