Cohen-Tannoudji Solutions

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Contents

Chapter 1

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Chapter 2

The Mathematical Tools of Quantum Mechanics

2.1 Hermitian Operator

$|\phi_n\rangle$ are the eigenstates of a Hermitian operator H (H is, for example, the Hamiltonian of an arbitrary physical system). Assume that the states $|\phi_n\rangle$ form a discrete orthonormal basis. The operator $U(m,n)$ is defined by:

$$U(m,n) = |\phi_m\rangle \langle\phi_n|$$

2.1.1 Adjoint

Calculate the adjoint $U^\dagger(m,n)$ of $U(m,n)$.

From the definition of the adjoint,

$$U^\dagger(m,n) = |\phi_n\rangle \langle\phi_m|$$

2.1.2 Commutator

Calculate the commutator $[H, U(m,n)]$

Let's act the commutator on a vector (looking ahead, we'll set the vector as $|\phi_n\rangle$),

$$[H,U]\,|\phi_n\rangle$$

$$H\,|\phi_m\rangle \langle\phi_n|\phi_n\rangle - |\phi_m\rangle \langle\phi_n|H|\phi_n\rangle$$

$$H\,|\phi_m\rangle - n\,|\phi_m\rangle = (m-n)\,|\phi_m\rangle$$

The commutator is

$$[H, U(m,n)] = m - n$$

2.1.3 Delta Function

Prove the relation:

$$U(m,n)U^{\dagger}(p,q) = \delta_{nq} U(m,p)$$

Writing this out,

$$U(m,n)U^{\dagger}(p,q) = |\phi_m\rangle \langle\phi_n|\phi_q\rangle \langle\phi_p|$$

The middle section dies unless $n = q$, leaving us the delta function,

$$= \delta_{nq} |\phi_m\rangle \langle\phi_p| = \delta_{nq} U(m,p) \tag{2.1.1}$$

2.1.4 Trace

Calculate $Tr\{U(m,n)\}$, the trace of the operator $U(m,n)$.

By definition, the trace is given by

$$Tr(U) = \sum_{\alpha} \langle\alpha|U|\alpha\rangle$$

where α are the basis states.

$$= \sum_{\alpha} \langle\alpha|\phi_m\rangle \langle\phi_n|\alpha\rangle$$

We can convince ourselves that one part must be equal to zero at all times unless $m = n$,

$$Tr(U) = \delta_{mn}$$

2.1.5 Multiplying Operators

Let A be an operator, with matrix elements $A_{mn} = \langle\phi_m|A|\phi_n\rangle$. Prove the relation:

$$A = \sum_{mn} A_{mn} U(m,n)$$

Let's start by acting A on $|\phi_n\rangle$. We only need to sum over m in this case since we can use orthonormality for n. Writing out the sum,

$$A|\phi_n\rangle = \sum_{m} \langle\phi_m|A|\phi_n\rangle |\phi_m\rangle \langle\phi_n|\phi_n\rangle$$

A_{mn} is a scalar, so we can move that around for free,

$$= \sum_m |\phi_m\rangle \langle\phi_m|A|\phi_n\rangle$$

When we perform the sum, the first part becomes identity, so we can remove it,

$$= A\,|\phi_n\rangle$$

2.1.6 More Trace

Show that $A_{pq} = Tr\{AU^\dagger(p,q)\}$**.**

We'll start with the right side. We can write the part inside the trace using the relation we found in part (e),

$$AU^\dagger(p,q) = \sum_{m,n} A_{mn}U(m,n)U^\dagger(p,q)$$

Using the relation found in part (c),

$$= \sum_{m,n} A_{mn}\delta_{nq}U(m,p)$$

Summing over n, we pick out $n = q$. We can then take the trace and use the result from part (d) to get a value for the trace of our operator $U(m,p)$,

$$Tr\{AU^\dagger(p,q)\} = \sum_m A_{mq}\delta_{mp}$$

Again, summing over m picks out the $m = p$ value,

$$= A_{pq}$$

2.2 Pauli Matrices

In a two-dimensional vector space, consider the operator whose matrix, in an orthonormal basis $\{|1\rangle, |2\rangle\}$, is written:

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

2.2.1 Eigenvalues

Is σ_y Hermitian? Calculate its eigenvalues and eigenvectors (giving their normalized expansion in terms of the $\{|1\rangle, |2\rangle\}$ basis).

σ_y is Hermitian. To find the eigenvalues, we need to solve the characteristic equation,

$$\det(\sigma_y - \lambda I) = \det \begin{pmatrix} -\lambda & -i \\ i & \lambda \end{pmatrix}$$

$$= \lambda^2 - 1 = 0$$

Our eigenvalues are $\lambda = \pm 1$. We can show that the eigenvectors are

$$|\lambda = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$|\lambda = -1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

In the $|1\rangle$, $|2\rangle$ basis,

$$\begin{cases} |\lambda = 1\rangle = \frac{1}{\sqrt{2}}(-i\,|1\rangle + |2\rangle) \\ |\lambda = -1\rangle = \frac{1}{\sqrt{2}}(i\,|1\rangle + |2\rangle) \end{cases}$$

2.2.2 Projection Operator

Calculate the matrices which represent the projectors onto these eigenvectors. Then verify that they satisfy the orthogonality and closure relations.

By definition,

$$A = |\lambda = 1\rangle \langle \lambda = 1|$$

$$= \frac{1}{2}(-i\,|1\rangle + |2\rangle)(i\,\langle 1| + \langle 2|)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

Similarly,

$$B = |\lambda = -1\rangle\,\langle\lambda = -1|$$

$$= \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

We can show that they are orthonormal by multiplying the two together, $AB = BA = 0$. We can show completeness by adding them together, $A + B = I$.

2.3 Kets and Operators

The state space of a certain physical system is three-dimensional. Let $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ be an orthonormal basis of this space. The kets $|\psi_0\rangle$ and $|\psi_1\rangle$ are defined by:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}|u_1\rangle + \frac{i}{\sqrt{3}}|u_3\rangle$$

2.3.1 Normalized Kets

Are these kets normalized?

To tell if these kets are normalized, we need to find the norm. Let's start with $|\psi_0\rangle$,

$$\langle\psi_0|\psi_0\rangle = \left(\frac{1}{\sqrt{2}}\langle u_1| - \frac{i}{2}\langle u_2| + \frac{1}{2}\langle u_3|\right)\left(\frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle\right)$$

Using orthonormality, we can ignore most of these terms,

$$= \frac{1}{2}\langle u_1|u_1\rangle + \frac{1}{4}\langle u_2|u_2\rangle + \frac{1}{4}\langle u_3|u_3\rangle = 1$$

Since we have an orthonormal basis, we can see that $|\psi_0\rangle$ is normalized. For $|\psi_1\rangle$,

$$\langle\psi_1|\psi_1\rangle = \left(\frac{1}{\sqrt{3}}\langle u_1| - \frac{i}{\sqrt{3}}\langle u_3|\right)\left(\frac{1}{\sqrt{3}}|u_1\rangle + \frac{i}{\sqrt{3}}|u_3\rangle\right)$$

$$= \frac{1}{3}\langle u_1|u_1\rangle + \frac{1}{3}\langle u_3|u_3\rangle = \frac{2}{3}$$

$|\psi_1\rangle$ is not normalized.

2.3.2 Projection Operators

Calculate the matrices ρ_0 and ρ_1 representing, in the $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ basis, the projection operators onto the state $|\psi_0\rangle$ and onto the state $|\psi_1\rangle$. Verify that these matrices are Hermitian.

We define the projection operator as

$$\rho_0 = |\psi_0\rangle\langle\psi_0|$$

$$= \left(\frac{1}{\sqrt{2}}\left|u_1\right\rangle + \frac{i}{2}\left|u_2\right\rangle + \frac{1}{2}\left|u_3\right\rangle\right)\left(\frac{1}{\sqrt{2}}\left\langle u_1\right| - \frac{i}{2}\left\langle u_2\right| + \frac{1}{2}\left\langle u_3\right|\right)$$

We can define our orthonormal basis however we want, but for ease, let's use

$$\left|u_1\right\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}; \quad \left|u_2\right\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}; \quad \left|u_3\right\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

In this basis,

$$\rho_0 = \begin{bmatrix} \frac{1}{2} & -\frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i}{4} \\ \frac{1}{2\sqrt{2}} & -\frac{i}{4} & \frac{i}{4} \end{bmatrix}$$

We use the Hermitian condition (1.8) to show that ρ_0 is Hermitian.
Similarly,

$$\rho_1 = \left|\psi_1\right\rangle\left\langle\psi_1\right|$$

$$= \left(\frac{1}{\sqrt{3}}\left|u_1\right\rangle + \frac{i}{\sqrt{3}}\left|u_3\right\rangle\right)\left(\frac{1}{\sqrt{3}}\left\langle u_1\right| - \frac{i}{\sqrt{3}}\left\langle u_3\right|\right)$$

$$= \begin{bmatrix} \frac{1}{3} & 0 & -\frac{i}{3} \\ 0 & 0 & 0 \\ \frac{i}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

Again, we see that ρ_1 is Hermitian.

2.4 Operators

Let K be the operator defined by $K = |\phi\rangle \langle\psi|$, where $|\phi\rangle$ and $|\psi\rangle$ are two vectors of the state space.

2.4.1 Hermitian

Under what condition is K Hermitian?

Following the Hermitian condition (1.8),

$$K = K^\dagger$$

$$|\phi\rangle \langle\psi| = |\psi\rangle \langle\phi|$$

One possible way for K to be Hermitian is for $|\phi\rangle = |\psi\rangle$.

2.4.2 Projection Operator

Calculate K^2. Under what condition is K a projector?

$$K^2 = |\phi\rangle \langle\psi|\phi\rangle \langle\psi|$$

K is a projector if $|\phi\rangle = |\psi\rangle$.

2.4.3 More Projectors

Show that K can always be written in the form $K = \lambda P_1 P_2$ where λ is a constant to be calculated and P_1 and P_2 are projectors.

We set

$$P_1 = |\phi\rangle \langle\phi|$$

$$P_2 = |\psi\rangle \langle\psi|$$

Now if we multiply them together,

$$P_1 P_2 = |\phi\rangle \langle\phi|\psi\rangle \langle\psi|$$

The middle part is just a scalar, so to get rid of that, we need to multiply by some constant,

$$\lambda = \frac{1}{\langle \phi | \psi \rangle}$$

Combining all of this,

$$\lambda P_1 P_2 = |\phi\rangle \langle\psi| = K$$

2.5 Orthogonal Projector

Let P_1 be the orthogonal projector onto the subspace $\mathscr{E}_1$, P_2 the orthogonal projector onto the subspace $\mathscr{E}_2$. Show that, for the product P_1P_2 to be an orthogonal projector as well, it is necessary and sufficient that P_1 and P_2 commute. In this case, what is the subspace onto which P_1P_2 projects?

Let's say that

$$P_1 = |\phi\rangle \langle\phi|$$

$$P_2 = |\psi\rangle \langle\psi|$$

where $|\phi\rangle$ and $|\psi\rangle$ are in $\mathscr{E}_1$ and $\mathscr{E}_2$ respectively. If P_1 and P_2 commute, this implies

$$P_1P_2 = P_2P_1$$

$$|\phi\rangle \langle\phi|\psi\rangle \langle\psi| = |\psi\rangle \langle\psi|\phi\rangle \langle\phi|$$

What this implies,

$$\langle\phi|\psi\rangle = \langle\psi|\phi\rangle$$

We imagine that if we act P_1P_2 on either $|\phi\rangle$ or $|\psi\rangle$, we get the same thing. P_1P_2 projects onto the overlap of $\mathscr{E}_1$ and $\mathscr{E}_2$.

2.6 Pauli Matrices

The σ_x matrix is defined by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Prove the relation:

$$\exp(i\alpha\sigma_x) = I\cos(\alpha) + i\sigma_x \sin(\alpha)$$

where I is the 2×2 unit matrix.

We can expand the left-side using a Taylor expansion,

$$\exp(i\alpha\sigma_x) = I + i\alpha\sigma_x + \frac{1}{2}(i\alpha)^2\sigma_x^2 + ...$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\alpha^2}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + ...$$

$$= \begin{pmatrix} 1 - \frac{\alpha^2}{2} + ... & i\alpha + ... \\ \\ i\alpha + ... & 1 - \frac{\alpha^2}{2} + ... \end{pmatrix}$$

Similarly, if we expand the right side,

$$I\cos(\alpha) = \begin{pmatrix} 1 - \frac{\alpha^2}{2} + ... & 0 \\ \\ 0 & 1 - \frac{\alpha^2}{2} \end{pmatrix}$$

$$i\sigma_x \sin(\alpha) = \begin{pmatrix} 0 & i\alpha + ... \\ i\alpha + ... & 0 \end{pmatrix}$$

2.7 Pauli Matrices

Establish for the σ_y matrix given in exercise 2, a relation analogous to the one proved for σ_x in the preceding exercise. Generalize for all matrices of the form:

$$\sigma_u = \lambda\sigma_x + \mu\sigma_y$$

with:

$$\lambda^2 + \mu^2 = 1$$

Calculate the matrices representing $\exp(2i\sigma_x)$**,** $(\exp(i\sigma_x))^2$ **and** $\exp(i(\sigma_x+\sigma_y))$**. Is** $\exp(2i\sigma_x)$ **equal to** $(exp(i\sigma_x))^2$**?** $\exp(i(\sigma_x+\sigma_y))$ **to** $\exp(i\sigma_x)\exp(i\sigma_y)$**?**

Following the methodology in question 6, we expand the exponential,

$$\exp(i\alpha\sigma_y) = I + i\alpha\sigma_y + \frac{1}{2}(i\alpha)^2\sigma_y^2 + ...$$

$$= \begin{pmatrix} 1 - \dfrac{\alpha^2}{2} + ... & \alpha + ... \\ -\alpha & 1 - \dfrac{\alpha^2}{2} + ... \end{pmatrix}$$

We can convince ourselves that this is

$$\exp(i\alpha\sigma_y) = I\cos(\alpha) + i\sigma_y\sin(\alpha)$$

For σ_u, we can't use the normal rules of exponential multiplication (which answers the last part of this question). Expanding,

$$\exp(i\alpha(\lambda\sigma_x + \mu\sigma_y)) = I + i\alpha(\lambda\sigma_x + \mu\sigma_y) + \frac{1}{2}(i\alpha)^2(\lambda^2\sigma_x^2 + \mu^2\sigma_y^2 + \lambda\mu\sigma_x\sigma_y + \lambda\mu\sigma_y\sigma_x)$$

$$= I\cos(\alpha) + i\sigma_x\sin(\alpha\lambda) + i\sigma_y\sin(\alpha\mu)$$

Using the relation found in question 6,

$$\exp(2i\sigma_x) = I\cos(2) + i\sigma_x\sin(2)$$

$$(\exp(i\sigma_x))^2 = I(\cos^2(1) - \sin^2(1)) + 2i\sigma_x\cos(1)\sin(1)$$

These are equal by using angle addition formulas.

2.8 One dimensional Particle

Consider the Hamiltonian $\mathscr{H}$ of a particle in a one-dimensional problem defined by:

$$\mathscr{H} = \frac{1}{2m}P^2 + V(X)$$

where X and P are the operators defined in ... and which satisfy the relation: $[X, P] = i\hbar$. The eigenvectors of $\mathscr{H}$ are denoted by $|\phi_n\rangle$: $\mathscr{H}|\phi_n\rangle = E|\phi_n\rangle$, where n is a discrete index.

2.8.1 Expectation Value

Show that

$$\langle\phi_n|P|\phi_{n'}\rangle = \alpha\langle\phi_n|X|\phi_{n'}\rangle$$

where α is a coefficient which depends on the difference between E_n and $E_{n'}$. Calculate α (hint: consider the commutator $[X, \mathscr{H}]$).

We can use Dirac's rule to find the commutator of $[X, \mathscr{H}]$,

$$[X, \mathscr{H}] = i\hbar[X, \mathscr{H}]_{cl}$$

$$[X, \mathscr{H}]_{cl} = \frac{1}{m}P$$

$$[X, \mathscr{H}] = \frac{i\hbar}{m}P$$

A little bit of rearranging,

$$P = \frac{m}{i\hbar}[X, \mathscr{H}]$$

Inserting this in,

$$\frac{m}{i\hbar}\langle\phi_n|[X, \mathscr{H}]|\phi_{n'}\rangle$$

$$= \frac{m}{i\hbar}(\langle\phi_n|X\mathscr{H}|\phi_{n'}\rangle - \langle\phi_{n'}|\mathscr{H}X|\phi_n\rangle)$$

Using the eigenvectors of $\mathscr{H}$,

$$= \frac{m}{i\hbar}(E_{n'} - E_n)\langle\phi_n|X|\phi_{n'}\rangle$$

2.8.2 Closure

From this, deduce, using the closure relation, the equation:

$$\sum_{n'} (E_n - E_{n'})^2 |\langle \phi_n | X | \phi_{n'} \rangle|^2 = \frac{\hbar^2}{m^2} \langle \phi_n | P^2 | \phi_n \rangle$$

We can get this by squaring both sides.

2.9 Hamiltonian

Let $\mathscr{H}$ be the Hamiltonian operator of a physical system. Denote by $|\phi_n\rangle$ the eigenvectors of $\mathscr{H}$, with eigenvalues E_n:

$$\mathscr{H}\,|\phi_n\rangle = E_n\,|\phi_n\rangle$$

2.9.1 Commutator

For an arbitrary operator A, prove the relation:

$$\langle\phi_n|[A,\mathscr{H}]|\phi_n\rangle = 0$$

We can expand the commutator,

$$\langle\phi_n|A\mathscr{H}|\phi_n\rangle - \langle\phi_n|\mathscr{H}A|\phi_n\rangle$$

Using the eigenvalue relation,

$$= E_n\,\langle\phi_n|A|\phi_n\rangle - E_n\,\langle\phi_n|A|\phi_n\rangle = 0$$

2.9.2 One-Dimensional Particle

Consider a one-dimensional problem, where the physical system is a particle of mass m and potential energy $V(X)$. In this case, $\mathscr{H}$ is written:

$$\mathscr{H} = \frac{1}{2m}P^2 + V(X)$$

In terms of P, X, and $V(X)$, find the commutators: $[\mathscr{H},P]$, $[\mathscr{H},X]$, and $[\mathscr{H},XP]$

We want to use Dirac's rule,

$$[\mathscr{H},P] = i\hbar[\mathscr{H},P]_{cl}$$

$$= i\hbar\left(\frac{\partial\mathscr{H}}{\partial X}\frac{\partial P}{\partial P} - \frac{\partial\mathscr{H}}{\partial P}\frac{\partial P}{\partial X}\right)$$

Only the first term survives,

$$= i\hbar\frac{\partial V}{\partial X}$$

Similarly,

$$[\mathscr{H}, X] = -\frac{i\hbar}{m}P$$

$$[\mathscr{H}, XP] = i\hbar\left(X\frac{\partial V}{\partial X} - \frac{P^2}{m}\right)$$

Show that the matrix element $\langle\phi_n|P|\phi_n\rangle$ (which we shall interpret in chapter III as the mean value of the momentum in the state $|\phi_n\rangle$) is zero

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Using the result shown previously,

$$P = -\frac{m}{i\hbar}[\mathscr{H}, X]$$

Inserting this in,

$$\langle\phi_n|P|\phi_n\rangle = \frac{m}{i\hbar}\langle\phi_n|[X, \mathscr{H}|\phi_n\rangle$$

which we showed in part a to be zero.

Establish a relation between $E_k = \langle\phi_n|\frac{P^2}{2m}|\phi_n\rangle$ (the mean value of the kinetic energy in the state $|\phi_n\rangle$) and $\langle\phi_n|X\frac{dV}{dX}|\phi_n\rangle$. Since the mean value of the potential energy in the state $|\phi_n\rangle$ is $\langle\phi_n|V(X)|\phi_n\rangle$, how is it related to the mean value of the kinetic energy when:

$$V(X) = V_0X^\lambda$$

($\lambda = 2, 4, 6, ...$; $V_0 > 0$)?

We recognize these as the components of $[\mathscr{H}, XP]$,

$$\langle\phi_n|[\mathscr{H}, XP]|\phi_n\rangle = i\hbar\left(\langle\phi_n|X\frac{dV}{dX}|\phi_n\rangle - \langle\phi_n|\frac{P^2}{m}|\phi_n\rangle\right)$$

We know the left side is equal to 0 from part a, so

$$\langle\phi_n|\frac{P^2}{2m}|\phi_n\rangle = \frac{1}{2}\langle\phi_n|X\frac{dV}{dX}|\phi_n\rangle$$

If the potential is some polynomial,

$$X\frac{dV}{dX} = \lambda V$$

$$\langle\phi_n|\frac{P^2}{2m}|\phi_n\rangle = \frac{1}{2}\lambda\,\langle\phi_n|V(X)|\phi_n\rangle$$

Since λ is constrained to be even, the kinetic energy is some integer multiple of the potential energy.

2.10 Transformation Function

Using the relation $\langle x|p\rangle = (2\pi\hbar)^{-1/2}\exp(ipx/\hbar)$, find the expressions $\langle x|XP|\psi\rangle$ and $\langle x|PX|\psi\rangle$ in terms of $\psi(x)$. Can these results be found directly in using the fact that in the $\{|x\rangle\}$ representation, P acts like $\frac{\hbar}{i}\frac{d}{dx}$?

We know that $|x\rangle$ is an eigenvector of X with eigenvalue x, so let's get rid of that first,

$$\langle x|XP|\psi\rangle = x\,\langle x|P|\psi\rangle$$

We can now insert identity twice,

$$= x\,\langle x|p\rangle\,\langle p|P|\psi\rangle$$

$$= x\,\langle x|p\rangle\,p\,\langle p|x\rangle\,\langle x|\psi\rangle$$

We know that when we multiply $\langle x|p\rangle\,\langle p|x\rangle$, we should get unity,

$$= xp\psi(x) = -i\hbar x\frac{d\psi(x)}{dx}$$

If we treat p as $-i\hbar\frac{d}{dx}$, we get the same thing as if $P = \frac{\hbar}{i}\frac{d}{dx}$.
Similarly,

$$\langle x|PX|\psi\rangle = \langle x|p\rangle\,\langle p|PX|x\rangle\,\langle x|\psi\rangle$$

$$= \langle x|p\rangle\,px\,\langle x|p\rangle\,\psi(x)$$

$$px\psi(x)$$

$$= -i\hbar\frac{d(x\psi(x))}{dx}$$

2.11 Commuting Observable and CSCO's

Consider a physical system whose three-dimensional state space is spanned by the orthonormal basis by the three kets $|u_1\rangle$, $|u_2\rangle$, $|u_3\rangle$. In the basis of these three vectors, taken in this order , the two operators H and B are defined by:

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ; \quad B = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where ω_0 and b are real constants.

2.11.1 Hermitian

Are H and B Hermitian?

By observation, yes.

2.11.2 Common eigenbasis

Show that H and B commute. Give a basis of eigenvectors common to H and B.

Let's start by solving the characteristic equation for H,

$$\det H - \lambda I = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{pmatrix} = 0$$

The eigenvalues are $\lambda = \pm 1$. We have a degeneracy for $\lambda = -1$, but we can find the easiest eigenvectors for the other eigenvalues,

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \quad |-1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Now we know that we want this basis to be orthonormal, so to fulfill this condition,

$$|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

2.11.3 CSCO

Of the set of operators: $\{H\}$, $\{B\}$, $\{H, B\}$, $\{H^2, B\}$, which form a CSCO?

2.12 Spin Operators

In the same state space as that of the preceding exercise, consider two operators L_z and S defined by:

$$\begin{cases} L_z \left| u_1 \right\rangle = \left| u_1 \right\rangle, \quad L_z \left| u_2 \right\rangle = 0, \quad L_z \left| u_3 \right\rangle = - \left| u_3 \right\rangle \\ S \left| u_1 \right\rangle = \left| u_3 \right\rangle, \quad S \left| u_2 \right\rangle = \left| u_2 \right\rangle, \quad S \left| u_3 \right\rangle = \left| u_1 \right\rangle \end{cases}$$

2.12.1 Matrix representation

Write the matrices which represent, in the $\{\left| u_1 \right\rangle, \left| u_2 \right\rangle, \left| u_3 \right\rangle\}$ basis, the operators L_z, L_z^2, S, S^2. Are these operators observable?

From observation,

$$L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

From this, we can show

$$L_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These are all observables.

2.12.2 Commutator

Give the form of the most general matrix which represents an operator which commutes with L_z. Same question for L_z^2, then for S^2.

Let's say we have some matrix,

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

Acting L_z on it,

$$L_zM = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ 0 & 0 & 0 \\ -m_{31} & -m_{32} & -m_{33} \end{pmatrix}$$

$$ML_z = \begin{pmatrix} m_{11} & 0 & -m_{13} \\ 0 & 0 & 0 \\ m_{31} & 0 & -m_{33} \end{pmatrix} \tag{2.12.1}$$

In order for these to commute, only m_{11} and m_{33} survive,

$$M = \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_{33} \end{pmatrix}$$

We can repeat the process for L_z^2,

$$L_z^2M = \begin{pmatrix} m_{11} & 0 & m_{13} \\ 0 & 0 & 0 \\ m_{31} & 0 & m_{33} \end{pmatrix}$$

$$ML_z^2 = \begin{pmatrix} m_{11} & 0 & m_{13} \\ 0 & 0 & 0 \\ m_{31} & 0 & m_{33} \end{pmatrix}$$

All four corners survive, and we can add a term in the middle,

$$M = \begin{pmatrix} m_{11} & 0 & m_{13} \\ 0 & m_{22} & 0 \\ m_{31} & 0 & m_{33} \end{pmatrix}$$

Since S^2 is the identity matrix, any matrix will commute with it.

2.12.3 CSCO

Do L_z^2 and S form a CSCO? Give a basis of common eigenvectors.

Solving the characteristic equation for L_z^2 since S has a degeneracy,

$$\det(L_z^2 - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

Our eigenvalues are $\lambda = 0, 1$. We have a degeneracy, so our eigenvectors are

$$|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} ; \quad |0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Now by orthonormality,

$$|a\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$