You had to remember Christmas, and postcards of the Crown Prince and his fiancée, and little cafés in Valence and beer gardens in Unter den Linden and weddings at the mairie, and going to the Derby, and your grandfather's whiskers... This kind of battle was invented by Lewis Carroll and Jules Verne and whoever wrote Undine, and country deacons bowling and marrianes in Marseilles and girls seduced in the back lanes of Wurtemburg and Westphalia. Why, this was a love battle – there was a century of middle-class love spent here. This was the last love battle – F. Scott Fitzgerald ("Tender is the Night", 1934)

Chapter 8: The Hamilton Equations of Motion

Section 1. Legendre Transformations and the Hamilton Equations of Motion

A reminder that Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \qquad (8.1)$$

Also remember that for each generalized coordinate q_i, we define the conjugate momentum p_i as

$$p_i = \frac{\partial \mathcal{L}(q_i, \dot{q}_i, t)}{\partial \dot{q}_i} \qquad \dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} \qquad (8.2)$$

(q, p) are known as the canonical variables and are used in Hamiltonian formulation. To convert from the $\mathcal{L}(q, \dot{q}, t)$ to $\mathcal{H}(q, p, t)$, we use a Legendre transformation.

Say we have a function $f(x, y)$ such that

$$df = u\,dx + v\,dy \qquad (8.3)$$

$$u = \frac{\partial f}{\partial x} \qquad v = \frac{\partial f}{\partial y} \qquad (8.4)$$

$$g = f - ux \qquad (8.5)$$

$$\begin{aligned} dg &= df - u\,dx - x\,du \\ &= u\,dx + v\,dy - u\,dx - x\,du \\ &= v\,dy - x\,du \end{aligned}$$

$$x = -\frac{\partial g}{\partial u} \qquad v = \frac{\partial g}{\partial y} \qquad (8.6)$$

Now let's apply this to the Lagrangian

$$\begin{aligned} d\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial q}\,dq + \frac{\partial \mathcal{L}}{\partial \dot{q}}\,d\dot{q} + \frac{\partial \mathcal{L}}{\partial t}\,dt \qquad (8.13) \\ &= \dot{p}\,dq + p\,d\dot{q} + \frac{\partial \mathcal{L}}{\partial t}\,dt \end{aligned}$$

$$\mathcal{H} = \dot{q}p - \mathcal{L}$$

$$\begin{aligned} d\mathcal{H} &= p\,d\dot{q} + \dot{q}\,dp - \dot{p}\,dq - p\,d\dot{q} - \frac{\partial \mathcal{L}}{\partial t}\,dt \\ &= \dot{q}\,dp - \dot{p}\,dq - \frac{\partial \mathcal{L}}{\partial t}\,dt \qquad (8.16) \\ &= \frac{\partial \mathcal{H}}{\partial q}\,dq + \frac{\partial \mathcal{H}}{\partial p}\,dp + \frac{\partial \mathcal{H}}{\partial t}\,dt \qquad (8.17) \end{aligned}$$

which gives us

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$

$$-\dot{p}_i = \frac{\partial \mathcal{H}}{\partial q_i} \qquad (8.18)$$

$$-\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial \mathcal{H}}{\partial t} \qquad (8.19)$$

(8.18) is known as the canonical equation of Hamilton

We can get from the Lagrangian to Hamiltonian by

$$\mathcal{H} = \dot{q}_i p_i - \mathcal{L} \qquad (8.20)$$

One thing to note is that if the equations defining the generalized coordinates don't depend on time explicitly and if the forces can be derived from a conservative potential

$$\mathcal{H} = T + V = E \qquad (8.21)$$

As an example,

$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - V(r)$$

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \qquad \dot{r} = p_r/m$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \dot{\theta} = p_\theta/mr^2$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta \qquad \dot{\phi} = p_\phi/mr^2\sin^2\theta$$

$$\mathcal{L} = p_r^2/2m + p_\theta^2/2mr^2 + p_\phi^2/2mr^2\sin^2\theta - V(r)$$

$$\begin{aligned} \mathcal{H} &= p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - \mathcal{L} \\ &= p_r^2/2m + p_\theta^2/2mr^2 + p_\phi^2/2mr^2\sin^2\theta + V(r) \end{aligned}$$

which if we had written $\mathcal{L} = T - V$, we would see $\mathcal{H} = T + V$.

$$\dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = -\left[-p_\theta^2/mr^3 - p_\phi^2/mr^3\sin^2\theta + \frac{\partial V}{\partial r}\right] = 1/mr^3\left[p_\theta^2 + p_\phi^2/\sin^2\theta\right] - \frac{\partial V}{\partial r}$$

$$\dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = -\left[-p_\phi^2/mr^2\sin^3\theta \cdot \cos\theta\right] = p_\phi^2\cos\theta/mr^2\sin^3\theta$$

$$p_\phi = 0$$

Section 2. Cyclic Coordinates and Conservation Theorems

Remember, a cyclic coordinate does not appear in the Lagrangian, meaning the conjugate momentum is a constant. We can convince ourselves that cyclic coordinates are also absent from the Hamiltonian.

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \qquad (8.41)$$

Derivations

1.

a. $d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial q} dq + \frac{\partial \mathcal{H}}{\partial p} dp + \frac{\partial \mathcal{H}}{\partial t} dt$

$\mathcal{L} = \dot{q} p - \mathcal{H}$

$d\mathcal{L} = p\,d\dot{q} + \dot{q}\,dp + \dot{p}\,dq - \dot{q}\,dp - \frac{\partial \mathcal{H}}{\partial t} dt$

$= p\,d\dot{q} + \dot{p}\,dq - \frac{\partial \mathcal{H}}{\partial t} dt$

$= \frac{\partial \mathcal{L}}{\partial q} dq + \frac{\partial \mathcal{L}}{\partial \dot{q}} d\dot{q} + \frac{\partial \mathcal{L}}{\partial t} dt$

$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$

$\dot{p} = \frac{\partial \mathcal{L}}{\partial q}$

$-\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial \mathcal{H}}{\partial t}$

b. $\mathcal{L}' = -\dot{p} q - \mathcal{H}$

$d\mathcal{L}' = -q\,d\dot{p} - \dot{p}\,dq + \dot{p}\,dq - \dot{q}\,dp - \frac{\partial \mathcal{H}}{\partial t} dt$

$= -q\,d\dot{p} - \dot{q}\,dp - \frac{\partial \mathcal{H}}{\partial t} dt$

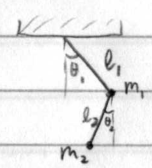
$d\mathcal{L}' = \frac{\partial \mathcal{L}'}{\partial p} dp + \frac{\partial \mathcal{L}'}{\partial \dot{p}} d\dot{p} + \frac{\partial \mathcal{L}'}{\partial t} dt$

$-\dot{q} = \frac{\partial \mathcal{L}'}{\partial p}$

$-q = \frac{\partial \mathcal{L}'}{\partial \dot{p}}$

$-\frac{\partial \mathcal{L}'}{\partial t} = \frac{\partial \mathcal{H}}{\partial t}$

Problems

13. 

From question 1.22,

$\mathcal{L} = \frac{m_1}{2}(l_1^2 \dot{\theta}_1^2) + \frac{m_2}{2}(l_1^2 \dot{\theta}_1^2 + l_2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2))$
$+ m_1 g l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$

$p_{\theta_1} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$

$p_{\theta_2} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$

$p_{\theta_1} - m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) = (m_1 l_1^2 + m_2 l_1^2)\dot{\theta}_1$

$p_{\theta_2} - m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) = m_2 l_2^2 \dot{\theta}_2$

$p_{\theta_1} - m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \cdot \left(\frac{p_{\theta_2}}{m_2 l_2^2} - \frac{l_1}{l_2} \cdot \cos(\theta_1 - \theta_2)\right) = (m_1 l_1^2 + m_2 l_1^2)\dot{\theta}_1$

14. $p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = 2a\dot{x} + c\dot{y} + f y^2 \dot{z}$

$p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = b/x + c\dot{x} + 2g\dot{y}$

$p_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = f y^2 \dot{x}$

$\mathcal{H} = \dot{x} p_x + \dot{y} p_y + \dot{z} p_z - \mathcal{L}$

$= 2a\dot{x}^2 + c\dot{x}\dot{y} + f y^2 \dot{x}\dot{z} + \cancel{b\dot{y}/x} + \cancel{c\dot{x}\dot{y}} + 2g\dot{y}^2 + \cancel{f y^2 \dot{x}\dot{z}}$

$- a\dot{x}^2 - \cancel{b\dot{y}/x} - \cancel{c\dot{x}\dot{y}} - \cancel{f y^2 \dot{x}\dot{z}} - g\dot{y}^2 + k\sqrt{x^2+y^2}$

$= a\dot{x}^2 + c\dot{x}\dot{y} + f y^2 \dot{x}\dot{z} - g\dot{y}^2 + k\sqrt{x^2+y^2}$

Lagrangian is so much easier