Goldstein, Classical Mechanics
Third Edition, International

BS

CLASS SCHEDULE

DATE ______________________

NAME ______________________ SCHOOL ______________________

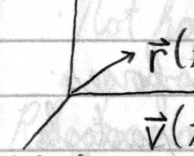
ADDRESS ______________________

PERIOD	Monday	ROOM	Tuesday	ROOM	Wednesday	ROOM	Thursday	ROOM	Friday	ROOM
1										
2										
3										
4										
5										
6										
7										
8										
9										

Do nothing out of selfish ambition or vain conceit. Rather, in humility value others above yourself, not looking to your own interests, but each of you to the interests of others. - Philippians 2:3-4 (NIV)

Chapter 1: Survey of the Elementary Principles

Section 1. Mechanics of a particle



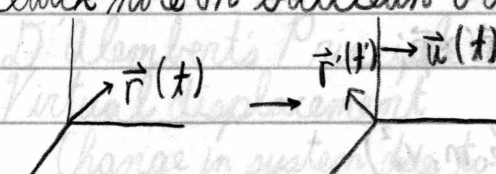
$\vec{r}(t) = (x(t), y(t), z(t))$

$\vec{v}(t) = \frac{d\vec{r}}{dt}$

- Define, given $\vec{r}(t)$ + $\vec{v}(t)$ for some particle:
  - Linear momentum: $\vec{p} = m\vec{v}$
  - Force: $\vec{F} = \frac{d\vec{p}}{dt} = \dot{\vec{p}}$
    - $\vec{F} = 0 \Rightarrow$ uniform motion
    - $\vec{F} = m\ddot{\vec{x}}$
    - $\vec{F}_{ij} = -\vec{F}_{ji}$

  $\Bigg\}$ Reformulation of Newton's Laws

- Assumptions in Classical Mechanics
  - Observer is in an inertial reference frame
  - Observer + instrument have no effect on the system and vice versa
- Quick note on Galilean Transformation

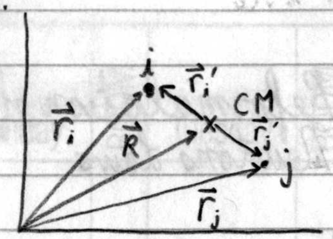


$\vec{r}'(t') = \vec{r}(t) + \vec{u}\cdot t$
$t' = t$

- Conservation Laws
  - Linear momentum
  - Angular momentum
    - $\vec{L} = \vec{r} \times \vec{p}$ angular momentum
    - $\vec{N} = \vec{r} \times \vec{F} = \dot{\vec{L}}$ torque
  - Energy
    - $W_{12} = \int_1^2 \vec{F}\cdot d\vec{s} = m\int_1^2 \dot{\vec{v}}\cdot\vec{v}\,dt$ Work
      $= \frac{m}{2}\int_1^2 \frac{d(\vec{v}\cdot\vec{v})}{dt}dt$
      $= \frac{1}{2}mv^2\Big|_1^2 = T_2 - T_1$ Kinetic energy
    - Does not depend on path
      $\int_1^2 \vec{F}\cdot d\vec{s} = \int_1^2 -\vec{\nabla}V\cdot d\vec{s} = -\int_1^2 dV \Rightarrow W_{12} = V_1 - V_2 = T_2 - T_1$
      $E_1 = T_1 + V_1 = T_2 + V_2 = E_2$ $E = T + V$ in a conservative field

Section 2. Mechanics of a system of particles

- Given n particles: $\{\vec{r}_i\}_{i=1,N}$
  - Distinguish between internal and external forces
  - $\sum_j \vec{F}_{ji} + \vec{F}_i^{(e)} = \dot{\vec{p}}_i$ equation of motion of i particle
  - $\vec{r}_i = \vec{R} + \vec{r}_i' \quad \forall i$
  - e.g.



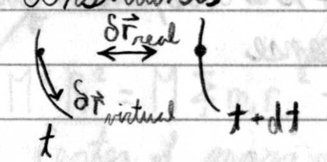
- Center of mass coordinates
  - $\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M}$
  - $\vec{P} = \sum_i m_i \dot{\vec{r}}_i = M\dot{\vec{R}}$
  - $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \vec{R} \times M\dot{\vec{R}} + \sum_i \vec{r}_i' \times \vec{p}_i'$
- Energy equations
  - $W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i = \sum_i \int_1^2 d(\tfrac{1}{2} m_i v_i^2)$
  - $W_{12} = T_2 - T_1$ $\qquad T = \tfrac{1}{2} M v^2 + \tfrac{1}{2} \sum_i m_i v_i'^2$
  - $W_{12}^{ext} = \sum_i \int_1^2 \vec{F}_i^{ext} \cdot d\vec{s}_i = -\sum_i \int_1^2 (\vec{\nabla}_i V) \cdot d\vec{s}_i = V_1 - V_2$
  - $W_{12}^{int} = \sum_{ij} \tfrac{1}{2}\int_1^2 \vec{F}_{ji} \cdot d\vec{s}_i = -\tfrac{1}{2} \sum_{ij} V_{ij} \Big|_1^2$
  - $E = \sum_i \tfrac{1}{2} m_i |\vec{v}_i|^2 + \sum_i V(\vec{r}_i) + \tfrac{1}{2} \sum_{ij} V(|\vec{r}_{ij}|)$

Section 3 Constraints

- Holonomic
  - $f(\vec{r}_1, \vec{r}_2, \ldots, t) = 0$
  - e.g. rigid body
  - e.g. particle constrained to move on a curve or given surface
- Non-holonomic
  - Not holonomic
  - e.g. walls of a gas container
- Rheonomous
  - Time-dependant
  - e.g. bead sliding on a moving wire
- Scleronomous
  - Time-independant
  - e.g. bead sliding on a rigid wire
- Generalized coordinates
  - Holonomic constraints
  - If no constraints, N particles will have 3N degrees of freedom
  - With constraints, 3N-k

Section 4. D'Alembert's Principle and Lagrange's Equations

- Virtual displacement
  - Change in system due to $\delta\vec{r}_i$. Maintain forces and constraints



At equilibrium, $\vec{F}_i = 0$

- $\sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0$
- $\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$ $\qquad$ applied force + force of constraint

$\sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0 \Rightarrow \sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i = 0$ $\qquad$ principle of virtual work

Section 4 · D'Alembert's principle

$\vec{F}_i = \dot{\vec{p}}_i$

$\vec{F}_i - \dot{\vec{p}}_i = 0$

$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$

· $\sum_i \vec{F}_i \cdot \delta \vec{r}_i = \sum_j Q_j \delta q_j$

· $\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots, q_n, t)$

· $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$ components of generalized force

· $\sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i$

· $\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right]$$

· Plugging back into D'Alembert's principle

$$\sum_j \left\{ \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j = 0$$

· $L = T - V$ Lagrangian

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0$$

Section 5 Velocity-dependant potentials and the dissipation function

· If no potential in the traditional sense

$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF}{dt}$

· $Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j} \right)$

· $L = T - U$ U is generalized potential

· Example: electric charge (q) with mass (m) moving at velocity ($\vec{v}$) in electric field ($\vec{E}$) + magnetic field ($\vec{B}$)

$\vec{F} = q[\vec{E} + (\vec{v} \times \vec{B})]$

· $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$ $\phi(x, y, z, t)$ scalar potential

· $\vec{B} = \nabla \times \vec{A}$ $\vec{A}(x, y, z, t)$ vector potential

$U = q\phi - q\vec{A} \cdot \vec{v}$

· $L = \frac{1}{2} mv^2 - q\phi + q\vec{A} \cdot \vec{v}$

· Frictional forces

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j$$

· $\mathcal{F} = \frac{1}{2} \sum_i (k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2)$ Rayleigh's dissipation function

· $\vec{F}_f = -\nabla_v \mathcal{F}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0$$

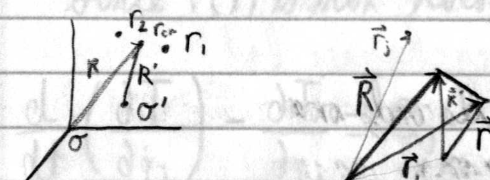
Section 6 Simple applications of the Lagrangian formulation

Derivations

1. $T = \frac{1}{2} m\vec{v} \cdot \vec{v}$

$\frac{dT}{dt} = \frac{m}{2} \cdot 2\dot{\vec{v}} \cdot \vec{v} = m\vec{a} \cdot \vec{v} = \vec{F} \cdot \vec{v}$

$\frac{d(mT)}{dt} = \frac{d}{dt}\left[\frac{1}{2} (m\vec{v})^2 \right] = \frac{d(m\vec{v})}{dt} \cdot m\vec{v} = \vec{F} \cdot \vec{p}$

2. 

$M^2 R^2 = M \sum_i m_i r_i^2 - \frac{1}{2} \sum_{i,j} m_i m_j r_{ij}^2$

Center of mass of each pair, then subtract out double-counted terms

3. $M\frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^{(e)}$

$\frac{dL}{dt} = \vec{N}^{(e)}$

Say two particles are at rest

$M\cdot 0 = m_1 a_1 + m_2 a_2$

$\Rightarrow m_1 a_1 = -m_2 a_2$ or

$\vec{F}_1^{(e)} = -\vec{F}_2^{(e)}$

4. $dx - a\sin\theta\, d\phi = 0$ (1)

$dy + a\cos\theta\, d\phi = 0$ (2)

(1): $\frac{\partial(f\cdot 1)}{\partial \phi} = \frac{\partial(f\cdot(-a\sin\theta))}{\partial x}$

$f = X(x)Q(\phi)$

$\frac{\partial f}{\partial \phi} = Q'(\phi)X(x)$

$\frac{\partial(-af\sin\theta)}{\partial x} = -a\sin\theta X'Q$

$Q'X = -a\sin\theta X'Q$

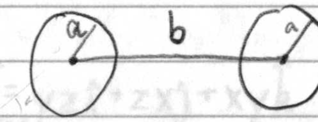
Can't find a Q that depends only on ø or an X dependant only on x.

Alternatively, can see that $g_\theta = 0$,

$\frac{\partial(-af\sin\theta)}{\partial\theta} = \frac{\partial(0)}{\partial\phi}$

the only solution is the trivial solution

Can show the same for (2)

5. 

$v = a\dot{\phi}$ $\quad v' = a\dot{\phi}'$

$\dot{x} = v\sin\theta$ $\quad \dot{x}' = v'\sin\theta$

$\dot{y} = -v\cos\theta$ $\quad \dot{y}' = -v'\cos\theta$

$\cos\theta\, dx + \sin\theta\, dy = 0$

$v\cos\theta\sin\theta - v\sin\theta\cos\theta = 0$

$\sin\theta\, dx - \cos\theta\, dy = \frac{1}{2}a(d\phi + d\phi')$

$\sin\theta\cdot v\sin\theta - \cos\theta\cdot(-v\cos\theta)$

$= v\sin^2\theta + v\cos^2\theta = v = \frac{1}{2}(v+v') = \frac{1}{2}a(d\phi + d\phi')$

$\theta = C - \frac{a}{b}(\phi - \phi')$

$\phi - \phi' = r - r'$

$\theta = C - \frac{a}{b}(r - r')$

6. $y\,dx - [f(t) - x]\,dy = 0$

Since f(t) is non-holonomic, cannot be solved

7. $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \left(\frac{\partial T}{\partial t}\right)\frac{\partial}{\partial \dot{q}_j} = \frac{\partial \dot{T}}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}$

$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j$

8. $\frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}'}{\partial q_j} = 0$

$\frac{d}{dt}\left(\frac{\partial(\mathcal{L})}{\partial \dot{q}_j} + \frac{\partial^2 F}{\partial \dot{q}_j \partial t}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_j} + \frac{\partial^2 F}{\partial q_j \partial t}\right) = 0$

$\partial \dot{q}_j = \frac{\partial q_j}{\partial t}$

$\partial \dot{q}_j \, \partial t = \partial q_j$

$\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j}\right] + \left[\frac{d}{dt}\left(\frac{\partial^2 F}{\partial \dot{q}_j \partial t}\right) - \frac{\partial^2 F}{\partial q_j \partial t}\right] = 0$

$\frac{\partial^2 F}{\partial q_j \partial t} - \frac{\partial^2 F}{\partial q_j \partial t} = 0$

9. $\mathcal{L} = \frac{1}{2} m v^2 - q\phi + q\vec{A}\cdot\vec{v}$

$\vec{A} \to \vec{A} + \vec{\nabla}\psi(\vec{r}, t)$

$\phi \to \phi - \frac{1}{c}\frac{\partial \psi}{\partial t}$

$\mathcal{L}' = \frac{1}{2} m v^2 - q\phi + \frac{q}{c}\frac{\partial \psi}{\partial t} + q\vec{A}\cdot\vec{v} + q\vec{\nabla}\psi\cdot\vec{v}$

$= \mathcal{L} + q\vec{\nabla}\psi\cdot\vec{v} + \frac{q}{c}\frac{\partial \psi}{\partial t}$

Using the previous, Lagrangian changes, but equations of motion do not

10. $\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \cdot \frac{\partial \dot{s}_j}{\partial \dot{s}_j} = \frac{\partial \mathcal{L}}{\partial \dot{s}_j} \cdot \frac{\partial \dot{s}_j}{\partial \dot{q}_j}$

$\frac{\partial \mathcal{L}}{\partial q_j} \cdot \frac{\partial s_j}{\partial s_j} = \frac{\partial \mathcal{L}}{\partial s_j} \cdot \frac{\partial s_j}{\partial q_j}$

Problems

11. $\vec{F} = yz\hat{\imath} + zx\hat{\jmath} + xy\hat{k}$

Conservative if $\nabla G = \vec{F}$ for some G

$\frac{\partial}{\partial x} G = yz$ $\qquad$ $G = xyz$

$\frac{\partial}{\partial y} G = zx$ $\qquad$ $\Rightarrow \vec{F}$ is conservative

$\frac{\partial}{\partial z} G = xy$

12. $F = \frac{mv^2}{r} = \frac{m\omega^2 r^2}{r}$ $\qquad$ $v = \omega r$, $T = 2\pi\omega$

$ma = m\omega^2 r$

$\omega^2 = a/r = \frac{9.83\ m/s^2}{6.72\times10^6} =$

$\omega = 1.2\times10^{-3}/s$

$T = 7.5\times10^{-3}/s$

13. $dv = -v' \frac{dm}{m} - g\,dt$

$v_f - v_0 = -v' \ln(m)\Big|_{m_0}^{m_f} - g\cdot t$

$11.2\ km/s = -2.1\ km/s\ \ln(m_f/m_0) - 9.8\ m/s^2 \cdot 60s$

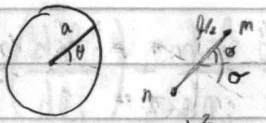
$\ln\left(\frac{m_f}{m_0}\right) = \ln\left(\frac{m_e}{m_e + m_f}\right) = \frac{-9.8\cdot60 - 11.2\times10^3}{2.1\times10^3}$

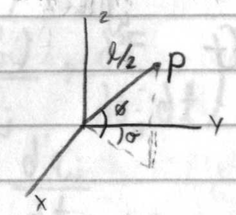
$\frac{m_e}{m_e + m_f} = 3.6\times10^{-3}$

$m_e = 3.6\times10^{-3}(m_e + m_f)$

$\frac{m_f}{m_e} = \frac{3.6\times10^{-3}}{1 - 3.6\times10^{-3}} = 3.6\times10^{-3}$

$\frac{m_e}{m_f} = 276.$

14. 

$\vec{x}$ 

$p_x = \frac{l}{2}\cos\phi\sin\sigma$

$p_y = \frac{l}{2}\cos\phi\cos\sigma$

$p_z = \frac{l}{2}\sin\phi$

$$\vec{x}_1 = \left(a\cos\theta + \tfrac{l}{2}\cos\phi\sin\sigma,\ a\sin\theta + \tfrac{l}{2}\cos\phi\cos\sigma,\ \tfrac{l}{2}\sin\sigma\right)$$
$$\vec{x}_2 = \left(a\cos\theta - \tfrac{l}{2}\cos\phi\sin\sigma,\ a\sin\theta - \tfrac{l}{2}\cos\phi\cos\sigma,\ -\tfrac{l}{2}\sin\sigma\right)$$
$$\dot{\vec{x}}_1 = \left(-a\dot\theta\sin\theta - \tfrac{l\dot\phi}{2}\sin\phi\sin\sigma + \tfrac{l\dot\sigma}{2}\cos\phi\cos\sigma,\right.$$
$$a\dot\theta\cos\theta - \tfrac{l\dot\phi}{2}\sin\phi\cos\sigma - \tfrac{l\dot\sigma}{2}\cos\phi\sin\sigma,$$
$$\left.\tfrac{l\dot\phi}{2}\cos\phi\right)$$
$$|\dot{\vec{x}}_1|^2 = a^2\dot\theta^2\sin^2\theta + \cancel{\tfrac{al\dot\theta\dot\phi}{2}\sin\theta\sin\phi\sin\sigma} - \cancel{\tfrac{al\dot\theta\dot\sigma}{2}\sin\theta\cos\phi\cos\sigma}$$
$$+\cancel{\tfrac{al\dot\theta\dot\phi}{2}\sin\theta\sin\phi\sin\sigma} + \tfrac{l^2\dot\phi^2}{4}\sin^2\phi\sin^2\sigma - \cancel{\tfrac{l^2\dot\phi\dot\sigma}{4}\sin\phi\sin\sigma\cos\phi\cos\sigma}$$
$$-\cancel{\tfrac{al\dot\theta\dot\sigma}{2}\sin\theta\cos\phi\cos\sigma} - \cancel{\tfrac{l^2\dot\phi\dot\sigma}{4}\sin\phi\sin\sigma\cos\phi\cos\sigma} + \tfrac{l^2\dot\sigma^2}{4}\cos^2\phi\cos^2\sigma$$
$$+a^2\dot\theta^2\cos^2\theta - \cancel{\tfrac{al\dot\theta\dot\phi}{2}\cos\theta\sin\phi\cos\sigma} - \cancel{\tfrac{al\dot\theta\dot\sigma}{2}\cos\theta\cos\phi\sin\sigma}$$
$$-\cancel{\tfrac{al\dot\theta\dot\phi}{2}\cos\theta\sin\phi\cos\sigma} + \tfrac{l^2\dot\phi^2}{4}\sin^2\phi\cos^2\sigma + \cancel{\tfrac{l^2\dot\phi\dot\sigma}{4}\sin\phi\cos\sigma\cos\phi\sin\sigma}$$
$$-\cancel{\tfrac{al\dot\theta\dot\sigma}{2}\cos\theta\cos\phi\sin\sigma} + \cancel{\tfrac{l^2\dot\phi\dot\sigma}{4}\sin\phi\cos\sigma\cos\phi\sin\sigma} + \tfrac{l^2\dot\sigma^2}{4}\cos^2\phi\sin^2\sigma$$
$$+\tfrac{l^2\dot\phi^2}{4}\cos^2\phi$$
$$\dot{\vec{x}}_2 = \left(-a\dot\theta\sin\theta + \tfrac{l\dot\phi}{2}\sin\phi\sin\sigma - \tfrac{l\dot\sigma}{2}\cos\phi\cos\sigma,\right.$$
$$a\dot\theta\cos\theta + \tfrac{l\dot\phi}{2}\sin\phi\cos\sigma + \tfrac{l\dot\sigma}{2}\cos\phi\sin\sigma,$$
$$\left.-\tfrac{l\dot\phi}{2}\cos\phi\right)$$
$$|\dot{\vec{x}}_2|^2 = a^2\dot\theta^2\sin^2\theta - \cancel{\tfrac{al\dot\theta\dot\phi}{2}\sin\theta\sin\phi\sin\sigma} + \cancel{\tfrac{al\dot\theta\dot\sigma}{2}\sin\theta\cos\phi\cos\sigma}$$
$$-\cancel{\tfrac{al\dot\theta\dot\phi}{2}\sin\theta\sin\phi\sin\sigma} + \tfrac{l^2\dot\phi^2}{4}\sin^2\phi\sin^2\sigma - \cancel{\tfrac{l^2\dot\phi\dot\sigma}{4}\sin\phi\sin\sigma\cos\phi\cos\sigma}$$
$$+\cancel{\tfrac{al\dot\theta\dot\sigma}{2}\sin\theta\cos\phi\cos\sigma} - \cancel{\tfrac{l^2\dot\phi\dot\sigma}{4}\sin\phi\sin\sigma\cos\phi\cos\sigma} + \tfrac{l^2\dot\sigma^2}{4}\cos^2\phi\cos^2\sigma$$
$$+a^2\dot\theta^2\cos^2\theta + \cancel{\tfrac{al\dot\theta\dot\phi}{2}\cos\theta\sin\phi\cos\sigma} + \cancel{\tfrac{al\dot\theta\dot\sigma}{2}\cos\theta\cos\phi\sin\sigma}$$
$$+\cancel{\tfrac{al\dot\theta\dot\phi}{2}\cos\theta\sin\phi\cos\sigma} + \tfrac{l^2\dot\phi^2}{4}\sin^2\phi\cos^2\sigma + \cancel{\tfrac{l^2\dot\phi\dot\sigma}{4}\sin\phi\cos\sigma\cos\phi\sin\sigma}$$
$$+\cancel{\tfrac{al\dot\theta\dot\sigma}{2}\cos\theta\cos\phi\sin\sigma} + \cancel{\tfrac{l^2\dot\phi\dot\sigma}{4}\sin\phi\cos\sigma\cos\phi\sin\sigma} + \tfrac{l^2\dot\sigma^2}{4}\cos^2\phi\sin^2\sigma$$
$$+\tfrac{l^2\dot\phi^2}{4}\cos^2\phi$$

$$T = \tfrac12 m(\dot{x}_1^2 + \dot{x}_2^2) = \tfrac12 m\left(a^2\dot\theta^2 + \tfrac{l^2\dot\phi^2}{4}\sin^2\phi + \tfrac{l^2\dot\sigma^2}{4}\cos^2\phi + \tfrac{l^2\dot\phi^2}{4}\cos^2\phi\right.$$
$$\left.+a^2\dot\theta^2 + \tfrac{l^2\dot\phi^2}{4}\sin^2\phi + \tfrac{l^2\dot\sigma^2}{4}\cos^2\phi + \tfrac{l^2\dot\phi^2}{4}\cos^2\phi\right)$$
$$= m\left(a^2\dot\theta^2 + \tfrac{l^2\dot\phi^2}{4} + \tfrac{l^2\dot\sigma^2}{4}\cos^2\phi\right)$$

15.

a. $\vec{r} = (x, y, z)$

$\vec{v} = (\dot x, \dot y, \dot z)$

$$\mathcal{U}(\vec r, \vec v) = V(\vec r) + \vec\sigma\cdot\vec L = V(\vec r) + \vec\sigma\cdot(\vec r\times m\vec v)$$
$$= V(\vec r) + \vec\sigma\cdot m\left(\hat\imath(y\dot z - \dot y z) - \hat\jmath(x\dot z - \dot x z) + \hat k(x\dot y - \dot x y)\right)$$
$$= V(\vec r) + m\left(\sigma_x(y\dot z - \dot y z) - \sigma_y(x\dot z - \dot x z) + \sigma_z(x\dot y - \dot x y)\right)$$
$$Q_j = -\frac{\partial\mathcal U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial \mathcal U}{\partial \dot q_j}\right)$$
$$Q_x = -\left(\frac{\partial V}{\partial x} + m(-\sigma_y\dot z + \sigma_z\dot y)\right) + \frac{d}{dt}\left(m(\sigma_y z - \sigma_z y)\right) \qquad \frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial x} = V'(r)\frac{\partial r}{\partial x}$$
$$= -V'(\vec r)\cdot\frac{\partial r}{\partial x} + 2m(\sigma_y\dot z - \dot y\sigma_z) \qquad r = (x^2+y^2+z^2)^{1/2}$$
$$Q_y = -V'(\vec r)\cdot\frac{\partial r}{\partial y} + 2m(\dot x\sigma_z - \sigma_x\dot z) \qquad \frac{\partial r}{\partial x} = \tfrac12(x^2+y^2+z^2)^{-1/2}\cdot 2x$$
$$Q_z = -V'(\vec r)\cdot\frac{\partial r}{\partial z} + 2m(\sigma_x\dot y - \dot x\sigma_y) \qquad = \frac{x}{r}$$

$x = r\sin\phi\cos\theta \qquad \dot x = \dot r\sin\phi\cos\theta + r\dot\phi\cos\phi\cos\theta - r\dot\theta\sin\phi\sin\theta$

$y = r\sin\phi\sin\theta \qquad \dot y = \dot r\sin\phi\sin\theta + r\dot\phi\cos\phi\sin\theta + r\dot\theta\sin\phi\cos\theta$

$z = r\cos\phi \qquad \dot z = \dot r\cos\phi - r\dot\phi\sin\phi$

A quick aside, we can set $\vec r$ + $\vec\sigma$ to something such that our calculations are simpler. Thus, we will set our fixed point as the origin and $\vec\sigma$ to the z-axis. Now the reason we can do this is because we can move + rotate our coordinate axis as we choose. Alternatively, you can do what I did originally and solve for the general case and spend several hours and dozens of pages doing algebra. And get it wrong anyways.

$$\mathcal U = V(\vec r) + m(x\dot y - \dot x y)\sigma$$

$$x\dot y-\dot x y=r\sin\phi\cos\theta(\dot r\sin\phi\sin\theta+r\dot\phi\cos\phi\sin\theta+r\dot\theta\sin\phi\cos\theta)$$
$$-r\sin\phi\sin\theta(\dot r\sin\phi\cos\theta+r\dot\phi\cos\phi\cos\theta-r\dot\theta\sin\phi\sin\theta)$$
$$=\cancel{r\dot r\sin^2\phi\sin\theta\cos\theta}+\cancel{r^2\dot\phi\sin\phi\cos\phi\sin\theta\cos\theta}+r^2\dot\theta\sin^2\phi\cos^2\theta$$
$$-\cancel{r\dot r\sin^2\phi\sin\theta\cos\theta}-\cancel{r^2\dot\phi\sin\phi\cos\phi\sin\theta\cos\theta}+r^2\dot\theta\sin^2\phi\sin^2\theta$$
$$=r^2\dot\theta\sin^2\phi$$
$$U=V(\vec r)+m\sigma r^2\dot\theta\sin^2\phi$$

$$Q_r=-(V'(\vec r)+2m\sigma r\dot\theta\sin^2\phi)+\tfrac{d}{dt}(0)$$
$$=-V'(\vec r)-2mr\sigma\dot\theta\sin^2\phi$$
$$Q_\phi=-(m\sigma r^2\dot\theta\cdot 2\cos\phi\sin\phi)$$
$$=-2m\sigma r^2\dot\theta\cos\phi\sin\phi$$
$$Q_\theta=\tfrac{d}{dt}(m\sigma r^2\sin^2\phi)$$
$$=2m\sigma r\dot r\sin^2\phi+2m\sigma r^2\dot\phi\sin\phi\cos\phi$$

b. $Q_j=\sum_i\vec F_i\cdot\frac{\partial\vec r_i}{\partial q_j}$ $\qquad\sigma_y=\sigma_x=0$

$\vec r=(r\sin\phi\cos\theta,\ r\sin\phi\sin\theta,\ r\cos\phi)$ $\qquad\sigma_z=\sigma$

$$Q_r=Q_x(\sin\phi\cos\theta)+Q_y(\sin\phi\sin\theta)+Q_z\cos\phi$$
$$=(-V'(r)\cdot\tfrac{x}{r}+2m(-\dot y\sigma))\sin\phi\cos\theta$$
$$+(-V'(r)\cdot\tfrac{y}{r}+2m(\dot x\sigma))\sin\phi\sin\theta$$
$$+(-V'(r)\cdot\tfrac{z}{r})\cos\phi$$
$$=-V'(r)\cdot\tfrac1r(r\sin^2\phi\cos^2\theta+r\sin^2\phi\sin^2\theta+\cos^2\phi)$$
$$+2m\sigma(-\cancel{\dot r\sin^2\phi\sin\theta\cos\theta}-\cancel{r\dot\phi\sin\phi\cos\phi\sin\theta\cos\theta}-r\dot\theta\sin^2\phi\cos^2\theta$$
$$+\cancel{\dot r\sin^2\phi\sin\theta\cos\theta}+\cancel{r\dot\phi\sin\phi\cos\phi\sin\theta\cos\theta}-r\dot\theta\sin^2\phi\sin^2\theta)$$
$$=-V'(r)-2m\sigma r\dot\theta\sin^2\phi$$

$$Q_\phi=Q_x(r\cos\phi\cos\theta)+Q_y(r\cos\phi\sin\theta)+Q_z(-r\sin\phi)$$
$$=(-V'(r)\cdot\tfrac{x}{r}+2m\sigma(-\dot y))r\cos\phi\cos\theta$$
$$+(-V'(r)\cdot\tfrac{y}{r}+2m\sigma(\dot x))r\cos\phi\sin\theta$$
$$+(-V'(r)\cdot\tfrac{z}{r})(-r\sin\phi)$$
$$=-V'(r)\cdot\tfrac1r(r^2\sin\phi\cos\phi\cos^2\theta+r^2\sin\phi\cos\phi\sin^2\theta-r^2\sin\phi\cos\phi)$$
$$+2m\sigma(-r\dot r\sin\phi\cos\phi\sin\theta\cos\theta-r^2\dot\phi\cos^2\phi\sin\theta\cos\theta-r^2\dot\theta\sin\phi\cos\phi\cos^2\theta$$
$$+r\dot r\sin\phi\cos\phi\sin\theta\cos\theta+r^2\dot\phi\cos^2\phi\sin\theta\cos\theta-r^2\dot\theta\sin\phi\cos\phi\sin^2\theta)$$
$$=-2m\sigma r^2\dot\theta\sin\phi\cos\phi$$

$$Q_\theta=Q_x(-r\sin\phi\sin\theta)+Q_y(r\sin\phi\cos\theta)$$
$$=(-V'(r)\cdot\tfrac{x}{r}+2m\sigma(-\dot y))(-r\sin\phi\sin\theta)$$
$$+(-V'(r)\cdot\tfrac{y}{r}+2m\sigma(\dot x))(r\sin\phi\cos\theta)$$
$$=-V'(r)\cdot\tfrac1r(-r^2\sin^2\phi\sin\theta\cos\theta+r^2\sin^2\phi\sin\theta\cos\theta)$$
$$+2m\sigma(+r\dot r\sin^2\phi\sin^2\theta+r^2\dot\phi\sin\phi\cos\phi\sin^2\theta+\cancel{r^2\dot\theta\sin^2\phi\sin\theta\cos\theta}$$
$$+r\dot r\sin^2\phi\cos^2\theta+r^2\dot\phi\sin\phi\cos\phi\cos^2\theta-\cancel{r^2\dot\theta\sin^2\phi\sin\theta\cos\theta})$$
$$=2m\sigma r\dot r\sin^2\phi+2m\sigma r^2\dot\phi\sin\phi\cos\phi$$

c. $\mathcal{L}=\frac12 m(\dot r^2+r^2\dot\phi^2+r^2\dot\theta^2\sin^2\phi)-U$

$$\tfrac{d}{dt}\left(\tfrac{\partial\mathcal L}{\partial\dot q_j}\right)-\tfrac{\partial T}{\partial q_j}-Q_j=0$$

$$\tfrac{d}{dt}(\tfrac12 m(2\dot r))-\tfrac12 m(2r\dot\phi^2+2r\dot\theta^2\sin^2\phi)-Q_r=0$$
$$m\ddot r=mr\dot\phi^2+mr\dot\theta^2\sin^2\phi+Q_r$$
$$\tfrac{d}{dt}(\tfrac12 m(2r^2\dot\phi))-\tfrac12 m(2r^2\dot\theta^2\sin\phi\cos\phi)-Q_\phi=0$$
$$2mr\dot r\dot\phi+mr^2\ddot\phi-mr^2\dot\theta^2\sin\phi\cos\phi-Q_\phi=0$$
$$\tfrac{d}{dt}(\tfrac12 m(2r^2\dot\theta\sin^2\phi))-Q_\theta=0$$
$$2mr\dot r\dot\theta\sin^2\phi+mr^2\ddot\theta\sin^2\phi+2mr^2\dot\theta\dot\phi\sin\phi\cos\phi-Q_\phi=0$$

16. $F = \frac{1}{r^2} - \frac{\dot r^2}{r^2c^2} + \frac{2\ddot r}{rc^2}$

$= -\frac{\partial \mathcal{U}}{\partial r} + \frac{d}{dt}\cdot\left(\frac{\partial \mathcal{U}}{\partial \dot r}\right) \qquad \frac{d}{dt} = \frac{\partial \dot r}{\partial r} + \frac{\partial \ddot r}{\partial \dot r}$

$= -\frac{\partial \mathcal{U}}{\partial r} + \frac{\partial}{\partial r}\left(\frac{\partial \mathcal{U}}{\partial \dot r}\right)\dot r + \frac{\partial}{\partial \dot r}\left(\frac{\partial \mathcal{U}}{\partial \dot r}\right)\ddot r$

$\frac{\partial}{\partial \dot r}\left(\frac{\partial \mathcal{U}}{\partial \dot r}\right)\ddot r = \frac{2\ddot r}{rc^2}$

$\frac{\partial}{\partial \dot r}\left(\frac{\partial \mathcal{U}}{\partial \dot r}\right) = \frac{2}{rc^2}$

$\frac{\partial \mathcal{U}}{\partial \dot r} = \frac{2\dot r}{rc^2} + \alpha$

$\mathcal{U} = \dot r^2/rc^2 + \alpha \dot r + \beta$

$\frac{\partial}{\partial r}\left(\frac{\partial \mathcal{U}}{\partial \dot r}\right)\dot r = \frac{\partial}{\partial r}\left(\frac{2\dot r}{rc^2} + \alpha\right)\dot r = \frac{-2\dot r^2}{r^2c^2} + \alpha' \dot r$

$\frac{\partial \mathcal{U}}{\partial r} = -\frac{\dot r^2}{r^2c^2} + \alpha'\dot r + \beta'$

$F = \frac{+\dot r^2}{r^2c^2} - \cancel{\alpha'\dot r} - \beta' - \frac{2\dot r^2}{r^2c^2} + \cancel{\alpha'\dot r} + \frac{2\ddot r}{rc^2}$

$= \frac{-\dot r^2}{r^2c^2} - \beta' + \frac{2\ddot r}{rc^2} = \frac{1}{r^2} - \frac{\dot r^2}{r^2c^2} + \frac{2\ddot r}{rc^2}$

$-\beta' = 1/r^2$

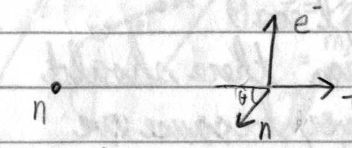
$\beta = 1/r$

$\frac{\partial \mathcal{U}}{\partial r} = \frac{-\dot r^2}{r^2c^2} - \frac{1}{r^2}$ $\quad \alpha = 0$ since it doesn't show up in our final equation

$\mathcal{U} = \dot r^2/rc^2 + 1/r$

$\mathcal{L} = \frac{1}{2}m(\dot r^2 + r^2\dot\phi^2) - \dot r^2/rc^2 - 1/r$

17. Two-dimensional



$p_e = 1.73\ \mathrm{MeV}/c$

$p_r = 1.00\ \mathrm{MeV}/c$

$n = (-p_r, -p_e)$

$\theta = \tan^{-1}(p_e/p_r) = 59.97°$

$p_n^2 = p_r^2 + p_e^2$

$p_n = 2.00\ \mathrm{MeV}/c$

$T_n = p_n^2/m_n = \frac{(2.00\ \mathrm{MeV}/c)^2}{3.90\times10^{-25}\,\mathrm{kg}} \cdot \frac{(5.34\times10^{-22})^2\ \mathrm{kg^2m^2/s^2}}{1(\mathrm{MeV}/c)^2}$

$= 2.92\times10^{-18}\ \mathrm{kg\,m^2/s^2} \cdot \frac{1\,\mathrm{MeV}}{1.6\times10^{-13}\,\mathrm{J}} = 1.8\times10^{-5}\ \mathrm{MeV}$

18. $\mathcal{L}' = \frac{m}{2}(a\dot x^2 + 2b\dot x\dot y + c\dot y^2) - \frac{K}{2}(ax^2 + 2bxy + cy^2)$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot x}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$

$\frac{d}{dt}(ma\dot x + mb\dot y) + K(ax + by) = 0$

$ma\ddot x + mb\ddot y = -(Kax + Kby)$

$mb\ddot x + mc\ddot y = -(Kbx + Kcy)$

$a = c = 0 \qquad\qquad b = 0,\ c = -a$

$mb\ddot y = -Kby \qquad\qquad ma\ddot x = -Kax$

$\ddot y = -K/m\,y \qquad\qquad \ddot x = -K/m\,x$

$mb\ddot x = -Kbx \qquad\qquad -ma\ddot y = Kay$

$\ddot x = -K/m\,x \qquad\qquad \ddot y = -K/m\,y$

This Lagrangian represents Harmonic motion in two-dimensions

19.

$\mathcal{L} = \frac{m}{2}(l^2\dot\theta^2 + l^2\dot\phi^2\sin^2\theta) + mgl\cos\theta$

N.B. In the full Lagrangian, there should be an $\dot{l}^2$ term. However, because we have a rigid rod, $\dot{l} = 0$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

$$\frac{d}{dt}(ml^2\dot\theta) - (ml^2\dot\phi^2\sin\theta\cos\theta - mgl\sin\theta) = 0$$

$$ml^2\ddot\theta = ml^2\dot\phi^2\sin\theta\cos\theta - mgl\sin\theta$$

$$\ddot\theta = \dot\phi^2\sin\theta\cos\theta - \frac{g}{l}\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

$$\frac{d}{dt}(ml^2\dot\phi\sin^2\theta) = 0$$

$$ml^2\ddot\phi\sin^2\theta + ml^2\dot\phi\cdot 2\sin\theta\cos\theta\cdot\dot\theta = 0$$

$$ml^2\ddot\phi\sin^2\theta = -2ml^2\dot\phi\dot\theta\sin\theta\cos\theta$$

$$\ddot\phi = -2\dot\phi\dot\theta\cot\theta$$

20. $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot x} - \frac{\partial\mathcal{L}}{\partial x} = 0$ $\qquad \frac{d}{dt}V(x) = \frac{dV(x)}{dx}\cdot\frac{dx}{dt} = V'(x)\cdot\dot x$

$$\mathcal{L} = \frac{m^2\dot x^4}{12} + m\dot x^2 V(x) - V^2(x)$$

$$\frac{d}{dt}\left(\frac{m^2\dot x^3}{3} + 2m\dot x V(x)\right) - (m\dot x^2 V'(x) - 2V(x)\cdot V'(x)) = 0$$

$$\frac{3m^2\dot x^2\ddot x}{3} + 2m\ddot x V(x) + 2m\dot x^2 V'(x) - m\dot x^2 V'(x) + 2V(x)\cdot V'(x) = 0$$

$$m\ddot x(m\dot x^2 + 2V(x)) + V'(x)(m\dot x^2 + 2V(x)) = 0$$

$$(m\ddot x + V'(x))(m\dot x^2 + 2V(x)) = 0$$

$$m\ddot x = -V'(x) \quad \text{or} \quad m\dot x^2 + 2V(x) = 0$$

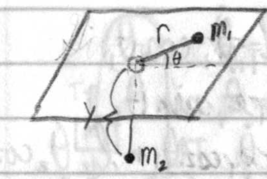
I think this is as close as you can get to $x(t)$ without knowing $V(x)$. However, we can see two things

$m\ddot x = -V'(x)$ is Newton's second law

$m\dot x^2 + 2V(x) = 0$

$\frac{1}{2}m\dot x^2 = -V(x)$ is conservation of energy

21.



$$\mathcal{L} = \frac{1}{2}m_1(\dot r^2 + r^2\dot\theta^2) + \frac{1}{2}m_2\dot y^2 + m_2 g y$$

$$= \frac{m_1}{2}(\dot r^2 + r^2\dot\theta^2) + \frac{m_2}{2}\dot r^2 + m_2 g(l - r)$$

The generalized coordinates are r + θ

$y + r = l$

$\dot r = -\dot y$

$$\frac{d}{dt}(m_1\dot r + m_2\dot r) - (m_1 r\dot\theta^2 - m_2 g) = 0$$

$$m_1\ddot r + m_2\ddot r = m_1 r\dot\theta^2 - m_2 g$$

$$\ddot r = \frac{m_1 r\dot\theta^2 - m_2 g}{m_1 + m_2}$$

$$\frac{d}{dt}(m_1 r^2\dot\theta) - 0 = 0$$

$$2m_1 r\dot r\dot\theta + m_1 r^2\ddot\theta = 0$$

$\frac{d}{dt}(m_1 r^2\dot\theta) = 0 \Rightarrow$ this is a conserved quantity i.e., angular momentum is conserved.

$$L = m_1 r^2\dot\theta$$

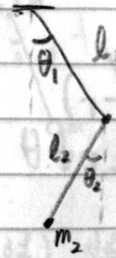
$$\ddot r = \frac{m_1 r\dot\theta^2 - m_2 g}{m_1 + m_2} = \frac{\frac{L^2}{m_1 r^3} - m_2 g}{m_1 + m_2}$$

$(m_1 + m_2)\ddot r\dot r\,dt = \frac{L^2}{m_1 r^3}\cdot\dot r\,dt - m_2 g\dot r\,dt \qquad \dot r\,dt = dr$

$\frac{m_1 + m_2}{2}\dot r^2 - \frac{L^2}{2m_1 r^2} + m_2 g r = E \qquad \ddot r\,dt = d\dot r$

This is simply conservation of energy.

22.



$x_1 = l_1 \sin\theta_1$ $\qquad \dot{x}_1 = l_1 \dot{\theta}_1 \cos\theta_1$

$y_1 = l_1 \cos\theta_1$ $\qquad \dot{y}_1 = l_1 \dot{\theta}_1 \sin\theta_1$

$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$ $\qquad \dot{x}_2 = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2$

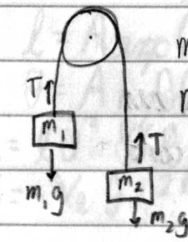
$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$ $\qquad \dot{y}_2 = l_1 \dot{\theta}_1 \sin\theta_1 + l_2 \dot{\theta}_2 \sin\theta_2$

$$\mathcal{L} = \tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) - m_1 g y_1 - m_2 g y_2$$
$$= \tfrac{1}{2} m_1 (l_1^2 \dot{\theta}_1^2 \cos^2\theta_1 + l_1^2 \dot{\theta}_1^2 \sin^2\theta_1)$$
$$+ \tfrac{m_2}{2} (l_1^2 \dot{\theta}_1^2 \cos^2\theta_1 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2 + l_2^2 \dot{\theta}_2^2 \cos^2\theta_2$$
$$+ l_1^2 \dot{\theta}_1^2 \sin^2\theta_1 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \sin\theta_2 + l_2^2 \dot{\theta}_2^2 \sin^2\theta_2)$$
$$+ g m_1 l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$
$$= \tfrac{m_1}{2} (l_1^2 \dot{\theta}_1^2) + \tfrac{m_2}{2} (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2))$$
$$+ m_1 g l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0$$
$$\frac{d}{dt}(m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2))$$
$$-(-m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_1 g l_1 \sin\theta_1 - m_2 g l_1 \sin\theta_1) = 0$$
$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$
$$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1 = 0$$
$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = 0$$
$$\frac{d}{dt}(m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)) - (m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(-1) - m_2 g l_2 \sin\theta_2) = 0$$
$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$
$$- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0$$
$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0$$

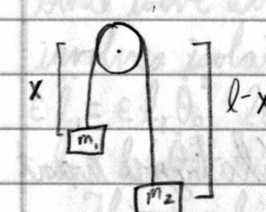
23.



$m_1 = 2\,kg$

$m_2 = 3\,kg$

$$-m_1 g + T = m_1 a \qquad T = m_1(a + g)$$
$$-m_2 g + T = -m_2 a \qquad T = m_2(-a + g)$$
$$m_1 a + m_1 g = -m_2 a + m_2 g$$
$$a = \frac{m_2 g - m_1 g}{m_1 + m_2}$$
$$T = m_1\left(\frac{m_2 - m_1}{m_1 + m_2} g + \frac{m_1 + m_2}{m_1 + m_2} g\right)$$
$$= m_1\left(\frac{2 m_2 g}{m_1 + m_2}\right) = \frac{2 m_1 m_2 g}{m_1 + m_2}$$

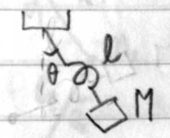


$$\mathcal{L} = \tfrac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1 g x + m_2 g (l - x)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$
$$\frac{d}{dt}((m_1 + m_2)\dot{x}) - (m_1 g - m_2 g) = 0$$
$$(m_1 + m_2)\ddot{x} = m_1 g - m_2 g$$
$$\ddot{x} = \frac{m_1 g - m_2 g}{m_1 + m_2}$$

Note that this is negative of the solution found using Newtonian methods. Also note that Tension cannot be found using Lagrangian methods.

24.

a.  $x = l\sin\theta \quad \dot x = \dot l\sin\theta + l\dot\theta\cos\theta$

$y = -l\cos\theta \quad \dot y = -\dot l\cos\theta + l\dot\theta\sin\theta$

$\mathcal{L} = \frac{1}{2}M(\dot l^2 + l^2\dot\theta^2) - \frac{1}{2}k(l-L_A)^2 + Mgl\cos\theta$

$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot l}\right) - \frac{\partial\mathcal{L}}{\partial l} = 0$

$\frac{d}{dt}(M\dot l) - (Ml\dot\theta^2 - k(l-L_A) + Mg\cos\theta) = 0$

$M\ddot l = Ml\dot\theta^2 - k(l-L_A) + Mg\cos\theta$

$\ddot l = l\dot\theta^2 - k/M\,(l-L_A) + g\cos\theta$

$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$

$\frac{d}{dt}(Ml^2\dot\theta) - (-Mgl\sin\theta) = 0$

$2Ml\dot l\dot\theta + Ml^2\ddot\theta = -Mgl\sin\theta$

$\ddot\theta = -g/l\,\sin\theta - 2\dot l\dot\theta/l$

b. At this point, things get a bit iffy. What you want to do is keep the ε (first-order) terms.

$\left.\begin{array}{l}\sin\theta \doteq \theta\\ \cos\theta = 1\\ l - L_A = x\end{array}\right\}$ each of these is order ε. Thus, if we multiply say θ^2, that term is order ε^2, which is smaller than what we're keeping

$\ddot l = \cancel{l\dot\theta^2} - k/m\,x + g = -k/m\,x + g$

$\ddot\theta = -g/l\,\theta - \cancel{2\dot l\dot\theta/l} = -g/l\,\theta$

which we recognize as the harmonic oscillator. Since l equation has some extra terms, need to offset the solution.

$l = A\exp(i\sqrt{k/m}\,t) + B\exp(-i\sqrt{k/m}\,t) + (L_A + mg/k)$

$\theta = A\exp(i\sqrt{g/l}\,t) + B\exp(-i\sqrt{g/l}\,t)$

c. $\ddot l = l\dot\theta^2 - k/m\,(l-L_A) + g(1-\theta^2/2)$

$\ddot\theta = -g/l\,\theta - 2\dot l\dot\theta/l$

$l = l_0 + \varepsilon l_1$ where l_0 + θ_0 are the solutions found above.

$\theta = \theta_0 + \varepsilon\theta_1$

$\ddot l = \ddot l_0 + \varepsilon\ddot l_1$

$\ddot\theta = \ddot\theta_0 + \varepsilon\ddot\theta_1$

$\ddot l = (l_0+\varepsilon l_1)(\dot\theta_0+\varepsilon\dot\theta_1)^2 - k/m\,(l_0+\varepsilon l_1 - L_A) + g(1 - (\theta_0+\varepsilon\theta_1)^2/2)$

$= (l_0+\varepsilon l_1)(\dot\theta_0^2 + 2\varepsilon\dot\theta_1 + \varepsilon^2\dot\theta_1^2) - k/m\,(l_0 - L_A) - \varepsilon k/m\,l_1 + g(1 - \frac{1}{2}(\theta_0^2 + 2\varepsilon\theta_0\theta_1 + \varepsilon^2\theta_1^2))$

Since we've already found l_0, we only need to find l_1, which involves isolating the ε terms.

$\varepsilon\ddot l_1 = \varepsilon l_1\dot\theta_0^2 + 2\varepsilon l_0\dot\theta_1 - \varepsilon k/m\,l_1 - \varepsilon g\theta_0\theta_1$

$\ddot l_1 = l_1(\dot\theta_0^2 - k/m) + 2l_0\dot\theta_1 - g\theta_0\theta_1$

This is a bit more complicated than using the simple harmonic oscillator equation as in the previous part.

$\ddot\theta = \frac{-g}{l_0+\varepsilon l_1}(\theta_0+\varepsilon\theta_1) - \frac{2}{l_0+\varepsilon l_1}(\dot l_0+\varepsilon\dot l_1)(\dot\theta_0+\varepsilon\dot\theta_1)$

$\ddot\theta_1 = \frac{-g\theta_1}{l_0+\varepsilon l_1} - \frac{2(\dot l_0\dot\theta_1 + \dot l_1\dot\theta_0)}{l_0+\varepsilon l_1}$

$= \frac{-(g\theta_1 - 2\dot l_0\dot\theta_1 - 2\dot l_1\dot\theta_0)}{l_0+\varepsilon l_1}$

d+e. are computer parts, so I'll probably leave those parts off.