Goldstein Solutions

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Contents

Chapter 1

Survey of the Elemntary Particles

1.1 Kinetic Energy Equation of Motion

Show that for a single particle with constant mass the equation of motion implies the following differential equation for the kinetic energy:

$$\frac{dT}{dt} = \vec{F} \cdot \vec{v}$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \vec{F} \cdot \vec{p}$$

We'll start with the definition of kinetic energy (1.9). If we take the time derivative, we only have to derivatify the velocity terms,

$$T = \frac{1}{2} m\vec{v} \cdot \vec{v}$$

$$\frac{dT}{dt} = \frac{m}{2} 2\dot{\vec{v}} \cdot \vec{v} = m\vec{a} \cdot \vec{v} = \vec{F} \cdot \vec{v}$$

If however our mass is time dependent,

$$\frac{d(mT)}{dt} = \frac{d}{dt}\left[\frac{1}{2}(m\vec{v})^2\right] = \frac{d(m\vec{v})}{dt} \cdot (m\vec{v}) = \vec{F} \cdot \vec{p}$$

1.2 Center of Mass

Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2$$

We'll start with the definition of the center of mass (1.12).

$$M\vec{R} = \sum_i m_i \vec{r}_i$$

$$M^2R^2 = \sum_{i,j} m_i m_j \vec{r}_i \cdot \vec{r}_j$$

Let's look at the distance between two arbitrary points,

$$r_{ij}^2 = (\vec{r}_i - \vec{r}_j)^2$$

$$= r_i^2 + r_j^2 - 2\vec{r}_i \cdot \vec{r}_j$$

Substituting this back in,

$$M^2R^2 = \frac{1}{2}\sum_{ij} m_i m_j r_i^2 + \frac{1}{2}\sum_{ij} m_i m_j r_j^2 - \frac{1}{2}\sum_{ij} m_i m_j r_{ij}^2$$

We note that the first two terms are identical since they will eventually all look at the same points.

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2$$

1.3 Newton's Third Law

Suppose a system of two particles is known to obey the equations of motion,

$$M\frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^{(e)} = \vec{F}^{(e)}$$

$$\frac{d\vec{L}}{dt} = \vec{N}^{(e)}$$

From the equations of the motion of the individual particles show that the internal forces between particles satisfy both the weak and the strong laws of action and reaction. The argument may be generalized to a system with arbitrary number of particles, thus proving the converse of the arguments leading to the equations of motion.

Say the two particles are at rest,

$$M \cdot 0 = m_1 a_1 + m_2 a_2$$

$$m_1 a_1 = -m_2 a_2$$

$$\vec{F}_1^{(e)} = -\vec{F}_2^{(e)}$$

1.4 Rolling Disk Constraint

The equations of constraint for the rolling disk,

$$\begin{cases} dx - a\sin(\theta)d\phi = 0 \\ dy + a\cos(\theta)d\phi = 0 \end{cases}$$

are special cases of general linear differential equations of constraint of the form

$$\sum_{i=1}^{n} g_i(x_1, ...x_n)dx_i = 0$$

A constraint condition of this type is holonomic only if an integrating function $f(x_1, ...x_n$ **can be found that turns it into an exact differential. Clearly the function must be such that**

$$\frac{\partial(fg_i)}{\partial x_j} = \frac{\partial(fg_j)}{\partial x_i}$$

for all $i \neq j$**. Show that no such integrating factor can be found for either of constraint equations.**

Let's start with the first constraint equation. We can write the g_i,

$$\begin{cases} g_x = 1 \\ g_\theta = 0 \\ g_\phi = -a\sin(\theta) \end{cases}$$

$$\frac{\partial f}{\partial \phi} = \frac{\partial - af\sin(\theta)}{\partial x}$$

Using separation of variables, we expect our solution to take the form $f = X(x)Q(\phi)$. Substituting this in,

$$Q'X = -a\sin(\theta)X'Q$$

There is no solution for Q and X that satisfy this equation. If we look at g_θ,

$$\frac{\partial f}{d\phi} = 0$$

The only solution is the trivial one. Similarly, we can follow the same steps for the second constraint equation.

1.5 Constraint Equations

Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a plane. Show that there are two nonholonomic equations of constraint

$$\begin{cases} \cos(\theta)dx + \sin(\theta)dy = 0 \\ \\ \sin(\theta)dx - \cos(\theta)dy = \dfrac{1}{2}a(d\phi + d\phi') \end{cases}$$

(where θ, ϕ, and ϕ' have meanings similar to those in the problem of a single vertical disk, and (x, y) are the coordinates of a point on the axle midway between the two wheels) and one holonomic equation of constraint,

$$\theta = C - \frac{a}{b}(\phi - \phi')$$

where C is a constant.

Let's call one wheel the unprimed system and the other, the primed system. Following the no-slipping condition,

$$\begin{cases} v = a\dot{\phi} \\ \dot{x} = v\sin(\theta) \\ \dot{y} = -v\cos(\theta) \end{cases}$$

$$\begin{cases} v' = a\dot{\phi}' \\ \dot{x}' = v'\sin(\theta) \\ \dot{y}' = -v'\cos(\theta) \end{cases}$$

Let's start by looking at the nonholonomic constraints,

$$\cos(\theta)dx + \sin(\theta)dy = 0$$

$$v\cos(\theta)\sin(\theta) - v\sin(\theta)\cos(\theta) = 0$$

$$\sin(\theta)dx - \cos(\theta)dy = v\sin^2(\theta) + v\cos^2(\theta) = \frac{1}{2}(v + v') = \frac{1}{2}a(d\phi + d\phi')$$

For the holonomic constraint,

$$\phi - \phi' = r - r'$$

$$\theta = C - \frac{a}{b}(r - r')$$

$$\theta = C - \frac{a}{b}(\phi - \phi')$$

1.6 Non-Holonomic Constraint

A particle moves int eh xy plane under the constraint that its velocity vector is always directed towards a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, the constraint is non-holonomic.

$$ydx - [f(t) - x]dy = 0$$

Since $f(t)$ is non-holonomic, cannot be solved.

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1.11 Conservative Forces

Check whether the force $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ is conservative or not.

We want to show that the potential energy obeys equation (1.10). That is, we need to find some V such that

$$\begin{cases} \dfrac{\partial}{\partial x}V = -yz \\ \\ \dfrac{\partial}{\partial y}V = -zx \\ \\ \dfrac{\partial}{\partial z}V = -xy \end{cases}$$

$$V = -xyz$$

Because we can find a potential energy that solves for the given force, the force is conservative.

1.12 Satellite Orbital Motion

Compute the orbital period and orbital angular velocity of a satellite revolving around the Earth at an altitude of 720km.

From undergraduate mechanics, we remember,

$$F = \frac{mv^2}{r} = \frac{m\omega^2 r^2}{r}$$

$$ma = m\omega^2 r$$

$$\omega^2 = \frac{a}{r} = \frac{9.83\ m/s^2}{6.72 \times 10^6\ m}$$

$$\omega = 1.2 \times 10^{-3}\ s^{-1}$$

The orbital period is given by $T = 2\pi\omega$,

$$T = 7.5 \times 10^{-3}\ s^{-1}$$

1.13 Rocket Propulsion

Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric friction, is

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg$$

where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m, assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to 2.1 km/s and a mass loss per second equal to 1/60th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300!

We want to use conservation of linear momentum. The change in momentum of the rocket is equal to the change in momentum of the gases plus the momentum due to gravitational force,

$$m\ dv = -v'\ dm - mg\ dt$$

$$dv = -v'\frac{dm}{m} - g\ dt$$

Integrating from initial to final mass of the rocket,

$$v_f - v_0 = -v' \ln(m)|_{m_0}^{m_f} - gt = -v' \ln\left(\frac{m_f}{m_0}\right) - gt$$

Looking up the escape velocity (11.2 km/s) and setting the initial mass m_0 to the mass of the rocket plus mass of the fuel, $m + m_s$ (s stands for sugar),

$$v_e = -v' \ln\left(\frac{m}{m + m_s}\right) - 60g$$

$$\frac{m}{m_s} = 276$$

1.14 Generalized Coordinates

Two points of mass m are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius a. Express the kinetic energy in generalized coordinates.

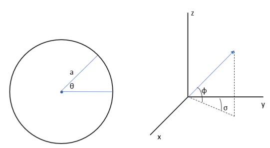

Figure 1.1: Generalized Coordinates

For this problem, we have three degrees of freedom. The rod provides one (θ) while the two masses have two (ϕ and σ) as seen in figure (1.1). From this, we can define the two masses as

$$\vec{x}_1 = \left(a\cos(\theta) + \frac{l}{2}\cos(\phi)\sin(\sigma),\ a\sin(\theta) + \frac{l}{2}\cos(\phi)\cos(\sigma),\ \frac{l}{2}\sin(\sigma)\right)$$

$$\vec{x}_2 = \left(a\cos(\theta) - \frac{l}{2}\cos(\phi)\sin(\sigma),\ a\sin(\theta) - \frac{l}{2}\cos(\phi)\cos(\sigma),\ -\frac{l}{2}\sin(\sigma)\right)$$

The kinetic energy is

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$$

Let's find the component pieces. First, we take the time derivative of the mass positions,

$$\begin{aligned}\dot{\vec{x}}_1 = \Big(& - a\dot{\theta}\sin(\theta) - \frac{l\dot{\theta}}{2}\sin(\theta)\sin(\sigma) + \frac{l\dot{\sigma}}{2}\cos(\phi)\cos(\sigma),\\ & a\dot{\theta}\cos(\theta) - \frac{l\dot{\phi}}{2}\sin(\phi)\cos(\sigma) - \frac{l\dot{\sigma}}{2}\cos(\phi)\sin(\sigma),\\ & \frac{l\dot{\phi}}{2}\cos(\phi)\Big)\end{aligned}$$

$$\begin{aligned}\dot{x}_1^2 &= a^2\dot{\theta}^2\sin^2(\theta) + \frac{l^2\dot{\phi}^2}{4}\sin^2(\phi)\sin^2(\sigma) + \frac{l^2\dot{\sigma}^2}{4}\cos^2(\phi)\cos^2(\sigma)\\ &+a^2\dot{\theta}^2\cos^2(\theta) + \frac{l^2\dot{\phi}^2}{4}\sin^2(\phi)\cos^2(\sigma) + \frac{l^2\dot{\sigma}^2}{4}\cos^2(\phi)\sin^2(\sigma) + \frac{l^2\dot{\phi}^2}{4}\cos^2(\phi)\\ &= a^2\dot{\theta}^2 + \frac{l^2\dot{\phi}^2}{4}\sin^2(\phi) + \frac{l^2\dot{\sigma}^2}{4}\cos^2(\phi) + \frac{l^2\dot{\phi}^2}{4}\cos^2(\phi)\end{aligned}$$

Similarly, for $\vec{x}_2$,

$$\begin{aligned}\dot{\vec{x}}_2 = \Big(& - a\dot{\theta}\sin(\theta) + \frac{l\dot{\theta}}{2}\sin(\theta)\sin(\sigma) - \frac{l\dot{\sigma}}{2}\cos(\phi)\cos(\sigma),\\ & a\dot{\theta}\cos(\theta) + \frac{l\dot{\phi}}{2}\sin(\phi)\cos(\sigma) + \frac{l\dot{\sigma}}{2}\cos(\phi)\sin(\sigma),\\ & -\frac{l\dot{\phi}}{2}\cos(\phi)\Big)\end{aligned}$$

$$\dot{x}_2^2 = a^2\dot{\theta}^2 + \frac{l^2\dot{\phi}^2}{4}\sin^2(\phi) + \frac{l^2\dot{\sigma}^2}{4}\cos^2(\phi) + \frac{l^2\dot{\phi}^2}{4}\cos^2(\phi)$$

Combining all of this, the total kinetic energy is

$$T = m\left(a^2\dot{\theta}^2 + \frac{l^2\dot{\phi}^2}{4} + \frac{l^2\dot{\sigma}^2}{4}\cos^2(\phi)\right)$$

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1.17 Conservation of Momentum

A nucleus, originally at rest, decays radioactively by emitting an electron of momentum 1.73MeV/c, and at right angles to the direction of the electron a neutrino with momentum 1.00MeV/c. (The MeV, million electron volt, is a unit of energy used in modern physics, equal to 1.60$\times 10^{-13}$J. Correspondingly, MeV/c is a unit of linear momentum equal to $5.34 \times 10^{-22} kg \cdot m/s$.) In what direction does the nucleus recoil? What is its momentum in MeV/c? If the mass of the residual nucleus is $3.90 \times 10^{-25} kg$ what is its kinetic energy, in electron volts?

We pretend like these are classical objects, ignoring relativistic and quantum effects. We can convince ourselves that the momentum of the neutron is given by $(-p_\nu, -p_e)$. From this, we can see that the direction is given by

$$\theta = \tan^{-1}\left(\frac{p_e}{p_\nu}\right) = 59.97^{\text{deg}}$$

To find the magnitude of the momentum, we use Pythagorean theorem,

$$p_n^2 = p_\nu^2 + p_e^2 = 2MeV/c$$

The kinetic energy is given by, assuming a neutron mass of $3.90 \times 10^{-25} kg$,

$$T = \frac{p_n^2}{m_n} = 1.8 \times 10^{-5} MeV$$

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1.24 Atwood Machine

Two masses $2kg$ and $3kg$, respectively, are tied to the two ends of a massless, inextensible string passing over a smooth pulley. When the system is released, calculated the acceleration of the masses and the tension in the string.

Atwood's machine is a problem that we cannot solve using Lagrangian formalism. By drawing force diagrams, we see that the force equations are

$$\begin{cases} -m_1 g + T = m_1 a \\ -m_2 g + T = -m_2 a \end{cases}$$

where I've defined $m_1 = 2kg$ and $m_2 = 3kg$. Solving, we find

$$a = \frac{m_2 g - m_1 g}{m_1 + m_2}$$

$$T = \frac{2 m_1 m_2 g}{m_1 + m_2}$$