Solutions to Classical Mechanics by H. Goldstein, C. Poole, and J. Safko, 3ed.

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Contents

Chapter 1

Survey of the Elementary Principles

1.1 Kinetic Energy and Force

Show that for a single particle with constant mass the equation of motion implies the following differential equation for the kinetic energy:

$$\frac{dT}{dt} = \vec{F} \cdot \vec{v},$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \vec{F} \cdot \vec{p}$$

The kinetic energy (1.14) is given by

$$T = \frac{m}{2} \vec{v} \cdot \vec{v} \tag{1}$$

Taking the time derivative,

$$\frac{dT}{dt} = \frac{m}{2} 2\dot{\vec{v}} \cdot \vec{v} = m\vec{a} \cdot \vec{v} \tag{2}$$

which we recognize from (1.5),

$$= \vec{F} \cdot \vec{v} \tag{3}$$

For a time-dependent mass,

$$\frac{d(mT)}{dt} = \frac{d}{dt}\left[\frac{1}{2}(m\vec{v})^2\right] = \frac{d(m\vec{v})}{dt} \cdot m\vec{v} \tag{4}$$

From (1.4) and (1.2),

$$= \vec{F} \cdot \vec{p} \tag{5}$$

1.2 Center of Mass

Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2$$

The vector pointing to the center of mass, (1.21),

$$M\vec{R} = \sum m_i \vec{r}_i$$

Squaring this,

$$M^2R^2 = \sum_{ij} m_i m_j \vec{r}_i \cdot \vec{r}_j \tag{1}$$

If we want to write this in non-vector notation,

$$r_{ij}^2 = (\vec{r}_i - \vec{r}_j)^2 \tag{2}$$

$$= r_i^2 + r_j^2 - 2\vec{r}_i \cdot \vec{r}_j \tag{3}$$

$$2\vec{r}_i \cdot \vec{r}_j = r_i^2 + r_j^2 - r_{ij}^2 \tag{4}$$

Inserting Equation 4 into Equation 1,

$$M^2R^2 = \frac{1}{2}\sum_{ij} m_i m_j (r_i^2 + r_j^2 - r_{ij}^2) \tag{5}$$

Since i and j are summing over the same range, the first two terms can combine. Note that we ignore repeating indices in the last term since r_{ij} is the distance between $\vec{r}_i$ and $\vec{r}_j$.

$$= M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2 \tag{6}$$

1.3 Internal Forces

Suppose a system of two particles is known to obey the equations of motion, Eqs. (1.22) and (1.26). From the equations of the motion of the individual particles show that the internal forces between particles satisfy both the weak and the strong laws of action and reaction. The argument may be generalized to a system with arbitrary number of particles, thus proving the converse of the arguments leading to Eqs. (1.22) and (1.26).

(1.22),

$$M\frac{d^2\vec{R}}{dt^2} = \vec{F}^{(e)}$$

(1.26),

$$\frac{d\vec{L}}{dt} = \vec{N}^{(e)}$$

Say the two particles are at rest, i.e., no external forces acting on them,

$$M \cdot 0 = m_1 a_1 + m_2 a_2 \tag{1}$$

This implies either,

$$\begin{cases} m_1 a_1 = -m_2 a_2 \\ \\ \vec{F}_1^{(e)} = -\vec{F}_2^{(e)} \end{cases} \tag{2}$$

1.4 Holonomic Constraints

The equations of constraint for the rolling disk, Eqs. (1.39), are special cases of general linear differential equations of constraint of the form

$$\sum_{i=1}^{n} g_i(x_1, ..., x_n)dx_i = 0$$

A constraint condition of this type is holonomic only if an integrating function $f(x_1, ..., x_n)$ can be found that turns it into an exact differential. Clearly the function must be such that

$$\frac{\partial(fg_i)}{\partial x_j} = \frac{\partial(fg_j)}{\partial x_i}$$

for all $i \neq j$. Show that no such integrating factor can be found for either of Eqs. (1.39).

(1.39),

$$\begin{cases} dx - a\sin(\theta)d\phi = 0 \\ dy + a\cos(\theta)d\phi = 0 \end{cases}$$

Let's look at the first equation and substitute it into the condition with $j = \phi$ and $i = x$,

$$\frac{\partial(f \cdot 1)}{\partial \phi} = \frac{\partial(f \cdot (-a\sin(\theta)))}{\partial x} \tag{1}$$

We can use separation of variables,

$$f = X(x)Q(\phi) \tag{2}$$

$$\frac{\partial f}{\partial \phi} = Q'(\phi)X(x) \tag{3}$$

$$\frac{\partial(-af\sin(\theta))}{\partial y} = -a\sin(\theta)X'Q \tag{4}$$

Matching solutions, setting Equation 3 equal to Equation 4,

$$Q'X = -a\sin(\theta)X'Q \tag{5}$$

We can convince ourselves that we cannot find a Q that depends only on ϕ or an X dependent only on x. We can also set $i = \theta$ and realize that there are no terms with $d\theta$,

$$\frac{\partial(-af\sin(\theta))}{\partial \theta} = \frac{\partial 0}{\partial \phi} \tag{6}$$

The only solution is the trivial solution. We can show the same for the second equation, replacing x with y.

1.5 Connected Wheels

Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a plane. Show that there are two nonholonomic equations of constraint,

$$\cos(\theta)dx + \sin(\theta)dy = 0$$

$$\sin(\theta)dx - \cos(\theta)dy = \frac{1}{2}a(d\phi + d\phi')$$

(where θ, ϕ, and ϕ' have meanings similar to those in the problem of a single vertical disk, and (x, y) are the coordinates of a point on the axle midway between the two wheels) and one holonomic equations of constraint,

$$\theta = C - \frac{a}{b}(\phi - \phi')$$

where C is a constant.

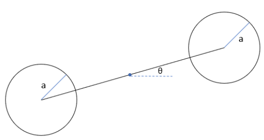

Figure 1.1:

In Figure 1.1, we label the left system 1 and the right system 2. If the wheels rotate individually, we can write how it moves the system,

$$\begin{cases} v_1 = ad\phi_1 \\ dx_1 = v_1 \sin(\theta) \\ dy_1 = -v_1 \cos(\theta) \end{cases} \tag{1}$$

$$\begin{cases} v_2 = ad\phi_2 \\ dx_2 = v_2 \sin(\theta) \\ dy_2 = -v_2 \cos(\theta) \end{cases} \tag{2}$$

Adding these together,

$$dx = \frac{1}{2}(dx_1 + dx_2) = \frac{a}{2}\sin(\theta)(d\phi_1 + d\phi_2) \tag{3}$$

$$dy = \frac{1}{2}(dy_1 + dy_2) = -\frac{a}{2}\cos(\theta)(d\phi_1 + d\phi_2) \tag{4}$$

We can show the first constraint by brute force,

$$\cos(\theta)dx + \sin(\theta)dy = v\cos(\theta)\sin(\theta) - v\sin(\theta)\cos(\theta) = 0 \tag{5}$$

And the same for the second constraint,

$$\sin(\theta)dx - \cos(\theta)dy = v\sin^2(\theta) + v\cos^2(\theta) = v \tag{6}$$

$$= \frac{1}{2}(v_1 + v_2) = \frac{a}{2}(d\phi_1 + d\phi_2) \tag{7}$$

We can write the positions of each wheel,

$$\begin{cases} x_1 = -\dfrac{b}{2}\cos(\theta) \\ \\ y_1 = -\dfrac{b}{2}\sin(\theta) \end{cases} \tag{8}$$

$$\begin{cases} x_2 = \dfrac{b}{2}\cos(\theta) \\ \\ y_2 = \dfrac{b}{2}\sin(\theta) \end{cases} \tag{9}$$

The distance between them $\vec{r}_{12}$,

$$\tan(\theta) = \frac{y_{12}}{x_{12}} \tag{10}$$

$$\sec^2(\theta)d\theta = -\frac{y_{12}}{x_{12}^2}dx_{12} + \frac{1}{x_{12}}dy_{12} \tag{11}$$

$$= a(d\phi_1 - d\phi_2)\left(-\frac{y_{12}}{x_{12}^2}\sin(\theta) - \frac{1}{x_{12}}\cos(\theta)\right) \tag{12}$$

$$= -a(d\phi_1 - d\phi_2)\frac{1}{x_{12}}(\tan(\theta)\sin(\theta) + \cos(\theta)) \tag{13}$$

$$d\theta = -a(d\phi_1 - d\phi_2)\frac{1}{x_{12}}\cos(\theta) = -a(d\phi_1 - d\phi_2)\frac{1}{x_{12}}\frac{x_{12}}{b} \tag{14}$$

$$= -\frac{a}{b}(d\phi_1 - d\phi_2) \tag{15}$$

From this, we can easily get the holonomic constraint.

1.6 Non-holonomic Constraint

A particle moves in the xy plane under the constraint that its velocity vector is always directed towards a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, the constraint is non-holonomic.

What this means is

$$y dx - [f(t) - x] dy = 0 \tag{1}$$

which since $f(t)$ is non-holonomic, cannot be solved.

1.7 Nielsen Form of the Lagrange Equations

Show that Lagrange's equations in the form of Eqs. (1.53) can also be written as

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j$$

These are sometimes known as the Nielsen form of the Lagrange equations.

Eq. (1.53),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

By definition,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial q}\frac{dq}{dt} + \frac{\partial}{\partial \dot{q}}\frac{d\dot{q}}{dt} \tag{1}$$

Applying this to the (1.53),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}}\right) + \frac{\partial}{\partial q}\left(\frac{\partial T}{\partial \dot{q}}\right)\dot{q} + \frac{\partial}{\partial \dot{q}}\left(\frac{\partial T}{\partial \dot{q}}\right)\ddot{q} \tag{2}$$

We now want to compare this to the Nielsen form, specifically,

$$\dot{T} = \frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial q}\dot{q} + \frac{\partial T}{\partial \dot{q}}\ddot{q} \tag{3}$$

$$\frac{\partial \dot{T}}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\left(\frac{\partial T}{\partial t}\right) + \frac{\partial}{\partial \dot{q}}\left(\frac{\partial T}{\partial q}\dot{q}\right) + \frac{\partial}{\partial \dot{q}}\left(\frac{\partial T}{\partial \dot{q}}\ddot{q}\right) \tag{4}$$

$$= \frac{\partial}{\partial \dot{q}}\left(\frac{\partial T}{\partial t}\right) + \frac{\partial}{\partial \dot{q}}\left(\frac{\partial T}{\partial q}\right)\dot{q} + \frac{\partial T}{\partial q} + \frac{\partial}{\partial \dot{q}}\left(\frac{\partial T}{\partial \dot{q}}\right)\ddot{q} \tag{5}$$

Comparing Equation 2 and Equation 5, we see that they differ by $\frac{\partial T}{\partial q}$ as we expect.

1.8 Lagrange Equation

If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L' = L + \frac{dF(q_1, ..., q_n, t)}{dt}$$

also satisfies Lagrange's equations where F is any arbitrary, but differentiable, function of its arguments.

Eq(1.57) gives us Lagrange's equation. Nothing doing, let's plug it in,

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) - \frac{\partial L'}{\partial q} = 0 \tag{1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} + \frac{\partial^2 F}{\partial \dot{q} \partial t}\right) - \left(\frac{\partial L}{\partial q} + \frac{\partial^2 F}{\partial q \partial t}\right) = 0 \tag{2}$$

Rearranging, we can return the normal Lagrange's equations and some change,

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial q}\right) - \frac{\partial L}{\partial q}\right] + \left[\frac{d}{dt}\left(\frac{\partial^2 F}{\partial \dot{q} \partial t}\right) - \frac{\partial^2 F}{\partial q \partial t}\right] = 0 \tag{3}$$

In the second term, we can show that these are equal by moving around derivatives, leaving us with (1.57).

1.9 Gauge Transformation

The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\vec{A} \to \vec{A} + \nabla\psi(\vec{r}, t)$$

$$\phi \to \phi - \frac{1}{c}\frac{\partial\psi}{\partial t}$$

where ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field? Is the motion affected?

(1.63) gives us the Lagrangian of a particle in an EM field,

$$\mathscr{L} = \frac{1}{2}mv^2 - q\phi + q\vec{A}\cdot\vec{v} \tag{1}$$

Applying the transformation,

$$\mathscr{L}' = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\frac{\partial\psi}{\partial t} + q\vec{A}\cdot\vec{v} + q\nabla\psi\cdot\vec{v} \tag{2}$$

$$= \mathscr{L} + q\nabla\psi\cdot\vec{v} + \frac{q}{c}\frac{\partial\psi}{\partial t} \tag{3}$$

We can convince ourselves that the additional terms are differentiable, which allows us to use the results of problem (1.8) to show that while the Lagrangian changes, the motion does not. This is probably a good thing since being able to change the laws of physics by adding some arbitrary constant would have unfortunate consequences.

1.10 Point Transformation

Let $q_1, ..., q_n$ be a set of independent generalized coordinates for a system of n degrees of freedom, with a Lagrangian $L(q, \dot{q}, t)$. Suppose we transform to another set of independent coordinates $s_1, ..., s_n$ by means of transformation equations

$$q_i = q_i(s_1, ...s_n, t), \quad i = 1, ..., n$$

(Such a transformation is called a point transformation.) Show that if the Lagrangian function is expressed as a function of $s_j, \dot{s}_j$, and t through the equations of transformation, then L satisfies Lagrange's equations with respect to the s coordinates:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_j}\right) - \frac{\partial L}{\partial s_j} = 0$$

In other words, the form of Lagrange's equations is invariant under a point transformation.

We start with the second term in Lagrange's equation, (1.57), and perform the point transformation,

$$\frac{\partial L}{\partial s} = \frac{\partial L}{\partial q} \cdot \frac{\partial q}{\partial s} + \frac{\partial L}{\partial \dot{q}} \cdot \frac{\partial \dot{q}}{\partial s} \tag{1}$$

We can now look at the first term,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = \frac{d}{dt}\left[\frac{\partial L}{\partial q}\frac{\partial q}{\partial \dot{s}} + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \dot{s}}\right] \tag{2}$$

The first term dies since q does not depend on $\dot{s}$,

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\frac{\partial q}{\partial s}\right] \tag{3}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\frac{\partial q}{\partial s} + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\left(\frac{\partial q}{\partial s}\right) \tag{4}$$

Lagrange's equation tell us $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$,

$$= \frac{\partial L}{\partial q}\frac{\partial q}{\partial s} + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial s} \tag{5}$$

Comparing Equation 1 and Equation 5, we see that Lagrange's equation holds true even after a point transformation.

1.11 Conservative Forces

Check whether the force $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ is conservative or not.

From (1.16), a force is conservative if we can find some potential V,

$$\vec{F} = -\nabla V(\vec{r})$$

This means we must find some V which satisfies,

$$\frac{\partial V}{\partial x} = -yz \tag{1}$$

$$\frac{\partial V}{\partial y} = -zx \tag{2}$$

$$\frac{\partial V}{\partial z} = -xy \tag{3}$$

We can convince ourselves there exists a solution,

$$V(\vec{r}) = -xyz \tag{4}$$

which means $\vec{F}$ is conservative.

1.12 Satellite Motion

Compute the orbital period and orbital angular velocity of a satellite revolving around the Earth at an altitude of 720km. [Given: radius of Earth $R = 6000km$ **and** $g = 9.83m/s^2$**.]**

From undergraduate mechanics,

$$F = \frac{mv^2}{r} = \frac{m\omega^2 r^2}{r} \tag{1}$$

$$mg = m\omega^2 r \tag{2}$$

$$\omega^2 = \frac{g}{r} = \frac{9.83m/s^2}{6.72 \times 10^6 m} \tag{3}$$

The angular velocity,

$$\omega = 1.2 \times 10^{-3} s^{-1} \tag{4}$$

From this we can find the orbital period,

$$T = \frac{2\pi}{\omega} = 5.2 \times 10^3 s \tag{5}$$

Or about an hour and a half. By comparison, the ISS at ∼400km has an orbital period of 92minutes.

1.13 Escape Velocity

Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric friction, is

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg,$$

where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m, assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to $2.1km/s$ and a mass loss per second equal to 1/60th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300!

From the statement of the problem, we have the following momentum relation

$$mdv = -v'\ dm - mg\ dt \tag{1}$$

We can easily rearrange this to get the desired result,

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg \tag{2}$$

For the second part, we integrate Equation 1,

$$v_f - v_0 = -v' \ln(m)|_{m_0}^{m_f} - gt \tag{3}$$

Looking this up, escape velocity of earth is $11.2km/s$. Here, $m_f = m_e$ and $m_0 = m_e + m_g$ where m_e is the mass of the empty rocket and m_g is the mass of the fuel. Solving for this quantity,

$$\ln\left(\frac{m_e}{m_e + m_g}\right) = \frac{-9.8 \cdot 60 - 11.2 \times 10^3}{2.1 \times 10^3} \tag{4}$$

$$\frac{m_e}{m_e + m_g} = 3.6 \times 10^{-3} \tag{5}$$

$$\frac{m_e}{m_g} = 276 \tag{6}$$

1.14 Generalized Coordinates

Two points of mass m are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius a. Express the kinetic energy in generalized coordinates.

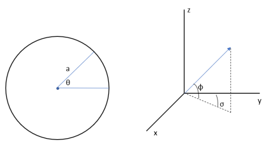

Figure 1.1:

We draw the system in Figure 1.1 with the circle in the $xy-$plane. From this, we can write the position vectors for both masses:

$$\begin{cases} \vec{x}_1 = (a\cos(\theta) + {}^l/_2\cos(\phi)\sin(\sigma), a\sin(\theta) + {}^l/_2\cos(\phi)\cos(\sigma), {}^l/_2\sin(\sigma)) \\ \vec{x}_2 = (a\cos(\theta) - {}^l/_2\cos(\phi)\sin(\sigma), a\sin(\theta) - {}^l/_2\cos(\phi)\cos(\sigma), -{}^l/_2\sin(\sigma)) \end{cases} \tag{1}$$

Taking the derivative and finding the magnitude of their velocities,

$$\begin{cases} \dot{x}_1^2 = a^2\dot{\theta}^2 + \dfrac{l^2\dot{\phi}^2}{4} + \dfrac{l^2\dot{\sigma}^2}{4}\cos^2(\phi) \\ \\ \dot{x}_2^2 = a^2\dot{\theta}^2 + \dfrac{l^2\dot{\phi}^2}{4} + \dfrac{l^2\dot{\sigma}^2}{4}\cos^2(\phi) \end{cases} \tag{2}$$

The kinetic energy,

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) = m\left(a^2\dot{\theta}^2 + \frac{l^2\dot{\phi}^2}{4} + \frac{l^2\dot{\sigma}^2}{4}\cos^2(\phi)\right) \tag{3}$$

1.15 Generalized Potential

A point particle moves in space under the influence of a force derivable from a generalized potential of the form

$$U(\vec{r}, \vec{v}) = V(r) + \vec{\sigma} \cdot \vec{L}$$

where $\vec{r}$ is the radius vector from a fixed point, $\vec{L}$ is the angular momentum about that point, and $\vec{\sigma}$ is a fixed vector in space.

1.15.a Find the components of the force on the particle in both Cartesian and spherical polar coordinates, on the basis of Eq. (1.58)

We'll start with Cartesian coordinates. Using (1.7), we rewrite the generalized potential in terms of $\vec{r}$,

$$U(\vec{r}, \vec{v}) = V(\vec{r}) + \vec{\sigma} \cdot (\vec{r} \times m\vec{v}) \tag{1}$$

$$= V(\vec{r}) + m(\sigma_x(y\dot{z} - \dot{y}z) - \sigma_y(x\dot{z} - \dot{x}z) + \sigma_z(x\dot{y} - \dot{x}y)) \tag{2}$$

Eq. (1.58),

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

Before plugging in, we want to take a look at $V(\vec{r})$ right quick.

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r} \cdot \frac{\partial r}{\partial x} = V'(\vec{r})\frac{\partial r}{\partial x} \tag{3}$$

Note the difference between vector $\vec{r}$ and scalar r,

$$r = (x^2 + y^2 + z^2)^{1/2} \tag{4}$$

Equation 3 thus becomes,

$$\frac{\partial V}{\partial x} = V'(\vec{r})\frac{x}{r} \tag{5}$$

Back to generalized force components,

$$Q_x = -\left(\frac{\partial V}{\partial x} + m(-\sigma_y\dot{z} + \sigma_z\dot{y})\right) + \frac{d}{dt}(m(\sigma_y z - \sigma_z y)) \tag{6}$$

$$= -V'(\vec{r})\frac{x}{r} + 2m(\sigma_y\dot{z} - \dot{y}\sigma_z) \tag{7}$$

In a similar manner,

$$Q_y = -V'(\vec{r})\frac{y}{r} + 2m(\dot{x}\sigma_z - \sigma_x\dot{z}) \tag{8}$$

$$Q_z = -V'(\vec{r})\frac{z}{r} + 2m(\sigma_x\dot{y} - \dot{x}\sigma_y) \tag{9}$$

For spherical coordinates, we want to use a little trick. Since we are free to move out coordinate axis freely, we can set $\vec{r}$ and $\vec{\sigma}$ to whatever we want. Let's choose $\vec{r}$ as the origin and $\vec{\sigma}$ pointing in the z-axis. Alternatively, you can do what I did originally and solve for the general case, spend several hours and dozens of pages doing algebra, and get it all wrong in the end (or hopefully, get it all right in the end) because I'm a bit of an idiot.

Choosing these particular vectors, we can start with Equation 2. We pick out σ_z,

$$U = V(\vec{r}) + m(x\dot{y} - \dot{x}y)\sigma \tag{10}$$

Converting to spherical coordinates,

$$= V(\vec{r}) + m\sigma r^2\dot{\theta}\sin^2(\phi) \tag{11}$$

The generalized force components (1.58),

$$Q_r = -V'(\vec{r}) - 2mr\sigma\dot{\theta}\sin^2(\theta) \tag{12}$$

$$Q_\phi = -2m\sigma r^2\dot{\theta}\cos(\phi)\sin(\phi) \tag{13}$$

$$Q_\theta = 2m\sigma r\dot{r}\sin^2(\phi) + 2m\sigma r^2\dot{\phi}\sin(\phi)\cos(\phi) \tag{14}$$

1.15.b Show that the components in the two coordinate systems are related to each other as in Eq. (1.49).

Eq. (1.49),

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

Let's start with Cartesian coordinates and try to recollect the spherical coordinates. We can write $\vec{r}$ and $\vec{\sigma}$,

$$\begin{cases} \vec{r} = (r\sin(\phi)\cos(\theta), r\sin(\phi)\sin(\theta), r\cos(\phi)) \\ \vec{\sigma} = (0, 0, \sigma) \end{cases} \tag{15}$$

Using (1.49),

$$Q_r = Q_x(\sin(\phi)\cos(\theta)) + Q_y(\sin(\phi)\sin(\theta)) + Q_z\cos(\phi) \tag{16}$$

$$Q_\phi = Q_x(r\cos(\phi)\cos(\theta)) + Q_y(r\cos(\phi)\sin(\theta)) + Q_z(-r\sin(\phi)) \tag{17}$$

$$Q_\theta = Q_x(-r\sin(\phi)\sin(\theta)) + Q_y(r\sin(\phi)\cos(\theta)) \tag{18}$$

We can insert Equation 7, Equation 8, Equation 9 and show that these match up with Equation 12, Equation 13, Equation 14.

1.15.c Obtain the equations of motion in spherical polar coordinates.

We can write the Lagrangian,

$$\mathscr{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{(\phi)}^2 + r^2\dot{\theta}^2\sin^2(\phi)) - U \tag{19}$$

The equation of motion,

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathscr{L}}{\partial q_j} = 0 \tag{20}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) + \frac{\partial U}{\partial q_j} = 0 \tag{21}$$

Using (1.58),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0 \tag{22}$$

From this, the equations of motion,

$$\begin{cases} m\ddot{r} = mr\dot{\phi}^2 + mr\dot{\theta}^2\sin^2(\phi) + Q_r \\ mr^2\ddot{\phi} = mr^2\dot{\theta}^2\sin(\phi)\cos(\phi) - 2mr\dot{r}\dot{\phi} + Q_\phi \\ mr^2\ddot{\theta}\sin^2(\phi) = -2mr\dot{r}\dot{\theta}\sin^2(\phi) - 2mr^2\dot{\theta}\dot{\phi}\sin(\phi)\cos(\phi) + Q_\phi \end{cases} \tag{23}$$

1.16 Weber Electrodymaics

A particle moves in a plane under the influence of a force, acting toward a center of force, whose magnitude is

$$F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2}\right)$$

where r is the distance of the particle to the center of force. Find the generalized potential that will result in such a force, and from that the Lagrangian for the motion in a plane. (The expression for F represents the force between two charges in Weber's electrodynamics.)

(1.58) gives us a relation to start with,

$$F = \frac{1}{r^2} - \frac{\dot{r}^2}{r^2c^2} + \frac{2\ddot{r}}{rc^2} = -\frac{\partial U}{\partial r} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right) \tag{1}$$

Since the generalized potential does not depend on time, we can rewrite the right side,

$$= -\frac{\partial U}{\partial r} + \frac{\partial}{\partial r}\left(\frac{\partial U}{\partial \dot{r}}\right)\dot{r} + \frac{\partial}{\partial \dot{r}}\left(\frac{\partial U}{\partial \dot{r}}\right)\ddot{r} \tag{2}$$

Let's start by matching solutions,

$$\frac{\partial}{\partial \dot{r}}\left(\frac{\partial U}{\partial \dot{r}}\right)\ddot{r} = \frac{2\ddot{r}}{rc^2} \tag{3}$$

$$\frac{\partial U}{\partial \dot{r}} = \frac{2\dot{r}}{rc^2} + \alpha \tag{4}$$

$$U = \frac{\dot{r}^2}{rc^2} + \alpha\dot{r} + \beta \tag{5}$$

Using this, we can look at the second and first terms,

$$\frac{\partial}{\partial r}\left(\frac{\partial U}{\partial \dot{r}}\right)\dot{r} = -\frac{2\dot{r}^2}{r^2c^2} + \alpha'\dot{r} \tag{6}$$

$$\frac{\partial U}{\partial r} = -\frac{\dot{r}^2}{r^2c^2} + \alpha'\dot{r} + \beta' \tag{7}$$

Substituting these in and comparing,

$$F = -\frac{\dot{r}^2}{r^2c^2} - \beta' + \frac{2\ddot{r}}{r^2c^2} \tag{8}$$

This tells us,

$$\beta' = -\frac{1}{r^2} \tag{9}$$

$$\beta = \frac{1}{r} \tag{10}$$

Since α doesn't show up, we are free to set it to whatever we want. $\alpha = 0$. Going back to Equation 5,

$$U = \frac{\dot{r}^2}{rc^2} + \frac{1}{r} \tag{11}$$

From this, we can show the Lagrangian,

$$\mathscr{L} = T - U = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{\dot{r}^2}{r^2c^2} - \frac{1}{r} \tag{12}$$

1.17 Radiation

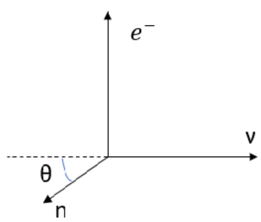

Figure 1.1:

A nucleus, originally at rest, decays radioactively by emitting an electron of momentum $1.73MeV/c$, and at right angles to the direction of the electron a neutrino with momentum $1.00Mev/c$. (The MeV, million electron volt, is a unit of energy used in modern physics, equal to $1.60 \times 10^{-13}J$. Correspondingly, MeV/c is a unit of linear momentum equal to $5.34 \times 10^{-22}kg \cdot m/s$.) In what direction does the nucleus recoil? What is its momentum in MeV/c? If the mass of the residual nucleus is $3.90 \times 10^{-25}kg$ what is its kinetic energy, in electron volts?

Referring to Figure 1.1, we work in two-dimensions. By conservation of momentum, the momentum vector of the neutron is,

$$p_n = (-p_\nu, -p_e) \tag{1}$$

Thus, the direction angle,

$$\theta = \tan^{-1}\left(\frac{p_e}{p_\nu}\right) = \tan^{-1}(1.73) = 59.97^{\circ} \tag{2}$$

To find the magnitude,

$$p_n^2 = p_\nu^2 + p_e^2 \tag{3}$$

$$p_n = 2.00MeV/c \tag{4}$$

Finally, the kinetic energy,

$$T_n = \frac{p_n^2}{m_n} = 1.8 \times 10^{-5}MeV \tag{5}$$

1.18 Two-dimensional Harmonic Motion Lagrangian

A Lagrangian for a particular physical system can be written as

$$L' = \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{K}{2}(ax^2 + 2bxy + cy^2)$$

where a, b, and c are arbitrary constants but subject to the condition that $b^2 - ac \neq 0$. What are the equations of motion? Examine particularly the two cases $a = 0 = c$ and $b = 0$, $c = -a$. What is the physical system described by the above Lagrangian? Show that the usual Lagrangian for this system as defined by Eq.(1.56) is related to L' by a point transformation (cf. Derivation 10). What is the significance of the condition on the value of $b^2 - ac$?

Following (1.57), we can show the equations of motion are,

$$\begin{cases} ma\ddot{x} + mb\ddot{y} = -(Kax + Kby) \\ mb\ddot{x} + mc\ddot{y} = -(Kbx + Kcy) \end{cases} \tag{1}$$

This is the harmonic motion in two-dimensions. For the special case, $a = c = 0$, our equations of motion simplify to,

$$\begin{cases} mb\ddot{y} = -Kby \\ mb\ddot{x} = -Kbx \end{cases} \tag{2}$$

$$\begin{cases} \ddot{y} = -{}^{K}\!/_{m}y \\ \ddot{x} = -{}^{K}\!/_{m}x \end{cases} \tag{3}$$

which we recognize as harmonic motion and can be solved as such. We can get the same thing if $b = 0$, $c = -a$. We note that $b^2 = ac$, we are able to move between the two equations of motion.

1.19 Spherical Pendulum

Obtain the Lagrange equations of motion for a spherical pendulum, i.e., a mass point suspended by a rigid weightless rod.

We can write the Lagrangian of this system,

$$\mathscr{L} = \frac{m}{2}(l^2\dot{\theta}^2 + l^2\dot{\phi}^2\sin^2(\theta)) + mgl\cos(\theta) \tag{1}$$

Note that in the full Lagrangian, there should be an $\dot{l}$ term. However, because we have a rigid rod, $\dot{l} = 0$. The equations of motion thus,

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathscr{L}}{\partial \theta} = 0 \tag{2}$$

$$\ddot{\theta} = \dot{\phi}^2\sin(\theta)\cos(\theta) - \frac{g}{l}\sin(\theta) \tag{3}$$

and

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathscr{L}}{\partial \phi} = 0 \tag{4}$$

$$\ddot{\phi} = -2\dot{\phi}\dot{\theta}\cot(\theta) \tag{5}$$

1.20 Lagrangian

A particle of mass m moves in one dimension such that it has the Lagrangian

$$\mathscr{L} = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2V(x) - V^2(x)$$

where V is some differential function of x. Find the equation of motion for $x(t)$ and describe the physical nature of the system on the basis of this equation.

The equation of motion,

$$\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{x}}\right) - \frac{\partial\mathscr{L}}{\partial x} = 0 \tag{1}$$

$$\frac{d}{dt}\left(\frac{m^2\dot{x}^3}{3} + 2m\dot{x}V(x)\right) - (m\dot{x}^2V'(x) - 2V(x)\cdot V'(x)) = 0 \tag{2}$$

As an aside,

$$\frac{d}{dt}V(x) = \frac{dV(x)}{dx}\cdot\frac{dx}{dt} = V'(x)\dot{x} \tag{3}$$

Substituting this back in,

$$m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V(x) + m\dot{x}^2V'(x) + 2V(x)\cdot V'(x) = 0 \tag{4}$$

$$m\ddot{x}(m\dot{x}^2 + 2V(x)) + V'(x)(m\dot{x}^2 + 2V(x)) = 0 \tag{5}$$

$$(m\ddot{x} + V'(x))(m\dot{x}^2 + 2V(x)) = 0 \tag{6}$$

What we're left with is,

$$\begin{cases} m\ddot{x} = -V'(x) \\ m\dot{x}^2 + 2V(x) = 0 \end{cases} \tag{7}$$

We note that the first equation is Newton's second law and the second equation is the conservation of energy.

1.21 Lagrangian

Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write the Lagrange equations for the system and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only until m_1 reaches the hole.

We can write the Lagrangian,

$$\mathscr{L} = \frac{m_1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m_2}{2}\dot{y}^2 + m_2 g y \tag{1}$$

However, we want to convert this to the generalized coordinates r and θ. To this, we have a constraint in the length of the string,

$$y + r = l \tag{2}$$

$$\dot{r} = -\dot{y} \tag{3}$$

Inserting this,

$$\mathscr{L} = \frac{m_1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m_2}{2}\dot{r}^2 + m_2 g(l - r) \tag{4}$$

The equation of motion (1.57),

$$\frac{d}{dt}(m_1\dot{r} + m_2\dot{r}) - (m_1 r\dot{\theta}^2 - m_2 g) = 0 \tag{5}$$

$$\ddot{r} = \frac{m_1 r\dot{\theta}^2 - m_2 g}{m_1 + m_2} \tag{6}$$

For θ,

$$\frac{d}{dt}(m_1 r^2\dot{\theta}) = 0 \tag{7}$$

This implies that $m_1 r^2\dot{\theta}$ or angular momentum(1.7) is conserved. Inserting this into Equation 6,

$$\ddot{r} = \frac{\frac{L^2}{m_1 r^3} - m_2 g}{m_1 + m_2} \tag{8}$$

Multiplying by dt,

$$(m_1 + m_2)\ddot{r}\dot{r}dt = \frac{L^2}{m_1 r^3}\dot{r}dt - m_2 g\dot{r}dt \tag{9}$$

We note,

$$\begin{cases} \dot{r}dt = dr \\ \ddot{d}rdt = d\dot{r} \end{cases} \tag{10}$$

This allows us to convert the integrals to a solvable form,

$$(m_1 + m_2)\dot{r}\ d\dot{r} - \left(\frac{L^2}{m_1 r^3} + m_2 g\right) dr = 0 \tag{11}$$

Integrating,

$$\frac{m_1 + m_2}{2}\dot{r}^2 - \frac{L^2}{2m_1 r^2} + m_2 g r = E \tag{12}$$

This is simply conservation of energy.

1.22 Double Pendulum

Obtain the Lagrangian and equations of motion for the double pendulum illustrated in Fig. 1.4, where the lengths of the pendula are l_1 and l_2 with corresponding masses m_1 and m_2.

The meat of this problem is writing out the positions of the masses in generalized coordinates. Past that, it's just algebra. The position of the upper mass,

$$\begin{cases} x_1 = l_1 \sin(\theta_1) \\ y_1 = -l_1 \cos(\theta_1) \end{cases} \tag{1}$$

And the lower mass,

$$\begin{cases} x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2) \\ y_2 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2) \end{cases} \tag{2}$$

From this, the Lagrangian,

$$\mathscr{L} = \frac{1}{2} m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2(\dot{x}_2^2 + \dot{y}_2^2) - m_1 g y - m_2 g y_2 \tag{3}$$

$$= \frac{m_1}{2}(l_1^2\dot{\theta}_1^2) + \frac{m_2}{2}(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)) + m_1 g l_1 \cos(\theta_1) + m_2 g(l_1 \cos(\theta_1) + l_2 \cos(\theta_2)) \tag{4}$$

Using (1.57), our generalized coordinates are θ_1 and θ_2. The equation of motion for θ_1,

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin(\theta_1) = 0 \tag{5}$$

The equations of motion for θ_2,

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0 \tag{6}$$

1.23 Atwood's Machine

Two masses $2kg$ and $3kg$, respectively, are tied to the two ends of a massless, inextensible string passing over a smooth pulley. When the system is released, calculate the acceleration of the masses and the tension in the string.

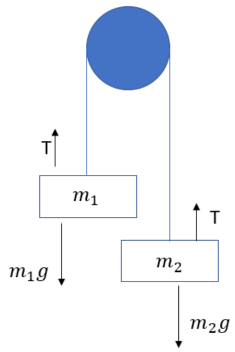

Figure 1.1:

Referring to Section 1.6 and Figure 1.1, we draw the forces existing in the system. From this, we can get two force equations. For this, we assume m_2 is heavier,

$$\begin{cases} -m_1g + T = m_1a \\ -m_2g + T = -m_2a \end{cases} \tag{1}$$

Setting tensions equal to each other,

$$m_1a + m_1g = -m_2a + m_2g \tag{2}$$

$$a = \frac{m_2g - m_1g}{m_1 + m_2} \tag{3}$$

Substituting this in,

$$T = m_1(a + g) = \frac{2m_1m_2g}{m_1 + m_2} \tag{4}$$

If we instead want to use the Lagrangian method, referring to Fig. 1.7 in the text, the Lagrangian is

$$\mathscr{L} = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1gx + m_2g(l - x) \tag{5}$$

(1.57),

$$\frac{d}{dt}((m_1 + m_2)\dot{x}) - (m_1g - m_2g) = 0 \tag{6}$$

$$\ddot{x} = \frac{m_1g - m_2g}{m_1 + m_2} \tag{7}$$

Note that this is the same magnitude but opposite direction of Equation 3, probably because I messed up some directions. In any case, we also want to note that the tension cannot be found using the Lagrangian method.

1.24 Spring Pendulum

A spring of rest length L_a (no tension) is connected to a support at one end and has a mass M attached at the other. Neglect the mass of the spring, the dimension of the mass M, and assume that the motion is confined to a vertical plane. Also, assume that the spring only stretches without bending but it can swing in the plane.

1.24.a Using the angular displacement of the mass from the vertical and the length that the string has stretched from its rest length (hanging with the mass m), find Lagrange's equations.

The position of the mass,

$$\begin{cases} x = l\sin(\theta) \\ y = -l\cos(\theta) \end{cases} \tag{1}$$

The Lagrangian is given by,

$$\mathscr{L} = \frac{1}{2}M(\dot{l}^2 + l^2\dot{\theta}^2) - \frac{1}{2}k(l - L_a)^2 + Mgl\cos(\theta) \tag{2}$$

The equation of motion for generalized coordinate l,

$$\ddot{l} = l\dot{\theta}^2 - \frac{k}{M}(l - L_a) + g\cos(\theta) \tag{3}$$

For θ,

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) - \frac{2\dot{l}\dot{\theta}}{l} \tag{4}$$

1.24.b Solve these equations for small stretching and angular displacements.

Now here, things get a bit iffy (as if they weren't already). I think what this is asking is what happens around $\theta = \theta_0$ and $l = l_0$ where $l_0 = L_a$ and $\theta = 0$. Simplifying,

$$\begin{cases} \sin(\theta) = \theta \\ \cos(\theta) = 1 \\ l - L_a = x \end{cases} \tag{5}$$

Here, the equations of motion simplify to,

$$\ddot{l} = -\frac{k}{M}x + g\ddot{\theta} = -\frac{g}{l}\theta \tag{6}$$

These are the harmonic oscillation equations, so we can pull those solutions. Since l has some extra terms, we do need to offset the solution by a little,

$$\begin{cases} l = A\exp\left(i\sqrt{\frac{k}{M}}t\right) + B\exp\left(-i\sqrt{\frac{k}{M}}t\right) + \left(L_a + \frac{Mg}{k}\right) \\ \\ \theta = A\exp\left(i\sqrt{\frac{g}{l}}t\right) + B\exp\left(-i\sqrt{\frac{g}{l}}t\right) \end{cases} \tag{7}$$

1.24.c Solve the equations in part (a) to the next order in both stretching and angular displacement. This part is amenable to hand calculations. Using some reasonable assumptions about the spring constant, the mass, and the rest length, discuss the motion. Is a resonance likely under the assumptions stated in the problem

We want to make the substitution,

$$\begin{cases} l = l_0 + \epsilon l_1 \\ \theta = \theta_0 + \epsilon\theta_1 \end{cases} \tag{8}$$

to the equations,

$$\begin{cases} \ddot{l} = l\dot{\theta}^2 - \frac{k}{M}(l - L_A) + g\left(1 - \frac{\theta^2}{2}\right) \\ \\ \ddot{\theta} = -\frac{g}{l}\theta - \frac{2\dot{l}\dot{\theta}}{l} \end{cases} \tag{9}$$

For the length,

$$\ddot{l} = (l_0 + \epsilon l_1)(\dot{\theta}_0 + \epsilon\dot{\theta}_1)^2 - \frac{k}{M}(l_0 + \epsilon l_1 - L_a) + g\left(1 - \frac{(\theta_0 + \epsilon\theta_1)^2}{2}\right) \tag{10}$$

Keeping only the terms that go out to ϵ and killing off any terms that were in the previous solution,

$$\epsilon\ddot{l}_1 = \epsilon l_1\dot{\theta}_0^2 + 2\epsilon l_0\dot{\theta}_1 - \epsilon\frac{k}{M}l_1 - \epsilon g\theta_0\theta_1 \tag{11}$$

$$\ddot{l}_1 = l_1\left(\dot{\theta}_0^2 - \frac{k}{M}\right) + 2l_0\dot{\theta}_1 - g\theta_0\theta_1 \tag{12}$$

For θ,

$$\ddot{\theta} = -\frac{g}{l_0 + \epsilon l_1}(\theta_0 + \epsilon\theta_1) - \frac{2}{l_0 + \epsilon l_1}(\dot{l}_0 + \epsilon\dot{l}_1)(\dot{\theta}_0 + \epsilon\dot{\theta}_1) \tag{13}$$

$$\ddot{\theta}_1 = -\frac{g\theta_1 - 2\dot{l}_0\dot{\theta}_1 - 2\dot{l}_1\dot{\theta}_0}{l_0 + \epsilon l_1} \tag{14}$$

We're going to skip parts (d) and (e) since they're computer parts.