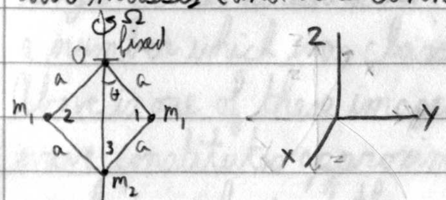
When you visualized a man or woman carefully, you could always begin to feel pity - that was a quality God's image carried with it. When you saw the lines at the corners of the eyes, the shape of the mouth, how the hair grew, it was impossible to hate. Hate was just a failure of imagination. - Graham Greene (The Power and the Glory)

2015-2016

X. Consider the system shown below, which consists of two point masses of mass m_1 and one of mass m_2. The masses are connected by massless rods of length a as shown, and the mass m_2 may slide frictionlessly on a central rod, around which the other two masses (and the connecting rods) rotate with frequency Ω.



a. Find the Lagrangian of the system in terms of the generalized coordinate θ, and the constants g, m_1, m_2, a, and Ω.

$x_1 = -a\sin\theta \cdot \sin(\Omega t)$ $\qquad \dot{x}_1 = -a\dot\theta\cos\theta\sin(\Omega t) - a\Omega\sin\theta\cos(\Omega t)$

$y_1 = a\sin\theta\cdot\cos(\Omega t)$ $\qquad \dot{y}_1 = a\dot\theta\cos\theta\cos(\Omega t) - a\Omega\sin\theta\sin(\Omega t)$

$z_1 = -a\cos\theta$ $\qquad \dot{z}_1 = a\dot\theta\sin\theta$

$x_2 = a\sin\theta\sin(\Omega t)$ $\qquad \dot{x}_2 = a\dot\theta\cos\theta\sin(\Omega t) + a\Omega\sin\theta\cos(\Omega t)$

$y_2 = -a\sin\theta\cos(\Omega t)$ $\qquad \dot{y}_2 = -a\dot\theta\cos\theta\cos(\Omega t) + a\Omega\sin\theta\sin(\Omega t)$

$z_2 = -a\cos\theta$ $\qquad \dot{z}_2 = a\dot\theta\sin\theta$

$x_3 = 0$

$y_3 = 0$

$z_3 = -2a\cos\theta$ $\qquad \dot{z}_3 = 2a\dot\theta\sin\theta$

$$\mathcal{L} = m_1(a^2\dot\theta^2 + a^2\Omega^2\sin^2\theta) + 2m_2a^2\dot\theta^2\sin^2\theta + 2ag\cos\theta(m_1+m_2)$$

b. Derive the equation of motion

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

$$\frac{d}{dt}(2m_1a^2\dot\theta + 4m_2a^2\dot\theta\sin^2\theta) - (2m_1a^2\Omega^2\sin\theta\cos\theta$$
$$+4m_2a^2\dot\theta^2\sin\theta\cos\theta - 2ag(m_1+m_2)\sin\theta) = 0$$

$$2m_1a^2\ddot\theta + 4m_2a^2\ddot\theta\sin^2\theta + 8m_2a^2\dot\theta^2\sin\theta\cos\theta - 2m_1a^2\Omega^2\sin\theta\cos\theta$$
$$-4m_2a^2\dot\theta^2\sin\theta\cos\theta + 2ag(m_1+m_2)\sin\theta = 0$$

$$\ddot\theta(m_1a + 2m_2a\sin^2\theta) + \dot\theta^2(2m_2a\sin\theta\cos\theta) + \sin\theta(-m_1a\Omega^2\cos\theta + 2g(m_1+m_2)) = 0$$

c. Find all positions of dynamical equilibrium

Set $\dot\theta = \ddot\theta = 0$

$$2g(m_1+m_2) = m_1a\Omega^2\cos\theta$$
$$\theta = \cos^{-1}\left(\frac{2g(m_1+m_2)}{m_1a\Omega^2}\right)$$

also $\theta = 0$

d. Classify the equilibrium positions as stable or unstable for different possible values of the system parameters. For each stable configuration, determine the frequency of small oscillations about the equilibrium position.

$$\sin\theta(-m_1a\Omega^2\cos\theta + 2g(m_1+m_2)) = 0$$

$$\frac{\partial}{\partial\theta} = \cos\theta(-m_1a\Omega^2\cos\theta + 2g(m_1+m_2)) + \sin\theta(m_1a\Omega^2\sin\theta)$$
$$= m_1a\Omega^2(\sin^2\theta - \cos^2\theta) + 2g\cos\theta(m_1+m_2)$$
$$= m_1a\Omega^2(1-2\cos^2\theta) + 2g\cos\theta(m_1+m_2)$$

$\theta = 0$: $-m_1a\Omega^2 + 2g(m_1+m_2) > 0$

$$2g(m_1+m_2) > m_1a\Omega^2$$
$$\Omega < \sqrt{\frac{2g(m_1+m_2)}{m_1a}}$$

$\theta = \cos^{-1}\left(2g(m_1+m_2)/m_1a\Omega^2\right)$:

$$m_1a\Omega^2\left(1 - \frac{2\cdot4g^2(m_1+m_2)^2}{m_1^2a^2\Omega^4}\right) + 2g(m_1+m_2)\cdot\frac{2g(m_1+m_2)}{m_1a\Omega^2}$$
$$= m_1a\Omega^2 - \frac{4g^2(m_1+m_2)^2}{m_1a\Omega^2} > 0$$
$$m_1^2a^2\Omega^4 > 4g^2(m_1+m_2)^2$$
$$\Omega > \sqrt{\frac{2g(m_1+m_2)}{m_1a}}$$

for $\Omega < \sqrt{\frac{2g(m_1+m_2)}{m_1 a}}$:

$$\ddot{\varepsilon}(m_1 a) + \varepsilon\left(-m_1 a\Omega^2 + 2g(m_1+m_2)\right) = 0$$

$$\ddot{\varepsilon} = -\varepsilon\left(\Omega^2 - \frac{2g(m_1+m_2)}{m_1 a}\right)$$

$$\omega = \sqrt{\Omega^2 - \frac{2g(m_1+m_2)}{m_1 a}}$$

2*. A particle of mass M starts at a point far away from the center of a repulsive potential center. The potential energy of the particle as a function of distance r from the center is $V = \beta/r^2$, with $\beta > 0$. It has an initial energy of E and impact parameter b.

a. Find the distance of the nearest point of approach to the potential center, R_{mn}.

$$E = V_{eff}(r_{min})$$

$$= \frac{l^2}{2mr_{min}^2} + \frac{\beta}{r_{min}^2}$$

$$= \frac{b^2\, 2mE}{2mr_{min}^2} + \frac{\beta}{r_{min}^2}$$

$$E r_{min}^2 = b^2 E + \beta$$

$$r_{min} = \sqrt{\frac{b^2E+\beta}{E}}$$

b. Find the speed of the particle at this nearest point of approach

$$l = b\sqrt{2mE} = m v_{min} r_{min}$$

$$v_{min} = \frac{b\sqrt{2mE}}{m r_{min}} = \frac{b}{m}\sqrt{\frac{(2mE)E}{b^2E+\beta}} = \frac{b}{m}\sqrt{\frac{2mE^2}{b^2E+\beta}}$$

3*. Consider a particle of mass m moving in a central potential of a three dimensional oscillator, $V(r) = \frac{1}{2} \cdot m\omega^2 r^2$. In terms of the energy E and angular momentum l.

a. Find the minimum and maximum distances (r_{min} and r_{max}) of the particle from the origin.

· Again, we use $E = V_{eff}(r_0)$

$$E = \frac{l^2}{2mr_0^2} + \frac{m\omega^2 r_0^2}{2}$$

$$2mEr_0^2 = l^2 + m^2\omega^2 r_0^4$$

$$m^2\omega^2 r_0^4 - 2mEr_0^2 + l^2 = 0$$

$$r_0^2 = \frac{2mE \pm \sqrt{4m^2E^2 - 4m^2\omega^2 l^2}}{2m^2\omega^2}$$

$$= \frac{E \pm \sqrt{E^2 - \omega^2 l^2}}{m\omega^2}$$

$$r_{min} = \left(\frac{E - \sqrt{E^2 - \omega^2 l^2}}{m\omega^2}\right)^{1/2}$$

$$r_{max} = \left(\frac{E + \sqrt{E^2 - \omega^2 l^2}}{m\omega^2}\right)^{1/2}$$

b. For the special case of circular motion, find the radius of the orbit.

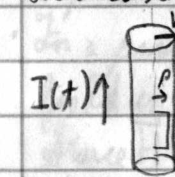
· For a circle, $E = V/2$,

$$\frac{m^2\omega^2 r_0^2}{4} = \frac{l^2}{2mr_0^2}$$

$$r_0^4 = \frac{2l^2}{m^2\omega^2}$$

$$r_0 = \left(\frac{\sqrt{2}\, l}{m\omega}\right)^{1/2}$$

4. A time-dependant current $I(t) = I_0 \exp(-t/\tau)$ flows in a long cylindrical wire of radius b, a segment of which is depicted below. The current density is uniform across the conductor.



a. Use Ampere's Law to determine the magnetic field at time t, $\vec{B}(\rho, t)$, at a radial distance ρ from the central axis, considering both the cases $\rho < b$ and $\rho > b$.

· $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Compare to Gauss law, $\nabla \cdot \vec{E} = \rho/\varepsilon_0$, which in scalar form is $4\pi r^2 E = q/\varepsilon_0$

$2\pi r \cdot B = \mu_0 I_{enc.}$ Also note, the magnetic field is always in the counterclockwise direction ($\hat{\phi}$?)

$\rho < b$:

$$2\pi\rho B = \mu_0 \cdot I_0 \exp(-t/\tau) \cdot (\pi r^2/\pi b^2)$$

$$B = \frac{\mu_0 I_0 \exp(-t/\tau)\, \rho}{2\pi b^2}$$

$\rho > b$:

$$2\pi\rho B = \mu_0 I_0 \exp(-t/\tau)$$

$$B = \frac{\mu_0 I_0 \exp(-t/\tau)}{2\pi\rho}$$

b. By applying Faraday's Law to the rectangular path shown, find an expression for the induced electric field $\vec{E}(\rho, t)$ at a distance ρ from the axis of the cylinder with $\rho < b$. In what direction is this $\vec{E}$ field oriented

- $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\frac{\partial \vec{B}}{\partial t} = \frac{-\mu_0 I_0 \exp(-t/\tau)\rho}{2\pi\tau b^2} \hat{\phi}$$

We know that $\vec{E}$ must be in the z- direction.

$$\vec{\nabla} \times \vec{E} \propto \left(\frac{\partial}{\partial \phi} E_z, -\frac{\partial}{\partial \rho} E_z, 0\right)$$

$$\frac{\partial}{\partial \rho} E_z = \frac{-\mu_0 I_0 \exp(-t/\tau)\rho}{2\pi\tau b^2}$$

$$E_z = -\frac{\mu_0 I_0 \exp(-t/\tau)\rho^2}{4\pi\tau b^2}$$

$$\vec{E} = \frac{-\mu_0 I_0 \exp(-t/\tau)\rho^2}{4\pi\tau b^2} \hat{z}$$

c. $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

$$\frac{\partial \vec{D}}{\partial t} = \vec{\nabla} \times \vec{H} - \vec{J}$$

$$\vec{H} = \vec{B}$$

$$\vec{\nabla} \times \vec{H} = \left(-\frac{\partial}{\partial z} B_\phi, 0, \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\phi)\right)$$

$$\frac{\partial}{\partial \rho}\left(\frac{\mu_0 I_0 \exp(-t/\tau)\rho^2}{2\pi b^2}\right) = \frac{\mu_0 I_0 \exp(-t/\tau)\rho}{\pi b^2} \hat{z}$$

$$\frac{\partial \vec{D}}{\partial t} = \frac{\mu_0 I_0 \exp(-t/\tau)}{\pi b^2} - I_0 \exp(-t/\tau)$$

$$= I_0 \exp(-t/\tau)(1/\pi b^2 - 1)\hat{z}$$

$$\vec{D} = \int_0^\infty \frac{\partial \vec{D}}{\partial t} dt$$

$$= I_0 (1/\pi b^2 - 1) \cdot (-\tau)\exp(-t/\tau)\Big|_0^\infty$$

$$= I_0 (1/\pi b^2 - 1)\tau(-0+1)$$

$$= I_0 \tau (1/\pi b^2 - 1)$$

d. For a paramagnetic wire, will the displacement current in part c be larger, smaller, or the same with respect to that calculated negleting its magnetic properties? Explain your reasoning

- In a paramagnetic wire, there will be an additional magnetic field in the direction of the already existing magnetic field. Since $\vec{H}$ increases, the displacement current should also increase.

5*. Consider two infinitely long concentric cylinders of radii a and b with $b>a$ and vacuum between them. If the electric potential, in cylindrical coordinates (ρ, ϕ, z), satisfies the boundary conditions $\Phi(\rho=a, \phi) = V\sin^2\phi$ and $\Phi(\rho=b, \phi) = V\cos^2\phi$, what is the potential for all values of ρ in the interior $(a<\rho<b)$?

· The general solution for the potential is

$$\Phi = A_0 + B_0 \ln\rho + \sum_{n=1}^{\infty} (A_n \rho^n + B_n \rho^{-n}) \cos(n\phi)$$

$$\Phi(\rho=a) = V/2\,(1-\cos(2\phi))$$

$$\Phi(\rho=b) = V/2\,(1+\cos(2\phi))$$

$$A_0 = V/2$$

$$-V/2 = A_2 a^2 + B_2 a^{-2}$$

$$V/2 = A_2 b^2 + B_2 b^{-2}$$

$$A_2 a^2 = -V/2 - B_2 a^{-2}$$

$$A_2 = -V/2a^2 - B_2 a^{-4}$$

$$\frac{V}{2} = \frac{-Vb^2}{2a^2} - \frac{B_2 b^2}{a^4} + \frac{B_2}{b^2}$$

$$\frac{V}{2}\left(\frac{a^2}{a^2}+\frac{b^2}{a^2}\right) = B_2\left(\frac{1}{b^2} - \frac{b^2}{a^4}\right)$$

$$B_2 = \frac{V}{2}\left(\frac{a^2+b^2}{a^2} \cdot \frac{a^4 b^2}{a^4-b^4}\right) = \frac{V}{2}\left(\frac{a^2 b^2}{a^2-b^2}\right)$$

$$A_2 = -V/2a^2 - B_2 a^{-4}$$

$$= \frac{-V}{2a^2} - \frac{V}{2}\left(\frac{b^2}{a^2(a^2-b^2)}\right) = \frac{-V}{2}\left(\frac{a^2-b^2+b^2}{a^2(a^2-b^2)}\right)$$

$$= \frac{-V}{2}\left(\frac{1}{a^2-b^2}\right)$$

$$\Phi = \frac{V}{2} - \frac{V\rho^2\cos(2\phi)}{2(a^2-b^2)} + \frac{Va^2b^2\cos(2\phi)}{2\rho^2(a^2-b^2)}$$

6*. Consider an ideal gas of N particles in a volume V, each atom of which has an excluded volume v_0 around which other atoms may not enter. The atoms do not otherwise interact with one another.

a. Write an expression for the entropy of a gas of these particles that is correct to lowest non-trivial order in Nv_0/V.

· The number of states available is V/v_0, thus the number of microstates is $\binom{V/v_0}{N} = \frac{(V/v_0)!}{N!(V/v_0-N)!}$ $\quad \mathcal{V} = V/v_0$

$$S = k_B \ln\Omega = k_B \ln\left(\frac{\mathcal{V}!}{N!(\mathcal{V}-N)!}\right)$$

$$= k_B[\ln\mathcal{V}! - \ln N! - \ln(\mathcal{V}-N)!] \quad \text{using Stirling approximation}$$

$$\approx k_B[\mathcal{V}\ln\mathcal{V} - N\ln N - (\mathcal{V}-N)\ln(\mathcal{V}-N)]$$

$$= k_B\left[\mathcal{V}\ln\left(\frac{\mathcal{V}}{\mathcal{V}-N}\right) + N\ln\left(\frac{\mathcal{V}-N}{N}\right)\right]$$

$$= k_B[N\ln(\mathcal{V}/N - 1) - \mathcal{V}\ln(1-N/\mathcal{V})]$$

$$= k_B[N(\ln(\mathcal{V}/N) + \ln(1-N/\mathcal{V})) - \mathcal{V}\ln(1-N/\mathcal{V})]$$

$$= k_B[N\ln(\mathcal{V}/N) + N(-N/\mathcal{V}) - \mathcal{V}(-N/\mathcal{V})]$$

$$= k_B[-N\ln(1+(N/\mathcal{V}-1)) - N^2/\mathcal{V} + N]$$

$$= k_B[-N(N/\mathcal{V}-1) - N^2/\mathcal{V} + N]$$

$$= k_B[-N^2/\mathcal{V} + 2N]$$

$$= Nk_B[2 - N/\mathcal{V}] = Nk_B[2 - Nv_0/V]$$

b. Use the result of part (a) to find the correction to the ideal gas law due to the excluded volume, to lowest non-trivial order in Nv_0/V.

· $\frac{P}{T} = \left(\frac{\partial S}{\partial V}\right) = N^2 k_B v_0/V^2$

$$PV = Nk_B(Nv_0/V)$$

7. Consider a gas of N identical non-interacting particles of mass m in an anharmonic potential. The Hamiltonian is given by:

$$\mathcal{H} = \frac{1}{2m}\sum_{i=1}^{N}(p_{x,i}^2 + p_{y,i}^2 + p_{z,i}^2) + K/s\sum_{i=1}^{N}(|x_i|^s + |y_i|^s + |z_i|^s)$$

where s and K are positive constants, and the positions and momenta are to be treated classically in this problem.

a. Find an expression for the canonical partition function for the system. Hint: You may find that you need to use the Euler-Gamma function, $\Gamma(\nu) = \int_0^\infty dx\, x^{\nu-1} \exp(-x)$

- $Z = \exp(-\beta \mathcal{H})^N$

$$= \left[\frac{1}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-\beta p^2/2m)\exp(-\beta K|x|^s/s)\,dx\,dp\right]^{3N} \frac{1}{N!}$$

$$= \left[\frac{2}{h}\sqrt{\frac{2\pi m}{\beta}}\int_0^\infty \exp(-\beta K x^s/s)\,dx\right]^{3N} = \left[2\left(\frac{1}{\lambda}\right)\int_0^\infty \exp(-\beta K x^s/s)\,dx\right]^{3N}$$

$$\int_0^\infty \exp(-\beta K x^s/s)\,dx = \frac{1}{\beta K}\int_0^\infty \frac{1}{x^{s-1}} \exp(-u)\,du$$

$u = \beta K x^s/s \qquad (us/\beta K)^{1/s} = x$

$du = \beta K x^{s-1}\,dx$

$$= \frac{1}{\beta K}\int_0^\infty \left(\frac{\beta K}{s}\right)^{1-1/s} \cdot \frac{1}{u^{1-1/s}} \exp(-u)\,du$$

$$= \frac{(\beta K)^{-1/s}}{s^{1-1/s}}\int_0^\infty u^{1/s-1}\exp(-u)\,du = \frac{(\beta K)^{-1/s}}{s^{1-1/s}} \cdot \Gamma(1/s)$$

$$Z = \left[2(1/\lambda)(\beta K)^{-1/s} s^{1/s-1}\Gamma(1/s)\right]^{3N} \cdot 1/N!$$

b. Find an expression for the average energy from the result of part (a).

- $\langle E \rangle = -\frac{1}{Z}\cdot\frac{\partial Z}{\partial \beta}$

$$\lambda = \left(\frac{2\pi m}{h^2 \beta}\right)^{1/2}$$

Since we only care about the β dependence, lets figure out what that dependance is

$$Z \propto [\beta^{1/2}\cdot\beta^{-1/s}]^{3N} = \beta^{3N/2 - 3N/s}$$

$$\langle E \rangle = \frac{(3N/2 - 3N/s)\beta^{3N/2 - 3N/s - 1}}{\beta^{3N/2 - 3N/s}} = (3N/2 + 3N/s)k_B T$$

c. Calculate the specific heat C_V of the system

- $C_V = \frac{\partial E}{\partial T} = (3N/2 + 3N/s)k_B$

8. Consider an ideal Fermi gas, with energy spectrum $\varepsilon(p) \propto p^s$, with p the momentum. The gas is contained in a box of volume V in a space of n dimensions. The gas is in thermal equilibrium at temperature T.

a. Write an integral expression for the Ω-potential, i.e., the logarithm of the grand partition function.

- $\Phi = -k_B T \ln Z_G$

$$Z_G = \prod_i (1 + z\exp(-\beta\varepsilon_i))$$

$$\ln Z_G = \sum_{i=1} \ln(1 + z\exp(-\beta\varepsilon_i))$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ln(1 + z\exp(-\beta p^s))\,d^3x\,d^3p$$

$$\Phi = -k_B T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ln(1 + z\exp(-\beta p^s))\,d^3x\,d^3p$$

b. How is the Ω-potential related to PV, the product of the pressure and the volume?

- $-PV = \Phi$

c. Write an expression for the average energy of the system $\mathcal{U}$ as an integral over momentum.

- $\langle E \rangle = -\frac{\partial}{\partial\beta}\ln Z_G$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial}{\partial\beta}\ln(1 + z\exp(-\beta p^s))\,d^3x\,d^3p$$

d. Use the result of (c) to show that $PV = \frac{s}{n}\mathcal{U}$

- $\langle E \rangle = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial}{\partial\beta}\ln(1 + z\exp(-\beta p^s))\,d^3x\,d^3p$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z p^s \exp(-\beta p^s)}{1 + z\exp(-\beta p^s)}\,d^3x\,d^3p$$

$u = p \qquad v = \ln(1 + z\exp(-\beta p^s))$

$du = 1 \qquad dv = \dfrac{z(-\beta s)p^{s-1}\exp(-\beta p^s)}{1 + z\exp(-\beta p^s)}$

$= p \ln(1+z\exp(-\beta p^s)) + \frac{1}{\beta s}\iint \ln(1+z\exp(-\beta p^s))\, d^n x\, d^n p$

$k_b T S \ln(1+z\exp(-\beta p^s))\, d^n x\, d^n p = \Phi$

$E = -\frac{n}{s}\Phi$

$PV = \frac{s}{n} E$

Not sure where n comes from. Possibly n-dimensions

9*. A uniform medium with dielectric constant ε and permeability μ occupies the half space $z>0$. The other half space ($z<0$) is vacuum. An electromagnetic plane wave

$\vec{E} = E_0 \exp(i(kz-\omega t))\hat{x}$

$\vec{H} = E_0/Z_0 \cdot \exp(i(kz-\omega t))\hat{y}$

travelling in the $\hat{z}$ direction is normally incident on the medium. Find the form of the electric and magnetic fields for the electromagnetic wave inside the medium.

$Z = \sqrt{\mu/\varepsilon}$

$v = 1/\sqrt{\mu\varepsilon}$

$\vec{E}_r = E_r \exp(i(-k_r z-\omega t))\hat{x}$ $\quad\quad \vec{H}_r = -E_r/Z_0 \cdot \exp(i(-k_r z-\omega t))\hat{y}$

$\vec{E}_t = E_t \exp(i(k_t z-\omega t))\hat{x}$ $\quad\quad \vec{H}_t = E_t/Z \cdot \exp(i(k_t z-\omega t))\hat{y}$

At z, we expect

$E_0 + E_r = E_t$ $\quad\quad k_r = k = \omega/v_1$ $\quad\quad \omega = k\cdot\frac{1}{\sqrt{\mu_0\varepsilon_0}}$

$E_0/Z_0 - E_r/Z_0 = E_t/Z$ $\quad\quad k_t = \omega/v_2$

$E_r = E_t - E_0$

$\frac{E_0 - E_t + E_0}{Z_0} = \frac{E_t}{Z}$

$2E_0 Z - E_t Z = E_t Z_0$

$E_t = \frac{2E_0 Z}{Z+Z_0}$

$\vec{E}_t = \frac{2E_0 Z}{Z+Z_0}\exp\left(i\left(k\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}\, z-\omega t\right)\right)\hat{x}$

$\vec{H}_t = \frac{2E_0}{Z+Z_0}\exp\left(i\left(k\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}\, z-\omega t\right)\right)\hat{y}$

10. In the radiation zone, the electric field is given by

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{\exp(ikr)}{r}\left[k^2(\hat{n}\times\vec{p})\times\hat{n}\right]$

for electric dipole radiation. A pair of positive point charges are moving in a circle around the $\hat{z}$ axis. The positive charge $+q$ has coordinates $x = d\cos(\omega t)$, $y = d\sin(\omega t)$, $z=0$, and the negative charge $-q$ has coordinates $x = -d\cos(\omega t)$, $y = -d\sin(\omega t)$, $z=0$.

a. What is the appropriate complex dipole moment $\vec{p}$ for use in the formula above for the electric field?

$\vec{p} = q\cdot 2d(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y})$

$p(t) = Re(\vec{p}\exp(-i\omega t))$

$\Rightarrow \vec{p}(t) = 2qd(\hat{x} + i\hat{y})$

$\hat{x} = \nabla(r\sin\theta\cos\phi) = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$

$\hat{y} = \nabla(r\sin\theta\sin\phi) = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$

$\vec{p}(t) = 2qd\left[\sin\theta(\cos\phi + i\sin\phi)\hat{r} + \cos\theta(\cos\phi + i\sin\phi)\hat{\theta} + (-\sin\phi + i\cos\phi)\hat{\phi}\right]$

b. What is the magnetic field $\vec{H}$ in the radiation zone?

· $\vec{H} = \frac{c k^2 \exp(ikr)}{4\pi r} [\hat{n} \times \vec{p}]$

$\hat{n} \times \vec{p} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ 1 & 0 & 0 \\ p_r & p_\theta & p_\phi \end{vmatrix} = (0, -p_\phi, p_\theta)$

$\vec{H} = \frac{ck^2 \exp(ikr)}{4\pi r} [(\sin\phi - i\cos\phi)\hat{\theta} + \cos\theta(\cos\phi + i\sin\phi)\hat{\phi}] \cdot 2qd$

c. $\frac{dP}{d\Omega} = \frac{|\vec{E}| \cdot |\vec{H}|}{2} = \frac{ck^4 \cdot 4q^2 d^2}{32\pi^2 \varepsilon_0} \cdot (\sin^2\phi + \cos^2\phi + \cos^2\theta(\cos^2\phi + \sin^2\phi))$

$= \frac{ck^4 q^2 d^2 (1 + \cos^2\theta)}{8\pi^2 \varepsilon_0}$

$P = 2\pi \int_{-1}^{1} \frac{ck^4 q^2 d^2}{8\pi^2 \varepsilon_0} (1 + \cos^2\theta)\, d(\cos\theta)$

$= \frac{ck^4 q^2 d^2}{4\pi\varepsilon_0} (\cos\theta + \tfrac{1}{3}\cos^3\theta)\Big|_{-1}^{1} = \frac{ck^4 q^2 d^2}{4\pi\varepsilon_0} \cdot (2 + 2/3)$

$= \frac{ck^4 q^2 d^2}{4\pi\varepsilon_0} \cdot \frac{8}{3} = \frac{2ck^4 q^2 d^2}{3\pi\varepsilon_0}$

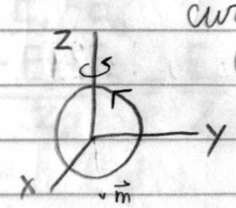
d. How should a circular current loop of radius a and constant current be situated to give the same time-averaged angular distribution, and what should be the magnitude of the constant current?

· We want the magnetic dipole to be the same magnitude and direction as the electric dipole.

$m = IA$

$2qd = \pi a^2 I$

$I = 2qd/\pi a^2$ Rotating vertically around the z-axis with current in the counterclockwise direction. Centered at the origin.



11.* Consider an electron which can be bound to one of three sites on an equilateral triangle. Let ε_0 be the energy of an electron bound to one site in isolation, and let t be the tunneling energy to move from one site to another.

a. Write down the Hamiltonian of this system in the state space of the three sites, $(|1\rangle, |2\rangle, |3\rangle)$.

· We can get the diagonals of the matrix fairly easily e.g. $\langle 1|\mathcal{H}|1\rangle = \varepsilon_0$, but how do we account for the tunneling energy? I don't know anything about tunneling, so there are almost certainly subtleties being missed. nevermind

$\mathcal{H} = \begin{bmatrix} \varepsilon_0 & t & t \\ t & \varepsilon_0 & t \\ t & t & \varepsilon_0 \end{bmatrix}$

$= \varepsilon_0(|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|) + t(|1\rangle\langle 2| + |1\rangle\langle 3| + |2\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 1| + |3\rangle\langle 2|)$

b. Find the energies and corresponding eigenvectors for states of this system.

· $\mathcal{H}|\psi\rangle = E|\psi\rangle$

$\det(\mathcal{H} - \lambda I) = \begin{bmatrix} \varepsilon_0 - \lambda & t & t \\ t & \varepsilon_0 - \lambda & t \\ t & t & \varepsilon_0 - \lambda \end{bmatrix}$

$= (\varepsilon_0 - \lambda)[(\varepsilon_0 - \lambda)^2 - t^2] - t[t(\varepsilon_0 - \lambda) - t^2] + t[t^2 - t(\varepsilon_0 - \lambda)]$

$= (\varepsilon_0 - \lambda)[(\varepsilon_0 - \lambda)^2 - t^2] - 2t^2[(\varepsilon_0 - \lambda) - t]$

$= (\varepsilon_0 - \lambda)^3 - t^2(\varepsilon_0 - \lambda) - 2t^2(\varepsilon_0 - \lambda) + 2t^3$

$= (\varepsilon_0 - \lambda)^3 - 3t^2(\varepsilon_0 - \lambda) + 2t^3$

$(\varepsilon_0 - \lambda) = x$

$x^3 - 3t^2 x + 2t^3 = (x - t)(x^2 + xt - 2t^2)$

$= (x - t)(x + 2t)(x - t) = (x - t)^2 (x + 2t)$

$= (\varepsilon_0 - \lambda - t)^2 (\varepsilon_0 - \lambda + 2t)$

$\lambda = \varepsilon_0 - t,\ \varepsilon_0 + 2t$

$|\varepsilon_0 - t\rangle$: $\begin{bmatrix} t & t & t \\ t & t & t \\ t & t & t \end{bmatrix}$

$$|\varepsilon_0 - t\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$|\varepsilon_0 + 2t\rangle$: $\begin{bmatrix} -2t & t & t \\ t & -2t & t \\ t & t & -2t \end{bmatrix}$

$$|\varepsilon_0 + 2t\rangle = \tfrac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

12.* Consider a one-dimensional particle of mass m in a harmonic oscillator potential, with Hamiltonian $\mathcal{H}_0 = p^2/2m + m\omega^2x^2/2$. The particle is initially prepared in the ground state of $\mathcal{H}_0$. At time $t=0$, the harmonic potential is suddenly turned off, so that the Hamiltonian becomes that of a free particle. Find the probability of finding the particle in an infinitesimal interval dx around a position x_0 at a time $t>0$.

At $t=0$ $E_0 = \hbar\omega/2$, $|\psi\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-m\omega x^2/2\hbar\right)$

$\mathcal{U} = \exp\left(-i\mathcal{H}t/\hbar\right) = \exp\left(-ip^2t/2m\hbar\right) \approx 1 - ip^2t/2m\hbar \approx 1 + \frac{i\hbar t}{2m}\frac{\partial^2}{\partial x^2}$

$\frac{\partial}{\partial x}|\psi\rangle = \left(m\omega/\pi\hbar\right)^{1/4} \cdot -\frac{m\omega}{\hbar}x \cdot \exp\left(-m\omega x^2/2\hbar\right)$

$\frac{\partial^2}{\partial x^2}|\psi\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\frac{m\omega}{\hbar}\exp\left(-m\omega x^2/2\hbar\right) + \frac{m^2\omega^2}{2\hbar^2}\cdot x^2 \exp\left(-m\omega x^2/2\hbar\right)\right]$

$\mathcal{U}|\psi\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[\exp\left(-m\omega x^2/2\hbar\right) - \frac{i\omega t}{2}\cdot\exp\left(-m\omega x^2/2\hbar\right) + \frac{im\omega^2 t}{4\hbar}\cdot x^2\exp\left(-m\omega x^2/2\hbar\right)\right]$

$\langle\psi_t|\mathcal{U}|\psi\rangle = \left(m\omega/\pi\hbar\right)^{1/4}\int_{-\infty}^{\infty}\exp\left(i(k(x-x_0)-\omega t)\right)\left[\exp\left(-m\omega(x-x_0)^2/2\hbar\right) - i\omega t/2\cdot\exp\left(-m\omega(x-x_0)^2/2\hbar\right) + im\omega^2t/4\hbar\,(x-x_0)^2\exp\left(-m\omega(x-x_0)^2/2\hbar\right)\right]dx$

$x - x_0 = u$

$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp(-i\omega t)\cdot\exp\left(-k\hbar/2m\omega\right)\cdot\int_{-\infty}^{\infty}\exp\left(-\left(\sqrt{\frac{m\omega}{2\hbar}}\,u + \right.\right.$

Evaluate this integral, then square solution and multiply by dx (p. 101 Sakurai).

$\langle x_0|\exp\left(-ip^2t/2m\hbar\right)|x\rangle N\exp\left(-\alpha x^2/2\right)dx$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\alpha}{\pi}\right)^{1/4}\cdot\frac{1}{\hbar}\,\frac{\exp\left(-ip(x-x_0)/\hbar\right)}{2\pi}\cdot\exp\left(-ip^2t/2m\hbar\right)\exp\left(-\alpha x^2/2\right)dx\,dp$

13. Consider a particle of mass M and energy $E = \hbar^2 k^2 / 2M$. The particle scatters off of a three-dimensional spherical square well potential,

$$V(r) = \begin{cases} -V_0 & , r \le R \\ 0 & , r > R \end{cases}$$

with $V_0 > 0$. In this problem, we consider near-threshold (i.e. low-energy) scattering.

a. Using a partial wave analysis, explain why such scattering is dominated by the s-wave ($l=0$) channel.

Because we have a low-energy, the particle cannot enter the potential region. In partial-wave analysis, we decompose the wave into a sum of Legendre polynomials. We only look at $l=0$. You get factorials of $1/2$, which quickly drop off as l increases

b. Find the s-wave scattering phase shift, $\delta_{l=0}$

The solution must satisfy

$$-\frac{\hbar^2}{2M} \frac{d^2\psi}{dr^2} + V(r)\psi = E\psi$$

For $r > R$: $-\frac{\hbar^2}{2m} \cdot \frac{d^2\psi}{dr^2} = E\psi$

$$\frac{d^2}{dr^2}\psi = -\frac{2ME}{\hbar^2} \cdot \psi$$

$$\psi = A \exp\left(i\sqrt{2ME/\hbar^2} \cdot r\right)$$

$$= A\cos\left(\sqrt{2ME/\hbar^2}\, r\right) + B\sin\left(\sqrt{2ME/\hbar^2} \cdot r\right) = \sin\left(\sqrt{2ME/\hbar^2} \cdot r\right)$$

For $r > R$: $-\frac{\hbar^2}{2M} \cdot \frac{d^2\psi}{dr^2} = (E+V_0)\psi$

$$\frac{d^2\psi}{dr^2} = \frac{-2M(E+V_0)}{\hbar^2}\psi$$

$$\psi' = C\cos\left(\sqrt{\frac{2M(E+V_0)}{\hbar^2}} \cdot r\right) + D\sin\left(\sqrt{\frac{2M(E+V_0)}{\hbar^2}} \cdot r\right)$$

$$= \sin\left(\sqrt{\frac{2M(E+V_0)}{\hbar^2}}\, r + \delta_0\right)$$

Note we get rid of constant in front because it's not relevant to following calculation

$\psi = \psi'|_{r=R}$ $\quad K_0 = \sqrt{2mE/\hbar^2}$

$\frac{\partial\psi}{\partial r} = \frac{\partial\psi'}{\partial r}\Big|_{r=R}$ $\quad K = \sqrt{2m(E+V_0)/\hbar^2}$

$$\sin(K_0 R) = \sin(KR + \delta_0)$$

$$K_0\cos(K_0 R) = K\cos(KR + \delta_0)$$

$$K\tan(K_0 R) = K_0\tan(KR + \delta_0)$$

$$\delta_0 = \tan^{-1}\left(\frac{K}{K_0}\tan(K_0 R) - KR\right)$$

$$= \tan^{-1}\left(\sqrt{\frac{2m(E+V_0)}{\hbar^2} \cdot \frac{\hbar^2}{2mE}}\tan\left(\sqrt{\frac{2mE}{\hbar^2}}R - \sqrt{\frac{2m(E+V_0)}{\hbar^2}} \cdot R\right)\right)$$

$$= \tan^{-1}\left(\sqrt{\frac{E+V_0}{E}}\tan\left(\sqrt{\frac{2mE}{\hbar^2}}R\right) - \sqrt{\frac{2m(E+V_0)}{\hbar^2}} \cdot R\right)$$

c. Find the total cross-section in the threshold limit, $\sigma(E \to 0)$

$$\sigma = \frac{4\pi}{k^2} \sin^2(\delta_0)$$

$$\approx \frac{4\pi \cdot \hbar^2}{2m(E+V_0)} \sin^2\left(\tan^{-1}\left(\sqrt{\frac{E+V_0}{E}}\sqrt{\frac{2mE}{\hbar^2}}R - \sqrt{\frac{2m(E+V_0)}{\hbar^2}} \cdot R\right)\right)$$

$$= \frac{2\pi\hbar^2}{2m(E+V_0)} \sin^2\left(\tan^{-1}\left(\sqrt{\frac{2m(E+V_0)}{\hbar^2}} \cdot R - \sqrt{\frac{2m(E+V_0)}{\hbar^2}}\, R\right)\right)$$

14. Consider a system of two spin-$\frac{1}{2}$ particles with spin degrees of freedom $\vec{S}_1$ and $\vec{S}_2$. Suppose we have two directions $\hat{a}$ and $\hat{a}'$ along which $\vec{S}_1$ can be measured, and two directions $\hat{b}$ and $\hat{b}'$ along which $\vec{S}_2$ can be measured. According to hidden variable theory, the average value over many equivalent experiments of the observable X, defined by

$$X = (\hat{a}\cdot\vec{S}_1)(\hat{b}\cdot\vec{S}_2) + (\hat{a}\cdot\vec{S}_1)(\hat{b}'\cdot\vec{S}_2) + (\hat{a}'\cdot\vec{S}_1)(\hat{b}'\cdot\vec{S}_2) - (\hat{a}'\cdot\vec{S}_1)(\hat{b}\cdot\vec{S}_2),$$

obeys the inequality $|\langle X\rangle| \le \hbar^2/2$ for any choice of the measurement axes. Suppose the spins are prepared in a singlet state $|\Phi\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$.

a. Compute the expectation value $\langle\Phi|(\hat{a}\cdot\vec{S}_1)(\hat{b}\cdot\vec{S}_2)|\Phi\rangle$ for arbitrary axes $\hat{a}, \hat{b}$.

$\hat{a} = (a_1, a_2, a_3)$

$\hat{b} = (b_1, b_2, b_3)$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{S}_1 = \hbar/2\,(\sigma_x, \sigma_y, \sigma_z)$$

$$\hat{a}\cdot\vec{S}_1 = \frac{\hbar}{2}\left[\begin{pmatrix} 0 & a_1 \\ a_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ia_2 \\ ia_2 & 0 \end{pmatrix} + \begin{pmatrix} a_3 & 0 \\ 0 & -a_3 \end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\begin{bmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{bmatrix}$$

$$\hat{b}\cdot\vec{S}_2 = \frac{\hbar}{2}\begin{bmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{bmatrix}$$

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can break $|\Phi\rangle$ into $\dfrac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}$ on the first and $\dfrac{|\downarrow\rangle - |\uparrow\rangle}{\sqrt{2}}$ on the second

$$\langle\Phi_1|\hat{a}\cdot\vec{S}_1|\Phi_1\rangle = \frac{\hbar}{4}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{bmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{bmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} -a_1 + ia_2 + a_3 \\ a_1 + ia_2 + a_3 \end{pmatrix} = \frac{\hbar}{4}\,2(-a_1)$$

expect the same for $\langle\Phi_2|\hat{b}\cdot\vec{S}_2|\Phi_2\rangle$:

$$\langle\Phi|(\hat{a}\cdot\vec{S}_1)(\hat{b}\cdot\vec{S}_2)|\Phi\rangle = \hbar^2/2\,[(a_1)(b_1)]$$

b. Use the result of (a) to show that there are choices for the directions $\hat{a}, \hat{b}, \hat{a}', \hat{b}'$ for which the hidden variable theory inequality above is violated.

- $\langle\Phi|X|\Phi\rangle = \hbar^2/2\,[a_1 b_1 + a_1 b_1' + a_1' b_1' - a_1' b_1]$

Want to find a_1 and a_1' and b_1 and b_1' such that

$$a_1 b_1 + a_1 b_1' + a_1' b_1' - a_1' b_1 < 1$$

$$\hat{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\quad \hat{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{a}' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\quad \hat{b}' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

15*.

a. Prove whether or not a relativistic electron traveling in a vacuum can spontaneously emit a photon.

- Electron before: $(E_B, \vec{p}_B) = p_b$

photon : $(E_\gamma, \vec{p}_\gamma) = p_\gamma$

Electron after: $(E_A, \vec{p}_A) = p_A$

$$p_B = p_\gamma + p_A$$
$$p_B^2 = p_\gamma^2 + p_A^2 + 2\vec{p}_\gamma \vec{p}_A$$
$$m_B^2 = 0 + m_A^2 + 2(E_\gamma E_A - |p_\gamma||p_A|\cos\theta)$$
$$E_A = \sqrt{E_A^2 - m_e^2}\cos\theta \quad \Rightarrow m_e = 0$$

which doesn't make sense (e.g. take $\cos\theta = 1$). Thus an electron cannot spontaneously emit a photon.

b. Prove whether or not a high energy photon traveling in a vacuum can spontaneously convert into an electron-positron pair.

- Initially, photon is moving in lab-frame. Then, it convert into a pair in the center of mass frame.

$\rightsquigarrow \vec{p}_\gamma$

$p_e \longleftrightarrow p_p$

$$p_\gamma = (E_\gamma, 0) \qquad p_\gamma^2 = p_e^2 + p_p^2 + 2p_e p_p$$
$$p_e = (E_e, -p) \qquad 0 = m_e^2 + m_e^2 + 2\cdot(E_e E_p + p^2)$$
$$p_p = (E_p, p) \qquad = 2m_e^2 + 2(E_e E_p + p^2)$$
$$= 2m_e^2 + 2\left(\sqrt{m_e^2 + p^2}\sqrt{m_e^2 + p^2} + p^2\right)$$
$$= 2m_e^2 + 2(m_e^2 + p^2 + p^2)$$
$$0 = 4m_e^2 + 4p^2$$
$$4m_e^2 = -4p^2$$

doesn't work. Thus, photon cannot spontaneously convert.

16.

a. A mass m is connected to a spring with force law $F = kx^n$, with $n > 0$ an integer and k a constant. How does the oscillator period τ depend on the amplitude x_0 of the motion?

- $kg\,m/s^2 = \frac{kg\cdot m^{1-n}}{s^2} m^n$

$$m\ddot{x} = kx^n$$

k has units $\frac{kg}{s^2 m^{n-1}}$

$$\tau = 2\pi\sqrt{\frac{m}{k x_0^{n-1}}}$$

b. The diffusion constant of gases in air is about $10^{-5}\,m^2/s$. Estimate the timescale for diffusion of a gas across a room. If a gas valve in the room were to start leaking, could this explain the time scale it would take to smell it?

- $$\frac{\text{molecules}}{m} = \left(10^{-5}\frac{m^2}{s}\right)\cdot\left(\rho\cdot\frac{\text{molecules}}{m^3}\right)$$
$$\frac{10^5\,s}{m^2} = \left(\frac{10^{24}\text{ molecules}}{m^3}\right).$$

c. Some theoretical physicists are trying to unify gravity, quantum mechanics, and relativity, which are respectively governed by the constants G, $\hbar$, and c. Estimate in meters the fundamental length scale D that appears in such theories.

$G = 7\times10^{-11}\,\frac{m^3}{kg\,s^2}$

$\hbar = 10^{-34}\,kg\,m^2/s$

$c = 3\times10^8\,m/s$

$$\sqrt{\frac{\hbar\cdot G}{c^3}} = \sqrt{\frac{10^{-34}\,kg\,m^2/s\cdot 7\times10^{-11}\,m^3/kg\,s^2}{(3\times10^8)^3\,m^3/s^3}}$$
$$= \sqrt{\frac{7\times10^{-45}}{2.7\times10^{24}}}\,m = \sqrt{10^{-69}}\,m = \sqrt{10\cdot10^{-70}}\,m$$
$$= 3\times10^{-35}\,m$$