There will come soft rains and the smell of the ground,
And swallows circling with their shimmering sound;
And frogs in the pools singing at night,
And wild plum trees in tremulous white;
Robins will wear their feathery fire
Whistling their whims on a low fence-wire;
And not one will know of the war, not one
Will care at last when it is done.

Chapter 5: The Rigid Body Equations of Motion

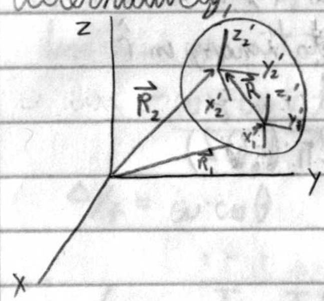
Section 1. Angular Momentum and Kinetic Energy of Motion about a Point

Chasles' Theorem (Chapter 4, Section 6) suggests that we should be able to split rigid body motion into a translational and rotational component. We have six coordinates to describe motion: 3 cartesian coordinates for translation and 3 Euler angles for rotation. If we set the origin at the center-of-mass, then we can divide the angular momentum and the kinetic energy into a translational and rotational component, like so

$$\vec{L} = \vec{R} \times M\vec{v} + \sum \vec{r}_i{}' \times \vec{p}_i{}' \qquad (1.28)$$

$$T = \tfrac{1}{2} M v^2 + \tfrac{1}{2} \sum m_i v_i{}'^2 \qquad (1.31)$$

In order to obtain angular momentum and kinetic energy of motion about some body point, we want to first show that the rotation angle of a rigid body displacement is independent of the origin. We can show this by remembering that all particles of the body move and rotate together. Alternatively,



$$\vec{R}_2 = \vec{R}_1 + \vec{R}$$

$$\left(\frac{d\vec{R}_2}{dt}\right)_s = \left(\frac{d\vec{R}_1}{dt}\right)_s + \left(\frac{d\vec{R}}{dt}\right)_s$$

$$= \left(\frac{d\vec{R}_1}{dt}\right)_s + \left(\frac{d\vec{R}}{dt}\right)_r + \vec{\omega}_1 \times \vec{R} \qquad (4.86)$$

$$= \left(\frac{d\vec{R}_1}{dt}\right)_s + \vec{\omega}_1 \times \vec{R}$$

$$\vec{R}_1 = \vec{R}_2 - \vec{R}$$

$$\left(\frac{d\vec{R}_1}{dt}\right)_s = \left(\frac{d\vec{R}_2}{dt}\right)_s - \vec{\omega}_2 \times \vec{R}$$

$$\vec{\omega}_1 \times \vec{R} - \vec{\omega}_2 \times \vec{R} = 0$$

$$(\vec{\omega}_1 - \vec{\omega}_2) \times \vec{R} = 0$$

$$\Rightarrow \vec{\omega}_1 = \vec{\omega}_2$$

Not one would mind, neither bird nor tree
If mankind perished utterly;
And Spring herself, when she woke at dawn,
Would scarcely know that we were gone. - Sara Teasdale (There will come soft rains, 1920)

From (1.28), if we keep one point stationary ($\vec{v} = 0$),

$$\vec{L} = m_i (\vec{r}_i \times \vec{v}_i) \qquad (5.1)$$

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i \qquad (5.2)$$

$$\vec{L} = m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

$$= m_i [\vec{\omega}(\vec{r}_i \cdot \vec{r}_i) - \vec{r}_i(\vec{r}_i \cdot \vec{\omega})]$$

$$= m_i [\vec{\omega} r_i^2 - \vec{r}_i (\vec{r}_i \cdot \vec{\omega})] \qquad (5.3)$$

Remembering the implicit summation

$$L_x = m_i [\omega_x r_i^2 - \omega_x x_i^2 - x_i y_i \omega_y - x_i z_i \omega_z]$$

$$= \omega_x m_i (r_i^2 - x_i^2) - \omega_y m_i x_i y_i - \omega_z m_i x_i z_i \qquad (5.4)$$

$$L_y = \omega_y m_i (r_i^2 - y_i^2) - \omega_x m_i y_i x_i - \omega_z m_i y_i z_i$$

$$L_z = \omega_z m_i (r_i^2 - z_i^2) - \omega_x m_i z_i x_i - \omega_y m_i z_i y_i$$

$$L_x = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z \qquad (5.5)$$

$$I_{xx} = m_i (r_i^2 - x_i^2) \qquad (5.6)$$

$$I_{xy} = -m_i x_i y_i$$

$$I_{xz} = -m_i x_i z_i \qquad (5.7)$$

The matrix $\overline{I}$ (uh oh, not identity matrix. Something, somethin context) is the inertia tensor. The diagonal terms are the moment of inertia coefficients and the off-diagonal terms are the products of inertia. The components can be written as

$$I_{jk} = \int_V \rho(\vec{r})(r^2 \delta_{jk} - x_j x_k)\, dV \qquad (5.8)$$

(5.5) can be rewritten as

$$\vec{L} = \overline{I}\, \vec{\omega} \qquad (5.9)$$

Section 2. Tensors.

We mention tensors in the previous section. In a Cartesian three-dimensional space, an N^{th}-rank tensor has 3^N components, and under an an orthogonal transformation $\overline{T}' = \overline{A}\,\overline{T}$,

$$T'_{ijk}(\vec{x}') = a_{il}\, a_{jm}\, a_{kn}\, T_{lmn}(\vec{x}) \qquad (5.10)$$

A scalar is a tensor of zero rank, a vector is a tensor of first rank, and a square matrix (assuming it satisfies (5.10)) is a tensor of second rank.

Section 3. The Inertia Tensor and the Moment of Inertia

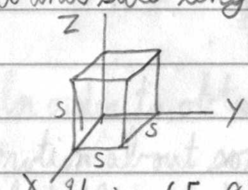
$$T = \tfrac{1}{2}\, m_i v_i^2$$
$$= \tfrac{1}{2}\, m_i \vec{v}_i \cdot (\vec{\omega} \times \vec{r}_i)$$
$$= \tfrac{\vec{\omega}}{2} \cdot m_i (\vec{r}_i \times \vec{v}_i)$$
$$= \tfrac{\vec{\omega} \cdot \vec{L}}{2} = \tfrac{1}{2}\, I \omega^2 \qquad (5.17)$$

where I is a scalar, defined as

$$I = \hat{n} \cdot \overleftrightarrow{I} \cdot \hat{n} = m_i \left[r_i^2 - (\vec{r}_i \cdot \hat{n})^2 \right] \qquad (5.18)$$

compare to (5.8).

As an example, consider we have a cube of uniform mass density μ and side length s with pivot at a corner



Using (5.8)

$$I_{11} = I_{22} = I_{33} = \int_0^s \int_0^s \int_0^s \mu \left(x_1^2 + x_2^2 + x_3^2 - x_1^2 \right) dx_1\, dx_2\, dx_3$$
$$= \mu \int_0^s \int_0^s \int_0^s (x_2^2 + x_3^2)\, dx_1\, dx_2\, dx_3$$
$$= \mu s \int_0^s \int_0^s (x_2^2 + x_3^2)\, dx_2\, dx_3$$
$$= \mu s \int_0^s \left(\tfrac{1}{3} \cdot s^3 + s x_3^2 \right) dx_3$$
$$= \mu s \left(\tfrac{1}{3} \cdot s^4 + \tfrac{1}{3} \cdot s^4 \right) = \tfrac{2}{3} \cdot \mu s^5$$

Off-diagonal terms are all equal

$$= -\mu \int_0^s \int_0^s \int_0^s x_1 x_2\, dx_1\, dx_2\, dx_3$$
$$= -\mu \int_0^s \int_0^s \tfrac{1}{2} \cdot s^2 x_2\, dx_2\, dx_3$$
$$= -\tfrac{1}{2} \cdot \mu s^2 \int_0^s \tfrac{1}{2}\, s^2\, dx_3$$
$$= -\tfrac{1}{4} \cdot \mu s^5$$

$$I = \tfrac{2}{3}(\mu s^5) \begin{bmatrix} 1 & -3/8 & -3/8 \\ -3/8 & 1 & -3/8 \\ -3/8 & -3/8 & 1 \end{bmatrix} = \tfrac{2}{3}\, \mu s^5 \begin{bmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{bmatrix}$$

Section 4. The Eigenvalues of the Inertia Tensor

We saw that the inertia tensor is symmetric,

$$I_{xy} = I_{yx} \qquad (5.29)$$

The eigenvalues of the inertia tensor are referred to as the components of the principal moment of inertia tensor.

In the previous example:

$$\begin{bmatrix} 1-\lambda & \alpha & \alpha \\ \alpha & 1-\lambda & \alpha \\ \alpha & \alpha & 1-\lambda \end{bmatrix} = (1-\lambda)\left[(1-\lambda)^2 - \alpha^2\right] - \alpha\left[\alpha(1-\lambda) - \alpha\right] + \alpha\left[\alpha^2 - \alpha(1-\lambda)\right]$$
$$= (1-\lambda)^3 - 3\alpha^2(1-\lambda) + 2\alpha^3$$
$$= (1-\lambda-\alpha)^2 (1-\lambda+2\alpha) = 0$$

$$\lambda_1 = \tfrac{1}{4}$$
$$\lambda_2 = \lambda_3 = \tfrac{11}{8}$$

$$|\lambda_1\rangle: \begin{bmatrix} 3/4 & -3/8 & -3/8 \\ -3/8 & 3/4 & -3/8 \\ -3/8 & -3/8 & 3/4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|\lambda_1\rangle = \tfrac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$|\lambda_2\rangle: \begin{bmatrix} -3/8 & -3/8 & -3/8 \\ -3/8 & -3/8 & -3/8 \\ -3/8 & -3/8 & -3/8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|\lambda_2\rangle = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$|\lambda_3\rangle = \tfrac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Section 5. Solving Rigid Body Problems and the Euler Equations of Motion

$$T = \tfrac{1}{2} M v^2 + \tfrac{1}{2} I \omega^2$$

The kinetic energy can be broken up into a translational and a rotational component

$$\mathcal{L} = \mathcal{L}_c(q_c, \dot{q}_c) + \mathcal{L}_b(q_b, \dot{q}_b)$$

$$\left(\frac{\partial \mathcal{L}}{\partial t}\right)_s = \left(\frac{\partial \mathcal{L}}{\partial t}\right)_b + \vec{\omega} \times \vec{L}$$

Compare to (4.82)

$$I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = N_1$$
$$I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) = N_2$$
$$I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = N_3 \qquad (5.39)$$

Euler's equations of motion for a rigid body with one point fixed

Section 6. Torque-Free Motion of a Rigid Body

If there are no outside forces, Euler's equations reduce to

$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3)$$
$$I_2 \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1)$$
$$I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2) \qquad (5.40)$$

Define a vector

$$\vec{\rho} = \vec{\omega}/\sqrt{2T} \qquad (5.41)$$

and a function

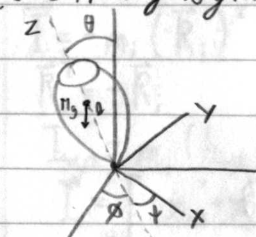
$$F(\rho) = \vec{\rho} \cdot \overleftrightarrow{I} \cdot \vec{\rho} = \rho_i^2 I_i \qquad (5.42)$$

If F is constant, it traces out an ellipse.

$$\nabla_{\rho} F = \sqrt{2/T}\, \vec{L} \qquad (5.43)$$

$$\vec{\rho} \cdot \vec{L}/L = \sqrt{2T}/L \qquad (5.44)$$

Section 7 The Heavy Symmetrical Top with One Point Fixed



$\dot{\psi}$ = rotation about body axis z

$\dot{\phi}$ = rotation of z about z'

$\dot{\theta}$ = bobbing up and down of z relative to z'

In general, $\dot{\psi} \gg \dot{\theta} \gg \dot{\phi}$ & $I_1 = I_2 \neq I_3$

Eulers equations become

$$I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) = N_1$$
$$I_2 \dot{\omega}_2 + \omega_1 \omega_3 (I_1 - I_3) = N_2$$
$$I_3 \dot{\omega}_3 = N_3$$

Using (4.87),

$$T = \tfrac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \tfrac{1}{2} I_3 \omega_3^2$$
$$= \tfrac{I_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \tfrac{I_3}{2} (\dot{\psi} + \dot{\phi} \cos\theta)^2 \qquad (5.50)$$

$$V = -M \vec{R} \cdot \vec{g}$$
$$= M g l \cos\theta \qquad (5.51)$$

$$\mathcal{L} = \tfrac{I_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \tfrac{I_3}{2} (\dot{\psi} + \dot{\phi}\cos\theta)^2 - M g l \cos\theta \qquad (5.52)$$

Since ϕ & ψ do not show up in the Lagrangian,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \phi}\right) - \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0$$

$$I_1 \dot{\phi} \sin^2\theta + I_3 (\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = 0$$

$$I_1 \dot{\phi} \sin^2\theta + I_3 \omega_3 \cos\theta = I_1 b \qquad (5.54)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \psi}\right) - \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = 0$$

$$I_3 (\dot{\psi} + \dot{\phi}\cos\theta) = I_3 \omega_3 = I_1 a \qquad (5.53)$$

$$I_1 \dot{\phi} \sin^2\theta + I_1 a \cos\theta = I_1 b$$

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \qquad (5.57)$$

$$I_3\dot{\psi} = I_1 a - I_3 \dot{\phi}\cos\theta$$

$$\dot{\psi} = \frac{I_1 a}{I_3} - \frac{b - a\cos\theta}{\sin^2\theta}\cos\theta \qquad (5.58)$$

$$E = {}^{I_1}\!/_2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + {}^{I_3}\!/_2\, \omega_3^2 + Mgl\cos\theta \qquad (5.55)$$

$$E - \frac{I_3\omega_3^2}{2} = \frac{I_1\dot{\theta}^2}{2} + \frac{I_1(b - a\cos\theta)^2}{2\sin^2\theta} + Mgl\cos\theta \qquad (5.59)$$

which is the one-dimensional problem with

$$V'(\theta) = \frac{I_1}{2}\left(\frac{b - a\cos\theta}{\sin\theta}\right)^2 + Mgl\cos\theta \qquad (5.60)$$

If we define the four constants of motion as

$$\alpha = \frac{2E - I_3\omega_3^2}{I_1}$$

$$\beta = \frac{2Mgl}{I_1}$$

$$a = \frac{p_\psi}{I_1}$$

$$b = \frac{p_\phi}{I_1} \qquad (5.61)$$

$$\alpha = \dot{\theta}^2 + \frac{(b - a\cos\theta)^2}{\sin^2\theta} + \beta\cos\theta \qquad (5.62)$$

let $u = \cos\theta$

$$\dot{u} = -\dot{\theta}\sin\theta$$

$$\alpha = \frac{\dot{u}^2}{\sin^2\theta} + \frac{(b - a u)^2}{\sin^2\theta} + \beta u$$

$$\sin^2\theta(\alpha - \beta u) = \dot{u}^2 + (b - au)^2$$

$$\dot{u}^2 = (1 - u^2)(\alpha - \beta u) - (b - au)^2$$

$$t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(1-u^2)(\alpha - \beta u) - (b - au)^2}} \qquad (5.63)$$

Derivations.

1. $\overleftrightarrow{I} = -m_i(\overleftrightarrow{R}_i)^2$

$$\overleftrightarrow{R}_i^T = -\overleftrightarrow{R}_i \qquad \text{antisymmetric}$$

$$I_{\alpha\beta} = m_i(\delta_{\alpha\beta} r_i^2 - r_{i\alpha} r_{i\beta}) \qquad (5.22)$$

$$\overleftrightarrow{I} = m_i \begin{bmatrix} x_2^2 + x_3^2 & -x_1x_2 & -x_1x_3 \\ -x_2x_1 & x_1^2 + x_3^2 & -x_2x_3 \\ -x_3x_1 & -x_3x_2 & x_2^2 + x_3^2 \end{bmatrix}$$

$$(\overleftrightarrow{R}_i)^2 = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -x_3^2 - x_2^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & -x_3^2 - x_1^2 & x_3x_2 \\ x_1x_3 & x_2x_3 & -x_2^2 - x_1^2 \end{bmatrix}$$

$$\overleftrightarrow{I} = -m_i(\overleftrightarrow{R}_i)^2$$

2. $I = m_i(\vec{r}_i \times \hat{n}) \cdot (\vec{r}_i \times \hat{n}) = m_i[(\vec{r}_i \cdot \vec{r}_i)(\hat{n}\cdot\hat{n}) - (\vec{r}_i \cdot \hat{n})(\hat{n}\cdot\vec{r}_i)]$

$$= m_i[r_i^2 - (\vec{r}_i \cdot \hat{n})^2] \qquad (5.18)$$

3. $T = \tfrac{1}{2} m_i v_i^2$

$$\frac{dT}{dt} = \frac{m_i}{4} v_i \frac{dv_i}{dt} = \frac{m_i(\vec{\omega} \times \vec{r}_i)}{2} \cdot \frac{d\vec{v}_i}{dt}$$

$$= \frac{m_i}{2} \vec{\omega} \cdot \left(\vec{r}_i \times \frac{d\vec{v}_i}{dt}\right) = \vec{\omega} \cdot \left(m\vec{r}_i \times \frac{d\vec{v}_i}{dt}\right) = \vec{\omega} \times \vec{N}$$

$$\vec{N} = \vec{r} \times \vec{F}$$

$$= \vec{r} \times m\frac{d\vec{v}}{dt}$$

4. $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\psi}_i}\right) - \dfrac{\partial T}{\partial \psi_i} = N_i$ (1.53)

$T = \frac{1}{2} \cdot I_i \omega_i^2$

$\dfrac{d}{dt}\left(I_i \omega_i \dfrac{d\omega_i}{d\dot{\psi}}\right) - \dfrac{\partial}{\partial \psi}\left(\dfrac{I_i \omega_i^2}{2}\right) = N_i$

$\dfrac{d}{dt}\left(I_i \omega_i \dfrac{d\omega_i}{d\dot{\psi}}\right) - \dfrac{I_i}{2} \cdot 2\omega_i \dfrac{d\omega_i}{d\psi} = N_i$

From (4.87),

$\dfrac{d\omega_1}{d\dot{\psi}} = 0 \qquad \dfrac{d\omega_1}{d\psi} = \omega_2$

$\dfrac{d\omega_2}{d\dot{\psi}} = 0 \qquad \dfrac{d\omega_2}{d\psi} = -\omega_1$

$\dfrac{d\omega_3}{d\dot{\psi}} = 1 \qquad \dfrac{d\omega_3}{d\psi} = 0$

$\frac{d}{dt}(I_3 \omega_3) - I_1 \omega_1 \omega_2 + I_2 \omega_2 \omega_1 = N_3$

$I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = N_3$

By permuting x, y, and z, we can return the rest of (5.39)

5. In (5.39), I_i is a constant

$I \dfrac{d\omega_i}{dt} = \dfrac{d(I\omega_i)}{dt} - \omega_i \dfrac{dI}{dt}$

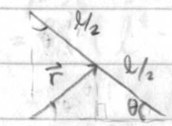
$u = I \qquad v = \omega_i$

$u' = \frac{dI}{dt} \qquad v' = \frac{d\omega_i}{dt}$

Problems.

13.

a.



$\vec{r} = \frac{l}{2} \cdot \cos\theta \hat{x} + \frac{l}{2} \cdot \sin\theta \cdot \hat{y}$

$\dot{\vec{r}} = \frac{l}{2} \cdot \dot{\theta} \sin\theta \, \hat{x} - \frac{l}{2} \cdot \dot{\theta} \cos\theta \, \hat{y}$

$T_{cm} = \frac{m}{2} \cdot \dot{r}^2 = \frac{m}{2} \cdot \left(\frac{l\dot{\theta}}{2}\right)^2 = \frac{ml^2\dot{\theta}^2}{8}$

$T_{rot} = \frac{1}{2} \, \underline{I} \, \dot{\theta}^2$

$\underline{I} = \int_{-l/2}^{l/2} \rho(r) \cdot r^2 \, dr = \int_{-l/2}^{l/2} \frac{mr^2}{l} \cdot dr$ (5.23)

$= \frac{m}{3l} \cdot r^3 \Big|_{-l/2}^{l/2} = \frac{m}{3l}\left(\frac{l^3}{8} + \frac{l^3}{8}\right)$

$= \frac{ml^2}{12}$

$T = T_{cm} + T_{rot} = \frac{ml^2\dot{\theta}^2}{8} + \frac{ml^2\dot{\theta}^2}{24} + \frac{ml^2\dot{\theta}^2}{6}$

Since this is the KE of one side total $T = \frac{ml^2\dot{\theta}^2}{3}$

$V = 2 \cdot mg \cdot \frac{l}{2} \cdot \sin\theta = mgl\sin\theta$

$mgl \sin(30) = \frac{ml^2\dot{\theta}^2}{3}$

$\frac{3g}{2} = l\dot{\theta}^2$

$\dot{\theta} = \sqrt{\frac{3g}{2l}}$

$v = l\dot{\theta} = \sqrt{\frac{3gl}{2}}$

b. $\dfrac{d\theta}{dt} = \sqrt{\dfrac{3g}{2l}}$

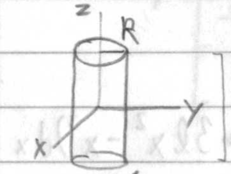
$d\theta = \sqrt{\frac{3g}{2l}} \, dt$

$\int_{\pi/6}^{0} d\theta = \sqrt{\frac{3g}{2l}} \, t$

$t = \frac{\pi}{6}\sqrt{\frac{2l}{3g}}$

14. For an inertia ellipsoid to be a sphere, $I_x = I_y = I_z$

$I_x = I_y$



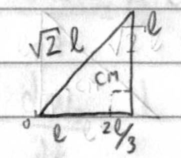
$\rho = \frac{m}{\pi R^2 h}$

$I_z = \int_0^R \rho \cdot 2\pi h r^3 \, dr$

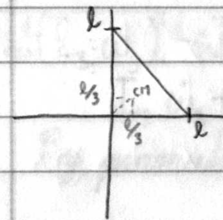
$= \dfrac{m \cdot 2\pi h}{\pi R^2 h} \cdot \dfrac{R^4}{4} = \dfrac{mR^2}{2}$

$$I_x = \frac{m}{h}\int_{-h/2}^{h/2}\left(\frac{R^2}{4} + z^2\right)dz$$
$$= \frac{m}{h}\left[\frac{R^2 z}{4} + \frac{z^3}{3}\right]\Big|_{-h/2}^{h/2}$$
$$= \frac{m}{h}\left[\frac{R^2 h}{4} + \frac{h^3}{24}\right]$$
$$= \frac{mR^2}{4} + \frac{mh^2}{24}$$
$$\frac{mR^2}{4} + \frac{mh^2}{24} = \frac{mR^2}{2}$$
$$h^2 = 24\cdot\frac{R^2}{4}$$
$$= 6R^2$$
$$h = \sqrt{6}\,R$$

15.



$$\rho = \frac{2M}{l^2}$$
$$x_{cm} = \frac{\int x\,dm}{M} = \frac{\int_0^l x^2\rho\,dx}{M} = \frac{2M}{l^2 M}\cdot\frac{1}{3}\cdot l^3$$
$$= \frac{2l}{3}$$



Using the parallel-axis theorem

$$I_{cm} = I - md^2$$

Since these are tensors, we want to figure out the inertia tensor of the figure on the left and the inertia tensor of a particle at the center of mass

$$I_{\alpha\beta} = \int_V \rho(\vec{r})(\delta_{\alpha\beta} r^2 - r_\alpha r_\beta)\,dV \qquad (5.23)$$

$$I_{11} = \rho\int_0^l\int_0^l y^2\,dx\,dy$$
$$= \rho\int_0^l\int_0^{l-x}(l-x)^2\,dx\,dx$$
$$= \rho\int_0^l l^2(l-x) - l(l-x)^2 + \frac{1}{3}(l-x)^3\,dx$$
$$= \rho\int_0^l l^3 - l^2x - l^3 + 2l^2x - lx^2 + \frac{1}{3}(l^3 - 3l^2x + 3lx^2 - x^3)\,dx$$
$$= \rho\int_0^l \frac{1}{3}(l^3 - x^3)\,dx$$
$$= \frac{\rho}{3}\left(l^4 - \frac{l^4}{4}\right) = \frac{\rho l^4}{4}$$
$$= \frac{Ml^2}{2}$$

$$I_{22} = \rho\int_0^l\int_0^{l-x} x^2\,dx\,dx$$
$$= \rho\int_0^l \frac{1}{3}(l-x)^3\,dx = \frac{\rho}{3}\int_0^l (l^3 - 3xl^2 + 3x^2l - x^3)\,dx$$
$$= \frac{\rho}{3}\left(l^4 - \frac{3}{2}l^4 + l^4 - \frac{l^4}{4}\right)$$
$$= \frac{\rho}{3}\left(\frac{l^4}{4}\right) = \frac{Ml^2}{6}$$

$$I_{33} = \rho\int_0^l\int_0^{l-x} x^2 + (l-x)^2\,dx\,dx$$
$$= \rho\int_0^l\int_0^{l-x} l^2 - 2lx + 2x^2\,dx\,dx$$
$$= \rho\int_0^l l^2(l-x) - l(l-x)^2 + \frac{2}{3}(l-x)^3\,dx$$
$$= \rho\int_0^l\left(\frac{2}{3}\cdot l^3 - xl^2 + x^2l - \frac{2}{3}\cdot x^3\right)dx$$
$$= \rho\left[\frac{2}{3} - \frac{1}{2} + \frac{1}{3} - \frac{2}{12}\right]l^4 = \rho\cdot\frac{l^4}{3} = \frac{2Ml^2}{3}$$

$$I_{12} = I_{21} = -\rho\int_0^l\int_0^{l-x} x(l-x)\,dx\,dx$$
$$= -\rho\int_0^l \frac{l}{2}(l-x)^2 - \frac{1}{3}(l-x)^3\,dx$$
$$= -\rho\int_0^l\left(\frac{l^3}{6} - \frac{x^2 l}{2} + \frac{x^3}{3}\right)dx$$
$$= -\rho\left[\frac{1}{6} - \frac{1}{6} + \frac{1}{12}\right]l^4$$
$$= -\frac{Ml^2}{6}$$

$$I_{cm} = \begin{bmatrix} \frac{Ml^2}{2} & -\frac{Ml^2}{6} & 0 \\ -\frac{Ml^2}{6} & \frac{Ml^2}{6} & 0 \\ 0 & 0 & \frac{2Ml^2}{3} \end{bmatrix}$$

For particle, $\rho = \delta(x - l/3)\,\delta(y - l/3)$.

$$I_{11} = I_{22} = M\cdot x^2 = \frac{Ml^2}{9}$$
$$I_{33} = M(x^2 + y^2) = \frac{2Ml^2}{9}$$
$$I_{12} = I_{21} = -M(xy) = -\frac{Ml^2}{9}$$

$$I_p = \begin{bmatrix} \frac{Ml^2}{9} & -\frac{Ml^2}{9} & 0 \\ -\frac{Ml^2}{9} & \frac{Ml^2}{9} & 0 \\ 0 & 0 & \frac{2Ml^2}{9} \end{bmatrix}$$

$$I_{cm} = I - I_p = Ml^2 \begin{bmatrix} 7/18 & -5/18 & 0 \\ -5/18 & 1/18 & 0 \\ 0 & 0 & 16/18 \end{bmatrix}$$

$$= \frac{Ml^2}{18} \begin{bmatrix} 7 & -5 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

$$\det(I_{cm} - \lambda I) = \det \begin{bmatrix} 7-\lambda & -5 & 0 \\ -5 & 1-\lambda & 0 \\ 0 & 0 & 16-\lambda \end{bmatrix}$$

$$= (7-\lambda)(1-\lambda)(16-\lambda) + 5(-5)(16-\lambda) = 0$$

$$= (16-\lambda)\left[(7-\lambda)(1-\lambda) - 25\right] = 0$$

$$= (16-\lambda)\left[7 - 8\lambda + \lambda^2 - 25\right] = 0$$

$$\lambda = 16, \frac{8 \pm \sqrt{64+72}}{2}$$

$$= 16,\ 4 \pm \sqrt{34}$$

16. $$2 = \frac{40x_1 + 50x_2 + 60x_3}{150}$$

$$0 = \frac{40x_1 + 50x_2 + 60x_3 + 70x_4}{220}$$

$$300 = -70x_4$$

$$x_4 = -30/7$$

place at $(-30/7, -30/7, -30/7)$

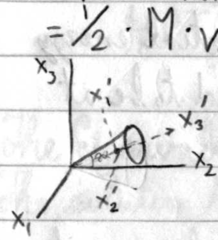
17. 

$$x_{cm} = \frac{\int x\,dm}{M} = \frac{\int_0^h x \cdot \rho \cdot \pi (x \tan\alpha)^2 \, dx}{\pi (h \tan\alpha)^2 \cdot h/3 \cdot \rho}$$

$$= \frac{1/4 \cdot h^4}{h^3/3} = 3h/4$$

$$T = T_{cm} + T_{rot}$$

$$= \tfrac{1}{2} \cdot M \cdot v_{cm}^2 + \tfrac{1}{2} \cdot I\omega^2$$



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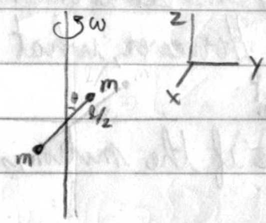
a. 

$$I_1 \dot\omega_1 - \omega_2 \omega_3 (I_2 - I_3) = N_1$$

$$I_2 \dot\omega_2 - \omega_3 \omega_1 (I_3 - I_1) = N_2$$

$$I_3 \dot\omega_3 - \omega_1 \omega_2 (I_1 - I_2) = N_3 \quad (5.39)$$

$$\omega_1 = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$

$$\omega_2 = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$$

$$\omega_3 = \dot\phi \cos\theta + \dot\psi \quad (4.87)$$

I'm so sorry