Jackson Classical Electrodynamics
Third Edition

CLASS SCHEDULE

DATE ______________________

NAME ______________________ SCHOOL ______________________

ADDRESS ______________________

PERIOD	Monday	ROOM	Tuesday	ROOM	Wednesday	ROOM	Thursday	ROOM	Friday	ROOM
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BS

Everyone must leave something behind when he dies, my grandfather said... Something your hand touches some way so your soul has somewhere to go when you die, and when people look at that tree or that flower you planted, you're there. It doesn't matter what you do, so long as you change something from the way it was before you touched it into something that's like you after you take your hands away. The difference between the man who just cuts lawns and a real gardener is in the touching. The lawn-cutter might just as well not have been there at all; the gardener will be there a lifetime.
– Granger (Ray Bradbury, Fahrenheit 451)

Chapter 1. Introduction to Electrostatics

Section 1. Coulomb's Law

For some reason, Jackson chooses to dance around the force between two charged particles, instead deciding to put it in the next section.

Section 2. Electric field

Denoted $\vec{E}$, the electric field is force per unit charge acting at a given point. To define $\vec{E}$, let's first define $\vec{F}$

$$\vec{F} = k q_1 q_2 \cdot \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3} = \frac{q_1 q_2}{4\pi\varepsilon_0} \cdot \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$$

$\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F/m}$

$k = 8.99 \times 10^{9}\ \mathrm{Nm/C^2}$

Since $\vec{F} = q\vec{E}$,

$$\vec{E} = \frac{q_1}{4\pi\varepsilon_0} \cdot \frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|^3}$$

Of course, these definitions only work for a single point charge (q_1), so what happens if we have multiple charges or even a distribution of charge?

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} q_i \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3}$$

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{x}') \cdot \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}\, d^3x'$$

N.B.: $\vec{x}'$ denotes the location of the source and $\vec{x}$ denotes the location of the observer.

- A short aside on Dirac delta function

$\delta(x-a) = 0$ for $x \neq a$
$\quad = 1$ for $x = a$

$\int \delta(x-a)\,dx = 1$ if region of integration includes $x = a$
$\quad = 0$ else

$\int f(x)\,\delta(x-a)\,dx = f(a)$

$\int f(x)\,\delta'(x-a)\,dx = -f'(a)$

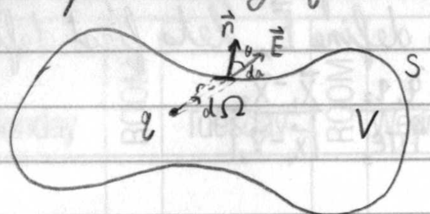
$$\delta(f(x)) = \sum_i \frac{1}{\left|\frac{df}{dx}(x_i)\right|}\,\delta(x - x_i)$$

$$\delta(\vec{x} - \vec{X}) = \delta(x_1 - X_1)\,\delta(x_2 - X_2)\,\delta(x_3 - X_3)$$

Section 3. Gauss's Law

· We've seen one version of $\vec{E}(\vec{x})$, but there also exists another form which leads to a differential equation

· Consider a point charge q contained within a closed surface S



$\hat{n}$: unit normal to S

da: element of surface area

$$\vec{E}\cdot\hat{n}\,da = \frac{q}{4\pi\varepsilon_0 r^2}\cos\theta\,da \qquad \cos\theta\,da = r^2\,d\Omega$$

$$= \frac{q}{4\pi\varepsilon_0}\,d\Omega$$

$$\oint_S \vec{E}\cdot\hat{n}\,da = \frac{q}{4\pi\varepsilon_0}\int d\Omega = \begin{cases} q/\varepsilon_0 & q \text{ inside } S \\ 0 & q \text{ outside } S. \end{cases}$$

For a continuous distribution of charges,

$$\oint_S \vec{E}\cdot\hat{n}\,da = \frac{1}{\varepsilon_0}\int_V \rho(\vec{x})\,d^3x$$

This implies,

· Inverse square law

· Central force

· Linear superposition

N.B. Gauss's Law also holds true for gravitational force

Section 4. Differential form of Gauss's Law

Obtained using divergence theorem:

$$\oint_S \vec{A}\cdot\hat{n}\,da = \int_V \vec{\nabla}\cdot\vec{A}\,d^3x$$

where $\vec{A}(\vec{x})$ is a well-behaved vector field inside a volume V surrounded by closed surface S

So now, taking the integral form of Gauss's Law,

$$\oint_S \vec{E}\cdot\hat{n}\,da = \frac{1}{\varepsilon_0}\int_V \rho(\vec{x})\,d^3x$$

$$\int_V \vec{\nabla}\cdot\vec{E}\,d^3x = \frac{1}{\varepsilon_0}\int_V \rho(\vec{x})\,d^3x$$

$$\int_V \left(\vec{\nabla}\cdot\vec{E} - \rho(\vec{x})/\varepsilon_0\right) d^3x = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \rho(\vec{x})/\varepsilon_0$$

A quick way to remember this is $\vec{E}\cdot\text{Area} = \text{charge enclosed}/\varepsilon_0$

Section 5 Another equation of electrostatics and the scalar potential

· If there is no $\vec{B}$, $\vec{\nabla}\times\vec{E} = 0$

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}')\,\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\,d^3x'$$

Since $\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$

$$\vec{E}(\vec{x}) = -\frac{1}{4\pi\varepsilon_0}\vec{\nabla}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x$$

$\vec{\nabla}\times\vec{\nabla}\vec{E}(\vec{x}) = 0$ since the curl of a divergence is 0

The above implies the electric field can be written as the gradient of some scalar function which will be named the scalar potential $\Phi(\vec{x})$.

$$E = -\vec{\nabla}\Phi(\vec{x})$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'$$

For at point charge q at $\vec{x}' = \vec{x}_1$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{q\,\delta(\vec{x}-\vec{x}_1)}{|\vec{x}-\vec{x}'|}\,d^3x'$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{q}{|\vec{x}-\vec{x}_1|}$$

· Consider the work done on a test charge q to move it from point A to point B.

$$\vec{F} = q\vec{E}$$

$$W = -\int_A^B \vec{F}\cdot d\vec{l} = -q\int_A^B \vec{E}\cdot d\vec{l}$$

$$= q\int_A^B \vec{\nabla}\Phi\cdot d\vec{l}$$

$$= q\int_A^B d\Phi = q(\Phi_B - \Phi_A)$$

This also implies that for a closed path (A=B),

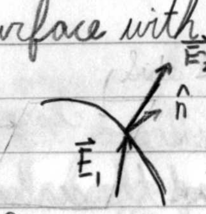
$$\oint \vec{E}\cdot d\vec{l} = 0$$

Using Stoke's theorem, we can return the equation of the curl of $\vec{E}$

$$\oint_C \vec{A}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,da$$

Section 6 Surface distributions of charges and dipoles and discontinuities in the electric field and potential

Imagine a surface with a charge density σ as shown below



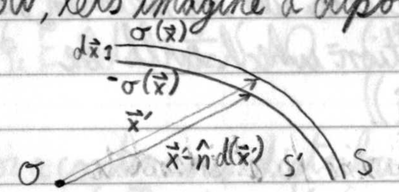
$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

$$(\vec{E}_2 - \vec{E}_1)\cdot\hat{n} = \sigma/\varepsilon_0$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'|}\,da$$

For volume and surface charge distributions, the potential is everywhere continuous

Now, let's imagine a dipole-layer distribution at S



let S' get infinitessimally close to S while $\lim_{d(\vec{x})\to 0}\sigma(\vec{x})\,d(\vec{x}) = D(\vec{x})$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'|}\,da' - \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'+\hat{n}d|}\,da''$$

expanding $|\vec{x}-\vec{x}'+\hat{n}d|^{-1}$ for small d,

$$\frac{1}{|\vec{x}-\vec{x}'+\hat{n}d|} = \frac{1}{|\vec{x}-\vec{x}'|} + \hat{n}d\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_S D(\vec{x}')\,\hat{n}\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)da'$$

$\vec{p} = \hat{n}D\,da'$ point dipole moment

$$\Phi(\vec{x}) = \int_S \frac{\vec{p}\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

Note that $\hat{n}\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)da' = \frac{-\cos\theta\,da'}{|\vec{x}-\vec{x}'|^2} = -d\Omega$,

$$\Phi(\vec{x}) = -\frac{1}{4\pi\varepsilon_0}\int_S D(\vec{x}')\,d\Omega$$

$$\Phi_2 - \Phi_1 = D/\varepsilon_0$$

Section 7. Poisson + Laplace equations

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

$$\vec{E} = -\vec{\nabla}\Phi$$

$$\vec{\nabla}\cdot\vec{\nabla}\Phi = -\rho/\varepsilon_0$$

$\vec{\nabla}^2\Phi = -\rho/\varepsilon_0$ Poisson's equation

$\vec{\nabla}^2\Phi = 0$ Laplace's equation

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')$$

Section 8 Green's Theorem

Imagine a well-behaved vector field $\vec{A}$ defined in V bouned by closed surface S.

Let $\vec{A} = \phi\vec{\nabla}\psi$

$$\vec{\nabla}(\vec{A}) = \vec{\nabla}(\phi\vec{\nabla}\psi) = \vec{\nabla}\phi\cdot\vec{\nabla}\psi + \phi\vec{\nabla}^2\psi$$

$$\vec{A}\cdot\hat{n} = \phi\frac{\partial\psi}{\partial n}$$

Using the divergence theorem:

$\int_V(\phi\vec{\nabla}^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi)\,d^3x = \oint_S \phi\frac{\partial\psi}{\partial n}\,da$ Green's first identity

Interchange $\phi + \psi$:

$$\int_V(\psi\vec{\nabla}^2\phi + \vec{\nabla}\psi\cdot\vec{\nabla}\phi)\,d^3x = \oint_S \psi\frac{\partial\phi}{\partial n}\,da$$

$\int_V(\phi\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\phi)\,d^3x = \oint_S\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)da$ Green's theorem

As an example, let's say $\psi = \frac{1}{|\vec{x}-\vec{x}'|}$, $\phi = \Phi$

$$\int_V(\phi\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\phi)\,d^3x = \oint_S\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)da$$

$$\int_V\left(\Phi\vec{\nabla}^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) - \frac{1}{|\vec{x}-\vec{x}'|}\vec{\nabla}^2\Phi\right)d^3x = \oint_S\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)da$$

$$\int_V\left(-4\pi\Phi\,\delta(\vec{x}-\vec{x}') - \frac{1}{|\vec{x}-\vec{x}'|}\cdot\frac{-\rho}{\varepsilon_0}\right)d^3x = \oint_S\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)da$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x' + \frac{1}{4\pi}\oint_S\left(\frac{1}{|\vec{x}-\vec{x}'|}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right)da'$$

Section 9 Uniqueness of the Solution with Dirichlet or Neumann Boundary Conditions

Dirichlet boundary conditions - specification of a potential on a closed surface

Neumann boundary conditions - specification of the normal derivatives

Uniqueness of $\vec\nabla^2\Phi = -\rho/\varepsilon_0$

Suppose there are two solutions Φ_1 + Φ_2

$$\vec\nabla^2\Phi_1 = -\rho/\varepsilon_0$$
$$\vec\nabla^2\Phi_2 = -\rho/\varepsilon_0$$
$$\vec\nabla^2 U = 0$$

$U = 0$ Dirichlet

$\frac{\partial U}{\partial n} = 0$ Neumann

$\int_V (U\vec\nabla^2 U + \vec\nabla U\cdot\vec\nabla U)\, d^3x = \oint_S U \frac{\partial U}{\partial n}\, da$ Green's first identity

$$\int_V (\vec\nabla U)^2\, d^3x = 0$$

$\vec\nabla U = 0 \Rightarrow U$ is a constant

N.B. Electrostatic problems are specified by either Dirichlet or Neumann boundary conditions

Section 10 Formal solution of electrostatic boundary-value problem with Green Function

· $\vec\nabla'^2 G(\vec x,\vec x') = -4\pi\delta(\vec x-\vec x')$ Green functions

$$G(\vec x,\vec x') = \frac{1}{|\vec x-\vec x'|} + F(\vec x,\vec x')$$
$$\vec\nabla'^2 F(\vec x,\vec x') = 0$$

$\phi = \Phi$, $\psi = G(\vec x,\vec x')$

$$\int_V (\phi\vec\nabla'^2\psi - \psi\vec\nabla'^2\phi)\, d^3x = \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] da$$

$$\int_V (\Phi\vec\nabla'^2 G(\vec x,\vec x') - G(\vec x,\vec x')\vec\nabla'^2\Phi)\, d^3x = \oint_S \left[\Phi\frac{\partial G(\vec x,\vec x')}{\partial n} - G(\vec x,\vec x')\frac{\partial\Phi}{\partial n}\right] da$$

$$\int_V -4\pi\Phi\delta(\vec x-\vec x')\, d^3x + \int_V \frac{\rho}{\varepsilon_0} G(\vec x,\vec x')\, d^3x = \text{''}$$

$$\Phi(\vec x) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec x')G(\vec x,\vec x')\, d^3x' + \frac{1}{4\pi}\oint_S \left[G(\vec x,\vec x')\frac{\partial\Phi}{\partial n'} - \Phi(\vec x')\frac{\partial G(\vec x,\vec x')}{\partial n'}\right] da'$$

For Dirichlet boundary conditions, $G_D(\vec x,\vec x') = 0$ for $\vec x'$ on S

$$\oint_S G_D(\vec x,\vec x')\frac{\partial\Phi}{\partial n}\, da = 0$$

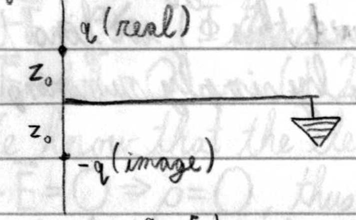
$$\Phi(\vec x) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec x')G_D(\vec x,\vec x')\, d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec x')\frac{\partial G_D}{\partial n'}\, da'$$

For Neumann boundary conditions, $\frac{\partial G_N}{\partial n'}(\vec x,\vec x') = -\frac{4\pi}{S}$ for $\vec x'$ on S

$$\Phi(\vec x) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec x')G_N(\vec x,\vec x')\, d^3x' + \frac{1}{4\pi}\oint_S G_N(\vec x,\vec x')\frac{\partial\Phi}{\partial n'}\, da' - \frac{1}{4\pi}\oint_S \Phi(\vec x')\cdot -\frac{4\pi}{S}\, da'$$

$$= \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec x')G_N(\vec x,\vec x')\, d^3x' + \frac{1}{4\pi}\oint_S G_N(\vec x,\vec x')\frac{\partial\Phi}{\partial n'}\, da' + \langle\Phi\rangle_S$$

A quick aside on method of images



Volume: $z > 0$

Surface: $z = 0$ + infinity

$$\rho = q\,\delta(x)\,\delta(y)\,\delta(z-z_0)$$

$$\Phi(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{[x^2+y^2+(z-z_0)^2]^{1/2}} - \frac{1}{[x^2+y^2+(z+z_0)^2]^{1/2}}\right]$$

Properties:

· $\Phi = 0$ on the boundary. As we can see, above solution satisfies this at $z=0$ + $z=\pm\infty$

· $\nabla^2\Phi$ gets no charge from the image charge inside V

· Discontinuity in normal derivative at $z=0$ is $\frac{d\Phi}{dz}\big|_0$.

Section 11. Electrostatic potential energy and energy density, capacitance

· Say we have a region containing an electric field described by some scalar potential Φ, and we want to bring some point charge q_i to a point $\vec x_i$ (in the given region). If we start that point charge at infinity, then the amount of work done on the charge is given by

$$W_i = q_i\Phi(\vec x_i) \qquad (1.47)$$

If we recall our introductory mechanics class, work is the change in potential energy. Now for those of you wondering why we labeled the charge with a subscript, we then turn to the question: What if we bring multiple charges into this arrangement?

Actually, let's rephrase this a little differently. We know that our potential Φ can be written as a configuration of point charges,

$$\Phi(\vec x_i) = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|\vec x_i-\vec x_j|} \qquad (1.48)$$

which then allows us to rewrite (1.47) as

$$W_i = \frac{q_i}{4\pi\varepsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|\vec x_i-\vec x_j|} \qquad (1.49)$$

If we then ask what the total potential energy is, (i.e., what would happen if we try to construct this Φ), we find that we can get the total potential energy by simply summing everything, or,

$$W = \frac{1}{8\pi\varepsilon_0} \sum_i \sum_{i \neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \qquad (1.51)$$

$$W = \frac{1}{8\pi\varepsilon_0} \iint \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x\, d^3x' \qquad (1.52)$$

$$= \frac{1}{2}\left[\frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'\right]\int \rho(\vec{x})\, d^3x$$

$$= \tfrac{1}{2}\int \rho(\vec{x})\Phi(\vec{x})\, d^3x \qquad (1.53)$$

Using Poisson's equation, ($\rho = -\varepsilon_0 \vec{\nabla}^2 \Phi$), we can write potential energy as a function of the electric field

$$W = -\tfrac{\varepsilon_0}{2}\int \Phi \nabla^2 \Phi\, d^3x$$

$u = \Phi$, $du = \nabla\Phi$; $v = \nabla\Phi$, $dv = \nabla^2\Phi$ Integration by parts

$$W = -\tfrac{\varepsilon_0}{2}\left(\Phi\nabla\Phi - \int \vec{\nabla}\Phi \cdot \vec{\nabla}\Phi\, d^3x\right)$$

$$= \tfrac{\varepsilon_0}{2}\int |\vec{\nabla}\Phi|^2 d^3x$$

$$= \tfrac{\varepsilon_0}{2}\int |\vec{E}|^2 d^3x \qquad (1.54)$$

Based on this, we can define the energy density as

$$w = \tfrac{\varepsilon_0}{2}|\vec{E}|^2 \qquad (1.55)$$

$$= \frac{\sigma^2}{2\varepsilon_0} \qquad (1.59)$$

Section 12 Variational approach to the solution of the Laplace and Poisson Equations

Variational methods are used more commonly in Classical and Quantum physics. The basic idea is that you try to extremize the value of certain functions (i.e. finding the path of least action). We'll come back to this and the next chapter as they weren't covered in Berger's EM class.

Problems 1.

a. $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ Gauss's theorem

$\oint \vec{E}\cdot d\vec{l} = 0$ (1.21)

We know that the electric field inside a conductor = 0.

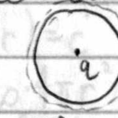
$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \rho = 0$, thus any excess surface must be distributed on the surface rather than inside the conductor.

b. In the first case, let's put a charge outside

$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$. Since there is no charge contained in the Gaussian surface,

$\vec{E} = 0$

Now let's consider a charge inside



Since there is a charge density contained in the Gaussian surface $\vec{E} \neq 0$

c. Gauss's theorem can essentially be written as

$E \cdot A = q_{enc}/\varepsilon_0$ (using our knowledge from baby's first EM)

$\underline{F} = q_{enc}/\varepsilon_0 A = \sigma/\varepsilon_0$

Also, we know that to write Gauss's law in this manner, the Gaussian surface must be normal to the conductor.

2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\,dx\,dy\,dz = 1$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lim_{\alpha\to 0} D(\alpha; x,y,z)\,dx\,dy\,dz = 1$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lim_{\alpha\to 0} D(\alpha; u,v,w)\,\frac{du\,dv\,dw}{UVW} = 1$

Note that since we don't have a map to get from $\{x,y,z\}\to\{u,v,w\}$, we can't rewrite D in terms of our new coordinates

The final equation looks like a delta function.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v,\omega)\,du\,dv\,d\omega = 1$$

$$\delta(\vec{x}-\vec{x}') = F(u,v,\omega)$$

$$= \delta(u-u')\delta(v-v')\delta(\omega-\omega')\,UVW$$

3.

a. $\rho = c\delta(r-R)$

$$\int \rho \cdot 4\pi r^2 dr = Q$$
$$4\pi c \int r^2 \delta(r-R) = Q$$
$$4\pi c \cdot R^2 = Q$$
$$c = Q/4\pi R^2$$
$$\rho = \frac{Q}{4\pi R^2}\,\delta(r-R)$$

b. $\rho = c\delta(r-b)$

$$\int \rho \cdot 2\pi r\, dr = \lambda$$
$$2\pi c \int r\,\delta(r-b)\,dr = \lambda$$
$$2\pi c \cdot b = \lambda$$
$$c = \lambda/2\pi b$$
$$\rho = \frac{\lambda}{2\pi b}\,\delta(r-b)$$

c. $\rho = c\,\delta(z)\,\Theta(R-r)$

where $\Theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$

$$\int \rho\, 2\pi r\, dr\, dz = Q$$
$$2\pi c \int \Theta(R-r)\, r\, dr = Q$$
$$2\pi c \int_0^R r\, dr = Q$$
$$2\pi c \cdot \tfrac{1}{2} R^2 = Q$$
$$c = Q/\pi R^2$$
$$\rho = \frac{Q}{\pi R^2}\,\delta(z)\,\Theta(R-r)$$

d. $\rho = f(r)\,\delta(\cos\theta)\,\Theta(R-r)$

$$2\pi \int \rho\, r^2 dr\, d(\cos\theta) = Q$$
$$2\pi \int_{-1}^{1}\int_0^R f(r)\, r^2\, \delta(\cos\theta)\, dr\, d(\cos\theta) = Q$$
$$2\pi \int_0^R f(r)\, r^2\, dr = Q$$

since Q is spread uniformly, $f(r) = c/r$

$$2\pi c \int_0^R r\, dr = Q$$
$$2\pi c \cdot \tfrac{1}{2} R^2 = Q$$
$$c = Q/\pi R^2$$
$$\rho = \frac{Q}{\pi R^2 r}\,\delta(\cos\theta)\,\Theta(R-r)$$

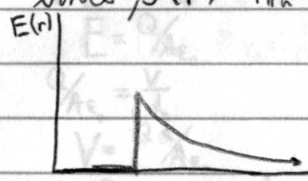
4. Outside the sphere, the electric field is the same regardless of charge distribution

$$4\pi r^2 E_{out} = Q/\varepsilon_0$$
$$E_{out} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r^2}$$

Conductor

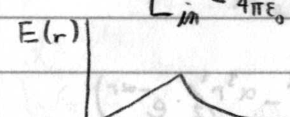
Charge is distributed entirely on the surface

Since $\rho(r) = \frac{Q}{4\pi a^2}\,\delta(r-a)$, $E_{in} = 0$



Uniform charge density

$$\rho(r) = c$$
$$\int_0^a \rho \cdot 4\pi r^2 dr = Q$$
$$\frac{4\pi c}{3}\cdot a^3 = Q$$
$$c = 3Q/4\pi a^3$$
$$4\pi r^2 E = \tfrac{4}{3}\pi r^3 \cdot \frac{3Q}{4\pi a^3 \varepsilon_0}$$
$$E_{in} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Qr}{a^3}$$



Spherically symmetric charge density

$$\rho = cr^n$$
$$\int_0^a \rho \cdot 4\pi r^2 dr = Q$$
$$4\pi c \int_0^a r^{2+n} dr = Q$$
$$\frac{4\pi c}{n+3}\, a^{n+3} = Q$$
$$c = \frac{(n+3)Q}{4\pi a^{n+3}}\, r^n$$

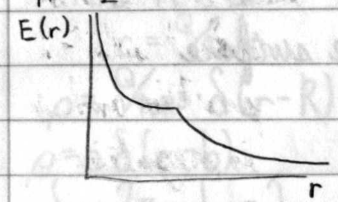
3. $\cancel{4}\pi r^2 E = \frac{1}{\varepsilon_0}\int_0^r \rho \cdot \cancel{4}\pi r^2\, dr$

$r^2 E = \frac{(n+3)Q}{4\pi a^{n+3}\varepsilon_0} \cdot \int_0^r r^{n+2}\, dr$

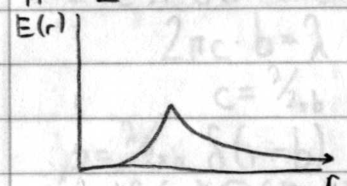
$= \frac{(n+3)Q}{4\pi a^{n+3}\varepsilon_0} \cdot \frac{1}{n+3} r^{n+3}$

$E = \frac{Q r^{n+1}}{4\pi\varepsilon_0 a^{n+3}}$

$n = -2$



$n = +2$



5. $\Phi = \frac{q}{4\pi\varepsilon_0}\left(\frac{e^{-\alpha r}}{r} + \frac{\alpha e^{-\alpha r}}{2}\right)$

$\nabla^2\Phi = -\rho/\varepsilon_0$

$\nabla^2\Phi = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)$

$\frac{\partial\Phi}{\partial r} = \frac{q}{4\pi\varepsilon_0}\left(-\alpha e^{-\alpha r}/r - e^{-\alpha r}/r^2 - \alpha^2 e^{-\alpha r}/2\right)$

$r^2\frac{\partial\Phi}{\partial r} = \frac{-q}{4\pi\varepsilon_0}\left(e^{-\alpha r} + \alpha r e^{-\alpha r} + \alpha^2 r^2/2 \cdot e^{-\alpha r}\right)$

$\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = \frac{-q}{4\pi\varepsilon_0}\left(\cancel{-\alpha e^{-\alpha r}} \cancel{+\alpha e^{-\alpha r}} \cancel{-\alpha^2 r e^{-\alpha r}} \cancel{+\alpha^2 r e^{-\alpha r}} - \alpha^3 r^2/2 \cdot e^{-\alpha r}\right)$

$= q\alpha^3 r^2 e^{-\alpha r}/8\pi\varepsilon_0$

$\rho = -q\alpha^3 e^{-\alpha r}/8\pi$

However, at $r=0$, we see that Φ explodes, so if we write $\Phi = \frac{q}{4\pi\varepsilon_0 r}$, we see that there is an additional $q\delta(r)$ term. Thus, the final density

$\rho = q\delta(0) - q\alpha^3 e^{-\alpha r}/8\pi$

which is a charge at the center (nucleus) surrounded by an electron cloud.

6.

a. $C = Q/V$

$E = -\nabla\Phi$

$E = V/d$ Since capacitance is a scalar quantity, I don't consider direction here. Also, it depends on which plate is positive and which is negative

$\nabla \cdot E = q_{enc}/\varepsilon_0$ Calculate E due to a plate

$A \cdot E = Q/\varepsilon_0$

$E = Q/A\varepsilon_0$

$Q/A\varepsilon_0 = \frac{V}{d}$

$V = Qd/A\varepsilon_0$

$C = Q/V = Q \cdot A\varepsilon_0/Qd = \frac{A\varepsilon_0}{d}$

b. $E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r^2}$

$\frac{d\Phi}{dr} = -E$

$V = -\int_a^b \frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\, dr$ Note that we integrate from a to b because we are interested in the region between the spheres

$= \frac{Q}{4\pi\varepsilon_0}\cdot 1/r\Big|_a^b = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right)$

$C = Q/V = 4\pi\varepsilon_0\left(\frac{1}{b} - \frac{1}{a}\right)^{-1}$

$= 4\pi\varepsilon_0\left(\frac{a-b}{ab}\right)^{-1}$

$= 4\pi\varepsilon_0 ab/b-a$

Probably dropped a negative somewhere, but since we want a positive C, I just multiplied by -1 until something positive came out

c. $E = \frac{Q}{2\pi\varepsilon_0 r L}$

$V = Q/2\pi\varepsilon_0 L \int_a^b 1/r\, dr$

$= Q/2\pi\varepsilon_0 L \ln r\Big|_a^b$

$= Q/2\pi\varepsilon_0 L \ln(b/a)$

$C = Q/V = 2\pi\varepsilon_0 L/\ln(b/a)$

d. Using the results from part c,

$C/L = 2\pi\varepsilon_0 / \ln(b/a)$

$\varepsilon_0 = 9\times10^{-12}\,F\,m^{-1}$

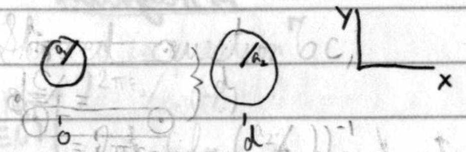
$C/L = 3\times10^{-11}\,F/m$, $a = .5\times10^{-3}\,m$

$b = 3mm$

$C/L = 3\times10^{-12}\,F/m$, $a = .5mm$

$b = 7.7\times10^{7}\,mm$

7.



$\vec{E}_1 = \frac{Q}{2\pi r\varepsilon_0 L}\hat{r}$ $\quad r^2 = x^2 + y^2 \quad ; \hat{r} = \frac{1}{r}(x\hat{i} + y\hat{j})$

$\vec{E}_2 = \frac{-Q}{2\pi r'\varepsilon_0 L}\hat{r}' \quad r'^2 = (x-d)^2 + y^2 ; \hat{r}' = \frac{1}{r'}((x-d)\hat{i} + y\hat{j})$

What we did here was find the electric field from each conductor individually, then we'll add them together to find the total electric field everywhere. Note that this falls apart inside the conductor, which is fine since we only care about the space between the conductor

$\vec{E} = \frac{Q}{2\pi\varepsilon_0 L}\left[\frac{x\hat{x} + y\hat{y}}{x^2+y^2} - \frac{(x-d)\hat{x} + y\hat{y}}{(x-d)^2+y^2}\right]$

Since we have symmetry, let's focus solely on x

$\frac{d\Phi}{dx} = E_x$

$\Phi = \frac{Q}{2\pi\varepsilon_0 L}\int_{a_1}^{d-a_2} \frac{x}{x^2+y^2} - \frac{x-d}{(x-d)^2+y^2}\,dx\,\Big|_{y=0}$

$= \frac{Q}{2\pi\varepsilon_0 L}\int_{a_1}^{d-a_2}\left(\frac{1}{x} - \frac{x-d}{(x-d)^2}\right)dx$

$= \frac{Q}{2\pi\varepsilon_0 L}\left[\int_{a_1}^{d-a_2}\frac{1}{x}\,dx - \int_{a_1}^{d-a_2}\frac{1}{x-d}\,dx\right]$

$= \frac{Q}{2\pi\varepsilon_0 L}\left[\ln\left(\frac{d-a_2}{a_1}\right) - \ln\left(\frac{-a_2}{a_1-d}\right)\right]$

$= \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{d-a_2}{a_1}\right)\left(\frac{d-a_1}{a_2}\right)$

$\approx \frac{Q}{\pi\varepsilon_0 L}\ln\left(\frac{d}{\sqrt{a_1a_2}}\right) = \frac{Q}{\pi\varepsilon_0 L}\ln\left(\frac{d}{a}\right)$

where $a = \sqrt{a_1a_2}$, the geometric mean

$C/L = Q/VL = \pi\varepsilon_0\left(\ln(d/a)\right)^{-1}$

$C/L = 1.2\times10^{-11}\,F/m$

$\varepsilon_0 = 9\times10^{-12}\,F/m$

$\ln\left(\frac{d}{a}\right) = \pi\varepsilon_0/C$

$\frac{d}{a} = \exp(\pi\varepsilon_0/C)$

$a = d\exp(-\pi\varepsilon_0/C)$

$d = .5\,cm \Rightarrow a = 1\,mm$

$d = 1.5\,cm \Rightarrow a = 3\,mm$

$d = 5.0\,cm \Rightarrow a = 10\,mm$

8.

a. Two large, flat, conducting sheets of area A, separated by a small distance d.

Looking at (1.53), or remembering intro physics, $W = \frac{1}{2}QV$.

$W = \frac{1}{2}\int\rho(x)\Phi(x)\,d^3x$

Since charge is located solely at the surface, we only need to care about the potential at the surface (i.e. potential difference).

$V = Qd/A\varepsilon_0$

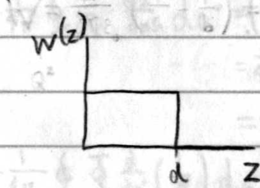
$Q = VA\varepsilon_0/d$

$W = \frac{1}{2}\cdot\frac{Q^2 d}{A\varepsilon_0} = V^2A\varepsilon_0/2d$

b. $w = \frac{\varepsilon_0}{2}|E|^2$

$Q = \frac{\varepsilon_0}{2}\frac{Q^2}{A^2\varepsilon_0^2} = \frac{Q^2}{2A^2\varepsilon_0}$

c.



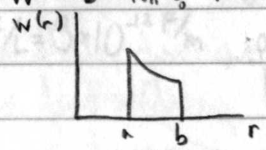
b. Two concentric conducting spheres with radii a, b.

$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$ $\quad Q = 4\pi\varepsilon_0 V\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}$

$W = \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) = 2\pi\varepsilon_0 V^2\left(\frac{ab}{b-a}\right)$

$E = \frac{Q}{4\pi\varepsilon_0 r^2}$

$w = \frac{\varepsilon_0}{2} \cdot \frac{Q^2}{16\pi^2\varepsilon_0^2} r^{-4} = \frac{Q^2}{32\pi^2\varepsilon_0 r^4}$



c. Two concentric conducting cylinders of length L, large compared to their radii a, b (b>a)

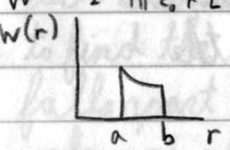
$V = \frac{Q}{2\pi\varepsilon_0 L}\ln(b/a)$

$Q = \frac{2\pi\varepsilon_0 LV}{\ln(b/a)}$

$W = \frac{Q^2\ln(b/a)}{4\pi\varepsilon_0 L} = \frac{\pi\varepsilon_0 LV^2}{\ln(b/a)}$

$E = \frac{Q}{2\pi\varepsilon_0 rL}$

$w = \frac{\varepsilon_0}{2} \cdot \frac{Q^2}{4\pi^2\varepsilon_0^2 r^2 L^2} = \frac{Q^2}{8\pi\varepsilon_0 r^2 L^2}$



9.

a. Conductors in parallel plate capacitor

For a single plate, $E = \frac{Q}{2A\varepsilon_0}$. Note that we want the electric field due to a single plate because the other plate is acting on this one (won't act on itself).

$F = \int \rho(\vec{x}) E(\vec{x})\, d^3x$

$\rho(\vec{x}) = Q/A\ \delta(z)$

$F = \frac{Q}{A} \cdot \frac{Q}{2A\varepsilon_0} \cdot A = \frac{Q^2}{2A\varepsilon_0}$

$Q = \frac{A\varepsilon_0 V}{d}$

$E = -\nabla\Phi$

$= -\frac{Q}{A\varepsilon_0}$ Remember to divide by 2

$F = \frac{Q^2}{A\varepsilon_0} = \frac{A^2\varepsilon_0^2 V^2}{2A\varepsilon_0 d^2}$

$= \frac{A\varepsilon_0 V^2}{2d^2}$

Note that we could have just used $W = Fd$

b. Parallel cylinder capacitor

$2\pi r L E = Q/\varepsilon_0$

$E = \frac{Q}{2\pi r\varepsilon_0 L}$

$= \frac{\lambda}{2\pi r\varepsilon_0 L}$

$\rho = \lambda\,\delta(y)\,\delta(x-d)$

$F = \frac{\lambda^2}{2\pi\varepsilon_0 d}$

$V = -\frac{\lambda}{\pi\varepsilon_0}\ln\left(\frac{d}{a}\right)$

$\lambda = -\frac{\pi\varepsilon_0 V}{\ln(d/a)}$

$F = \frac{\pi^2\varepsilon_0^2 V^2}{\ln^2(d/a)} \cdot \frac{1}{2\pi\varepsilon_0 d}$

$= \frac{\pi\varepsilon_0 V}{2d\ln(d/a)}$

10. $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{x}')}{R}\, d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] da'$ (1.36)

Let's look at each term individually

$\frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{x}')}{R}\, d^3x' = 0$ since charge-free space

$\frac{1}{4\pi}\oint_S \frac{1}{R} \cdot \frac{\partial\Phi}{\partial n'}\, da' = \frac{1}{4\pi R}\oint_S \vec{E}\cdot\hat{n}\, da$

$= \frac{1}{4\pi\varepsilon_0 R}\int_V \rho(\vec{x})\, d^3x$ (1.11)

$= 0$ again, because of charge-free space

$-\frac{1}{4\pi}\oint_S \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) da' = -\frac{1}{4\pi}\oint_S \Phi \cdot \frac{\partial}{\partial R}\left(\frac{1}{R}\right) da'$

$= -\frac{1}{4\pi}\oint_S \cdot -\frac{1}{R^2}\Phi\, da'$

$= \frac{1}{4\pi R^2}\oint_S \Phi\, da'$

Thus, we're left with $\underbrace{\Phi(\vec{x})}_{\text{value of electrostatic potential at any point}} = \underbrace{\frac{1}{4\pi R^2}\oint_s \Phi(\vec{x})\,da'}_{\text{average of the potential over the surface of a sphere}}$

11. $\frac{\partial E}{\partial n} = \lim_{\Delta r \to 0} \frac{E(\vec{x}+\Delta r\hat{n}) - E(\vec{x})}{\Delta r}$

Let's create a Gaussian pillbox just above the surface. Most notably, our pillbox will contain no charge, so

$$\oint_s \vec{E}\cdot\hat{n}\,da = 0$$

$$E_{top}\int_{top} da = E_{bot}\int_{bot} da$$

$$E(\vec{x}+\Delta r\hat{n})(R_1+\Delta r)(R_2+\Delta r) = E(\vec{x})R_1R_2$$

$$\frac{\partial E}{\partial n} = \lim_{\Delta r\to 0} \frac{E(\vec{x})\,{}^{R_1R_2}/_{(R_1+\Delta r)(R_2+\Delta r)} - E(\vec{x})}{\Delta r}$$

$$= \lim_{\Delta r \to 0} \frac{R_1R_2 - (R_1+\Delta r)(R_2+\Delta r)}{\Delta r(R_1+\Delta r)(R_2+\Delta r)} E(\vec{x})$$

$$= \lim_{\Delta r\to 0} \frac{-R_1-R_2-\Delta r}{(R_1+\Delta r)(R_2+\Delta r)} E(\vec{x})$$

$$= \frac{-R_1-R_2}{R_1R_2} E(\vec{x})$$

$$\frac{1}{E}\cdot\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

12. $\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,d^3x = \oint_s \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] da \qquad (1.35)$

$$\phi = \Phi$$

$$\psi = \Phi'$$

$$\int_V (\Phi\nabla^2\Phi' - \Phi'\nabla^2\Phi)\,d^3x = \oint_s \left[\Phi\frac{\partial\Phi'}{\partial r} - \Phi'\frac{\partial\Phi}{\partial r}\right] da$$

From problem 1c, $E = \sigma/\varepsilon_0$

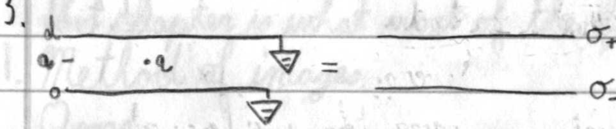
$$\Rightarrow \frac{\partial\Phi}{\partial r} = -\sigma/\varepsilon_0$$

$$\nabla^2\Phi = -\rho/\varepsilon_0$$

$$\int_V \Phi\cdot(-\rho'/\varepsilon_0)\,d^3x - \int_V \Phi'(-\rho/\varepsilon_0)\,d^3x = \oint_s \Phi(-\sigma'/\varepsilon_0)\,da - \oint_s \Phi'\cdot(-\sigma/\varepsilon_0)\,da$$

$$\int_V \Phi'\rho\,d^3x + \oint_s \Phi'\sigma\,da = \int_V \Phi\rho'\,d^3x + \oint_s \Phi\sigma'\,da$$

13.



We can treat this system as either a charge between two conducting plates or as two plates with some unknown induced surface charge density. Let's call the left the original, and the right as the primed system

$\Phi = {}^{qx}/_{A\varepsilon_0}$ $\qquad\qquad$ $\Phi' = {}^{\sigma x}/_{\varepsilon_0}$

$\rho = q\delta(a)$ $\qquad\qquad$ $\rho' = 0$

$\sigma = 0$ $\qquad\qquad$ $\sigma' = \sigma_+$ on top + σ_- on bottom, $|\sigma_+| = |\sigma_-|$

Now let's plug this into Green's reciprocal theorem:

$$\int q\delta(a)\cdot{}^{\sigma x}/_{\varepsilon_0}\cdot d^3x + 0 = 0 + \oint_s \sigma\cdot{}^{qx}/_{A\varepsilon_0}\,dx$$

$$q\cdot{}^{\sigma a}/_{\varepsilon_0} = \sigma\cdot(-q_{top}\cdot{}^{d}/_{A\varepsilon_0})\cdot A$$

$$qa = -q_{top}\cdot d$$

$$q_{top} = -q\cdot\frac{a}{d}$$

14. This problem is a little wierd since there is a part before a, so don't get tricked like I did.

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,d^3x = \oint_s \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) da \qquad (1.35)$$

Say $\phi = G(\vec{x},\vec{y})$

$\psi = G(\vec{x}',\vec{y})$ which gives

$$\int_V G(\vec{x},\vec{y})\nabla^2 G(\vec{x}',\vec{y})\,d^3x - \int G(\vec{x}',\vec{y})\nabla^2 G(\vec{x},\vec{y})\,d^3x = \oint_s \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) da$$

$$-4\pi G(\vec{x},\vec{x}') + 4\pi G(\vec{x}',\vec{x}) = \oint_s G(\vec{x},\vec{y})\frac{\partial G(\vec{x}',\vec{y})}{\partial n}\,da - \oint_s G(\vec{x}',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n}\,da$$

$$G(\vec{x},\vec{x}') - G(\vec{x}',\vec{x}) = \frac{-1}{4\pi}\left[\oint_s \left(G(\vec{x},\vec{y})\frac{\partial G(\vec{x}',\vec{y})}{\partial n} - G(\vec{x}',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n}\right) da\right]$$

a. $G_D(\vec{x},\vec{x}')=0$ on the surface (1.43)

$$G_D(\vec{x},\vec{x}')-G_D(\vec{x}',\vec{x})=0$$

$$G_D(\vec{x},\vec{x}')=G_D(\vec{x}',\vec{x})$$

b. $\frac{\partial G_N}{\partial n'}(\vec{x},\vec{x}')=-\frac{4\pi}{S}$ on the surface (1.45)

$$G_N(\vec{x},\vec{x}')-G_N(\vec{x}',\vec{x})=-\frac{1}{4\pi}\left[\oint G_N(\vec{x},\vec{y})\cdot\frac{-4\pi}{S}\,da-\oint_S G_N(\vec{x}',\vec{y})\cdot\frac{-4\pi}{S}\,da\right]$$

$$=\frac{1}{S}\oint_S G_N(\vec{x},\vec{y})\,da-\frac{1}{S}\oint_S G_N(\vec{x}',\vec{y})\,da$$

$$G_N(\vec{x},\vec{x}')-\frac{1}{S}\oint_S G_N(\vec{x},\vec{y})\,da=G_N(\vec{x}',\vec{x})-\frac{1}{S}\oint_S G_N(\vec{x}',\vec{y})\,da$$

c. $\Phi(\vec{x})=\langle\Phi\rangle_S+\frac{1}{4\pi\varepsilon_0}\int_V\rho(\vec{x}')G_N(\vec{x},\vec{x}')d^3x'+\frac{1}{4\pi}\oint_S\frac{\partial\Phi}{\partial n'}G_N\,da'$ (1.46)

$$\Phi'(\vec{x})=\langle\Phi\rangle_S+\frac{1}{4\pi\varepsilon_0}\int_V\rho(\vec{x}')(G_N(\vec{x},\vec{x}')-F(\vec{x}))d^3x'$$
$$+\frac{1}{4\pi}\oint_S\frac{\partial\Phi}{\partial n'}(G_N(\vec{x},\vec{x}')-F(\vec{x}))\,da'$$

$$=\langle\Phi\rangle_S+\frac{1}{4\pi\varepsilon_0}\int_V\rho(\vec{x}')G_N(\vec{x},\vec{x}')d^3x'+\frac{1}{4\pi}\oint_S\frac{\partial\Phi}{\partial n'}G_N(\vec{x},\vec{x}')\,da'$$
$$-\frac{1}{4\pi\varepsilon_0}\int_V\rho(\vec{x}')F(\vec{x})d^3x'-\frac{1}{4\pi}\oint_S\frac{\partial\Phi}{\partial n'}F(\vec{x})\,da'$$

$$=\Phi(\vec{x})-\frac{1}{4\pi}F(\vec{x})\left[\int_V\frac{\rho(\vec{x}')}{\varepsilon_0}d^3x'+\oint_S\frac{\partial\Phi}{\partial n'}\,da'\right]$$

$$=\Phi(\vec{x})-\frac{1}{4\pi}F(\vec{x})\left[\oint_S\vec{E}\cdot\hat{n}'\,da'-\oint_S\vec{E}\cdot\hat{n}'\,da'\right] \quad (1.12+1.16)$$

$$\Phi'(\vec{x})=\Phi(\vec{x})$$

15. The remaining problems require the use of variational methods or relaxation methods to solve, and so they will not be done here.