"Feynman liked to quote his first wife's advice to stop caring what other people think. That is not a license to be mean to people, but really another way to practice integrity." -Dr. Roxanne Springer

Chapter 2, Quantum dynamics

Section 1. Time evolution and the Schrodinger equation

Subsection Time evolution operator

- $|\alpha, t_0; t\rangle$ denotes a state at some t that started as $|\alpha\rangle$ at time t_0
  - $|\alpha, t_0; t\rangle = \mathcal{U}(t, t_0)|\alpha, t_0\rangle$ time-evolution operator
  - $|\alpha, t_0\rangle = \sum_{a'} c_{a'}(t_0)|a'\rangle$
  - $|\alpha, t_0; t\rangle = \sum_{a'} c_{a'}(t)|a'\rangle$
- $\mathcal{U}^\dagger(t, t_0)\mathcal{U}(t, t_0) = 1$
  - This implies that probability is conserved. As we move forward in time, we don't see things appearing or disappearing from nothing
- $\mathcal{U}(t_2, t_0) = \mathcal{U}(t_2, t_1)\mathcal{U}(t_1, t_0)$
  - Starting at t_0, we get the same result if we go to t_1 then t_2 as if we had gone straight to t_2
- $\mathcal{U}(t_0 + dt, t_0) = 1 - \frac{iHdt}{\hbar}$ infinitesimal time-evolution operator

Subsection The Schrodinger equation

- $i\hbar \frac{\partial}{\partial t}\mathcal{U}(t, t_0) = H\mathcal{U}(t, t_0)$

  $i\hbar \frac{\partial}{\partial t}|\alpha, t_0; t\rangle = H|\alpha, t_0; t\rangle$

Solutions to Schrodinger equation 1. Hamiltonian operator is time-independent

$\mathcal{U}(t, t_0) = \exp\left(\frac{-iH(t-t_0)}{\hbar}\right)$

2. Hamiltonian operator is time-dependent, H at different times commute

$\mathcal{U}(t, t_0) = \exp\left(\frac{-i}{\hbar}\int_{t_0}^{t} dt' H(t')\right)$

3. Hamiltonian operator is time-dependent, H does not commute

- $\mathcal{U}(t, t_0) = 1 + \sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n H(t_1)H(t_2)\cdots H(t_n)$

  Dyson series

  N.B. we are leaving this for chapter 5

Subsection Energy eigenkets

- $H|a'\rangle = E_{a'}|a'\rangle$

  $\exp\left(\frac{-iHt}{\hbar}\right) = \sum_{a'}|a'\rangle \exp\left(\frac{-iE_{a'}t}{\hbar}\right)\langle a'|$
- Going back to the expansion of $|\alpha, t_0; t\rangle$,

  $c_{a'}(t) = c_{a'}(t=0)\exp\left(\frac{-iE_{a'}t}{\hbar}\right)$
- Basic task of quantum dynamics is to find an observable that commutes with H and evaluate the eigenvalues

Subsection Time dependence of expectation values

- $\langle B\rangle = \langle a'|B|a'\rangle$ where $|a', t_0 = 0; t\rangle = \mathcal{U}(t, 0)|a'\rangle$

  independent of t
  - Energy eigenstates are often referred to as stationary states
- Nonstationary state where $|\alpha, t = 0\rangle = \sum_{a'} c_{a'}|a'\rangle$

  $\langle B\rangle = \sum_{a'}\sum_{a''} c_{a'}^{*} c_{a''}\langle a'|B|a''\rangle \exp\left(\frac{-i(E_{a''} - E_{a'})t}{\hbar}\right)$

Subsection Spin precession

- Spin-$\frac{1}{2}$ system with magnetic moment $\frac{e\hbar}{2m_e c}$ subjected to external magnetic field B

  $H = -\left(\frac{e}{m_e c}\right)\vec{S}\cdot\vec{B}$
- If $\vec{B}$ is constant in z-direction

  $H = -\left(\frac{eB}{m_e c}\right)S_z$

  $E_{\pm} = \mp\frac{e\hbar B}{2m_e c}$ are eigenvalues for $|S_z; \pm\rangle$

Subsection Correlation amplitude and the energy-time uncertainty relation

- Given initial state $|\alpha\rangle$ that with time changes into $|\alpha, t_0 = 0; t\rangle$,

  $C(t) = \langle\alpha|\alpha, t_0 = 0; t\rangle$ Correlation amplitude

  $= \langle\alpha|\mathcal{U}(t, 0)|\alpha\rangle$
- Modulus of $C(t)$ returns a measurement of the "resemblance" as the state evolves over time
- As an example, if $|\alpha\rangle$ is an eigenket of H

  $C(t) = \langle a'|\mathcal{U}(t, 0)|a'\rangle = \exp\left(\frac{-iE_{a'}t}{\hbar}\right)$

  $|C(t)| = 1$

  makes sense since stationary state

More generally, $C(t)=\sum_{a'}|c_{a'}|^2 \exp(-iE_{a'}t/\hbar)$

expect correlation amplitude that starts with unity at $t=0$ to decrease in magnitude over time

- If exhibiting a quasi-continuous spectrum

$$\sum_{a'} \to \int dE\,\rho(E)\ ;\ c_{a'} \to g(E)\big|_{E=E_{a'}}$$

$$C(t)=\int dE\,|g(E)|^2\rho(E)\exp(-iEt/\hbar)$$

$$\int dE\,|g(E)|^2\rho(E)=1$$

$$C(t)=\exp(-iE_0t/\hbar)\int dE\,|g(E)|^2\rho(E)\exp(-i(E-E_0)t/\hbar)$$

The characteristic time for which $|C(t)|$ deviates appreciably from 1 is given by

$$t\approx\frac{\hbar}{\Delta E}$$

- $\Delta t\Delta E\simeq\hbar$ time-energy uncertainty relation

Section 2. The Schrodinger versus the Heisenberg Picture

Subsection Unitary Operators

- Schrodinger picture: time-evolution operator affects state kets
- Heisenberg picture: observables vary with time
- Since $\langle\beta|\alpha\rangle\to\langle\beta|\mathcal{U}^\dagger\mathcal{U}|\alpha\rangle=\langle\beta|\alpha\rangle$

$$\langle\beta|X|\alpha\rangle\to(\langle\beta|\mathcal{U}^\dagger)\cdot X\cdot(\mathcal{U}|\alpha\rangle)=\langle\beta|\mathcal{U}^\dagger X\mathcal{U}|\alpha\rangle$$

This gives us two approaches to unitary transformations

- $|\alpha\rangle\to\mathcal{U}|\alpha\rangle$ operator unchanged
- $X\to\mathcal{U}^\dagger X\mathcal{U}$ state kets unchanged

As an example, look at translation

$$|\alpha\rangle\Rightarrow(1-i\vec{p}\cdot d\vec{x}'/\hbar)|\alpha\rangle$$

$$\vec{x}\to\vec{x}$$

$$\langle\vec{x}\rangle=\langle\alpha|\mathcal{U}^\dagger x\,\mathcal{U}|\alpha\rangle=\langle\alpha|(1+i\vec{p}\cdot d\vec{x}'/\hbar)\,x\,(1-i\vec{p}\cdot d\vec{x}'/\hbar)|\alpha\rangle$$

$$=\langle\alpha|x|\alpha\rangle+\langle\alpha|x\,p^2dx'^2/\hbar^2|\alpha\rangle$$

$$=\langle\vec{x}\rangle+\langle d\vec{x}'\rangle$$

$$|\alpha\rangle\Rightarrow|\alpha\rangle$$

$$\vec{x}\to\vec{x}+d\vec{x}'$$

$$\langle\vec{x}\rangle=\langle\alpha|\vec{x}+d\vec{x}'|\alpha\rangle=\langle\alpha|\vec{x}|\alpha\rangle+\langle\alpha|d\vec{x}'|\alpha\rangle$$

$$=\langle\vec{x}\rangle+\langle d\vec{x}'\rangle$$

Subsection State kets and observables in the Schrodinger and Heisenberg pictures

$$\mathcal{U}(t)=\exp(-i\mathcal{H}t/\hbar)$$

$A^{(H)}(t)=\mathcal{U}^\dagger(t)A^{(S)}\mathcal{U}(t)$ Heisenberg picture observable

$$A^{(H)}(0)=A^{(S)}$$

$$|\alpha,t_0=0;t\rangle_H=|\alpha,t_0=0\rangle$$

$$|\alpha,t_0=0;t\rangle_S=\mathcal{U}(t)|\alpha,t_0=0\rangle$$

- Expectation value of A remains the same in both pictures

Subsection The Heisenberg equation of motion

$$\frac{dA^{(H)}}{dt}=\frac{\partial\mathcal{U}^\dagger}{\partial t}A^{(S)}\mathcal{U}+\mathcal{U}^\dagger A^{(S)}\frac{\partial\mathcal{U}}{\partial t}$$

$$=\frac{i\mathcal{H}}{\hbar}\exp(+i\mathcal{H}t/\hbar)A^{(S)}\mathcal{U}+\mathcal{U}^\dagger A^{(S)}\cdot\left(-\frac{i\mathcal{H}}{\hbar}\right)\exp(-i\mathcal{H}t/\hbar)$$

$$=-\frac{1}{i\hbar}\mathcal{U}^\dagger\mathcal{H}A^{(S)}\mathcal{U}+\frac{1}{i\hbar}\mathcal{U}^\dagger A^{(S)}\cdot\mathcal{H}\mathcal{U}$$

$$=-\frac{1}{i\hbar}\mathcal{U}^\dagger\mathcal{H}\mathcal{U}\mathcal{U}^\dagger A^{(S)}\mathcal{U}+\frac{1}{i\hbar}\mathcal{U}^\dagger A^{(S)}\mathcal{U}\mathcal{U}^\dagger\mathcal{H}\mathcal{U}$$

$$=-\frac{1}{i\hbar}\mathcal{U}^\dagger\mathcal{H}\mathcal{U}A^{(H)}+\frac{1}{i\hbar}A^{(H)}\mathcal{U}^\dagger\mathcal{H}\mathcal{U}$$

$$=\frac{1}{i\hbar}[A^{(H)},\mathcal{U}^\dagger\mathcal{H}\mathcal{U}]$$

$$=\frac{1}{i\hbar}[A^{(H)},\mathcal{H}]$$ Heisenberg equation of motion

- $\frac{dA}{dt}=[A,H]_{\text{classical}}$
- $\frac{[\ ,\ ]}{i\hbar}=[\ ,\ ]_{\text{classical}}$

Subsection Free particles; Ehrenfest's theorem

- $[x_i, F(\vec{p})] = i\hbar \frac{\partial F}{\partial p_i}$
  $[p_i, G(\vec{x})] = -i\hbar \frac{\partial G}{\partial x_i}$

- Example: free particle of mass m
  $\mathcal{H} = \frac{\vec{p}^2}{2m} = (p_x^2 + p_y^2 + p_z^2)/2m$
  $\frac{1}{i\hbar}[p_i, \mathcal{H}] = 0 = \frac{dp_i}{dt} \Rightarrow p_i(t) = p_i(0)$
  $\frac{dx_i}{dt} = \frac{1}{i\hbar}[x_i, \mathcal{H}] = \frac{1}{i\hbar} \cdot i\hbar \cdot \frac{1}{2m} \cdot 2p_i$
  $= \frac{p_i}{m} = \frac{p_i(0)}{m}$
  $x_i(t) = x_i(0) + \left(\frac{p_i(0)}{m}\right) t$
  Note that $[x_i(0), x_j(0)] = 0$
  $[x_i(t), x_i(0)] = \left[\frac{p_i(0)t}{m}, x_i(0)\right] = -i\hbar t/m$
  $\langle(\Delta x_i)^2\rangle_t \langle(\Delta x_i)^2\rangle_{t=0} \geq \hbar^2 t^2/4m^2$

- If we then add in some potential $V(\vec{x})$ so $\mathcal{H} = \vec{p}^2/2m + V(\vec{x})$
  $\frac{dp_i}{dt} = \frac{1}{i\hbar}[p_i, \mathcal{H}] = -\frac{\partial}{\partial x_i} V(x)$
  $\frac{dx_i}{dt} = p_i/m$
  $\frac{d^2 x_i}{dt^2} = \frac{1}{i\hbar}\left[\frac{dx_i}{dt}, \mathcal{H}\right] = \frac{1}{i\hbar}[p_i/m, \mathcal{H}] = \frac{1}{m}\left(\frac{1}{i\hbar}[p_i, \mathcal{H}]\right) = \frac{1}{m} \cdot \frac{dp_i}{dt}$
  $m\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla} V(\vec{x})$
  $m\frac{d^2\langle x\rangle}{dt^2} = \frac{d\langle p\rangle}{dt} = -\langle\vec{\nabla} V(\vec{x})\rangle$ Ehrenfest theorem

Subsection Base Kets and Transition Amplitudes

- Start with $A|a'\rangle = a'|a'\rangle$
  In the Schrodinger picture, base ket $|a'\rangle$ does not change since observables are stationary.
- In the Heisenberg picture,
  $\mathcal{U}^\dagger A(0)\mathcal{U}\mathcal{U}^\dagger|a'\rangle = a'\mathcal{U}^\dagger|a'\rangle$
  $A^{(H)}(\mathcal{U}^\dagger|a'\rangle) = a'(\mathcal{U}^\dagger|a'\rangle)$
  $\mathcal{U}^\dagger|a'\rangle$ is used as the base ket
  $|a', t\rangle_H = \mathcal{U}^\dagger|a'\rangle$

$\mathcal{U}^\dagger$ implies the base ket moves backwards in time

- Transition amplitude: probability amplitude for the system to be found in an eigenstate of B with eigenvalue b' given that it started in A with eigenvalue a'
  $\langle b'|\mathcal{U}(t,0)|a'\rangle$

A handy table

	Schrodinger picture	Heisenberg picture
State ket	$\|\alpha, t_0; t\rangle = \mathcal{U}(t,t_0)\|\alpha, t_0\rangle$ $i\hbar\frac{\partial}{\partial t}\|\alpha, t_0; t\rangle = \mathcal{H}\|\alpha, t_0; t\rangle$	Stationary
Observable	Stationary	$A^{(H)}(t) = \mathcal{U}^\dagger(t) A^{(S)} \mathcal{U}(t)$ $\frac{dA^{(H)}}{dt} = \frac{1}{i\hbar}[A^{(H)}, \mathcal{H}]$
Base ket	Stationary	$\|a', t\rangle_H = \mathcal{U}^\dagger\|a'\rangle$ $i\hbar\frac{\partial}{\partial t}\|a', t\rangle_H = -\mathcal{H}\|a', t\rangle_H$

Section 3 Simple Harmonic Oscillator

Subsection Energy Eigenkets and Energy Eigenvalues

- $\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$
  - $\omega = \sqrt{k/m}$ angular frequency
- $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$ annihilation operator
- $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$ creation operator
- $[a, a^\dagger] = \frac{m\omega}{2\hbar}\left[x + \frac{ip}{m\omega}, x - \frac{ip}{m\omega}\right]$
  $= \frac{m\omega}{2\hbar}\left(\cancel{[x,x]} - \frac{i}{m\omega}[x,p] + \frac{i}{m\omega}[p,x] + \cancel{\frac{1}{m^2\omega^2}[p,p]}\right)$
  $= \frac{1}{2\hbar}(-i[x,p] + i[p,x]) = \frac{1}{2\hbar}(\hbar + \hbar) = 1$
- $N = a^\dagger a = \frac{m\omega}{2\hbar}\left(x - \frac{ip}{m\omega}\right)\left(x + \frac{ip}{m\omega}\right) = \frac{m\omega}{2\hbar}\left(x^2 - \frac{ixp}{m\omega} + \frac{ipx}{m\omega} + p^2/m^2\omega^2\right)$
  $= \frac{m\omega}{2\hbar}\left(x^2 + p^2/m^2\omega^2\right) + \frac{i}{2\hbar}[x,p]$
  $= \mathcal{H}/\hbar\omega - 1/2$ Number operator
- $\mathcal{H} = \hbar\omega(N + 1/2)$
- $N|n\rangle = n|n\rangle$
  $\mathcal{H}|n\rangle = (n + 1/2)\hbar\omega|n\rangle$
- $E_n = (n + 1/2)\hbar\omega$ Energy eigenvalues of SHO

Subsection · $[N,a]=[a^{\dagger}a,a]=a^{\dagger}[a,a]+[a^{\dagger},a]a=-a$

· $[N,a^{\dagger}]=[a^{\dagger}a,a^{\dagger}]=a^{\dagger}[a,a^{\dagger}]+[a^{\dagger},a^{\dagger}]a=a^{\dagger}$

$$Na^{\dagger}|n\rangle=([N,a^{\dagger}]+a^{\dagger}N)|n\rangle$$
$$=(a^{\dagger}+a^{\dagger}N)|n\rangle$$
$$=(n+1)a^{\dagger}|n\rangle$$

$$Na|n\rangle=([N,a]+aN)|n\rangle$$
$$=(-a+aN)|n\rangle$$
$$=(n-1)a|n\rangle$$

· $a|n\rangle=\sqrt{n}\,|n-1\rangle$

· $a^{\dagger}|n\rangle=\sqrt{n+1}\,|n+1\rangle$

$n=0$ gives $E_0=\frac{1}{2}\hbar\omega$, the grand state

· $x=\sqrt{\frac{\hbar}{2m\omega}}(a+a^{\dagger})$

· $p=i\sqrt{\frac{m\hbar\omega}{2}}(-a+a^{\dagger})$

· $\langle n'|x|n\rangle=\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n',n-1}+\sqrt{n+1}\,\delta_{n',n+1}\right)$

· $\langle n'|p|n\rangle=i\sqrt{\frac{m\hbar\omega}{2}}\left(-\sqrt{n}\,\delta_{n',n-1}+\sqrt{n+1}\,\delta_{n',n+1}\right)$

· $\langle x'|n\rangle=\left(\frac{1}{\pi^{1/4}\sqrt{2^n n!}}\right)\left(\frac{1}{x_0^{n+1/2}}\right)\left(x'-x_0^2\frac{d}{dx'}\right)^n\exp\left(-\frac{1}{2}(x'/x_0)^2\right)$

$x_0=\sqrt{\frac{\hbar}{m\omega}}$

· $\langle x^2\rangle=x_0^2/2$ $\quad\langle x\rangle=0$

· $\langle p^2\rangle=\hbar m\omega/2$ $\quad\langle p\rangle=0$

· $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle=(n+\frac{1}{2})^2\hbar^2$

Subsection Time Development of the Oscillator

· Work in the Heisenberg picture

$\frac{dp}{dt}=-\frac{\partial}{\partial x_i}V(\vec{x})=-\frac{\partial}{\partial x_i}\left(\frac{m\omega^2x^2}{2}\right)=-m\omega^2x$

$\frac{dx}{dt}=p/m$

$$\frac{da}{dt}=\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{dx}{dt}+\frac{i}{m\omega}\frac{dp}{dt}\right)$$
$$=\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{p}{m}+\frac{i}{m\omega}(-m\omega^2x)\right)$$
$$=\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{p}{m}-i\omega x\right)=-i\omega a$$

$$\frac{da^{\dagger}}{dt}=\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{dx}{dt}-\frac{i}{m\omega}\frac{dp}{dt}\right)$$
$$=\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{p}{m}-\frac{i}{m\omega}(-m\omega^2x)\right)$$
$$=\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{p}{m}+i\omega x\right)=i\omega a^{\dagger}$$

$a(t)=a(0)\exp(-i\omega t)$

$a^{\dagger}(t)=a^{\dagger}(0)\exp(i\omega t)$

$x(t)=x(0)\cos\omega t+\frac{p(0)}{m\omega}\sin\omega t$

$p(t)=-m\omega x(0)\sin\omega t+p(0)\cos\omega t$

$\exp(iG\lambda)A\exp(-iG\lambda)=A+i\lambda[G,A]+\left(\frac{i^n\lambda^n}{n!}\right)[G,[G,\ldots[G,A]]\ldots]$

Baker-Hausdorff lemma

Section 4 Schrodinger's Wave Equation

Subsection Time-dependent wave equation

· $\psi(\vec{x}',t)=\langle\vec{x}'|\alpha,t_0;t\rangle$

$\mathcal{H}=\frac{p^2}{2m}+V(\vec{x})$

where $V(\vec{x})$ is Hermitian and $\langle\vec{x}''|V(\vec{x})|\vec{x}'\rangle=V(\vec{x}')\delta^3(\vec{x}'-\vec{x}'')$

· $i\hbar\frac{\partial}{\partial t}\psi(\vec{x}',t)=\mathcal{H}\psi(\vec{x}',t)$

$$\mathcal{H}\psi(\vec{x}',t)=\langle\vec{x}'|\mathcal{H}|\alpha,t_0;t\rangle$$
$$=\langle\vec{x}'|\frac{p^2}{2m}+V(\vec{x})|\alpha,t_0;t\rangle$$
$$=\langle\vec{x}'|\frac{p^2}{2m}|\alpha,t_0;t\rangle+\langle\vec{x}'|V(\vec{x})|\alpha,t_0;t\rangle$$
$$=\frac{(-i\hbar)^2}{2m}\frac{\partial^2}{\partial x'^2}\langle\vec{x}'|\alpha,t_0;t\rangle+V(\vec{x}')\langle\vec{x}'|\alpha,t_0;t\rangle$$
$$=-\left(\frac{\hbar^2}{2m}\right)\nabla'^2\psi(\vec{x}',t)+V(\vec{x}')\psi(\vec{x}',t)$$

$i\hbar\frac{\partial}{\partial t}\psi(\vec{x}',t)=-\left(\frac{\hbar^2}{2m}\right)\nabla'^2\psi(\vec{x}',t)+V(\vec{x}')\psi(\vec{x}',t)$ wave mechanics

Subsection The time-independent wave equation

· $\langle\vec{x}'|a',t_0;t\rangle=\langle\vec{x}'|a'\rangle\exp(-iE_{a'}t/\hbar)$

Plugging into Schrodinger's equation

$$i\hbar\frac{\partial}{\partial t}\langle\vec{x}'|a'\rangle\exp(-iE_{a'}t/\hbar)=-\left(\frac{\hbar^2}{2m}\right)\nabla'^2\langle\vec{x}'|a'\rangle\exp(-iE_{a'}t/\hbar)+V(\vec{x}')\langle\vec{x}'|a'\rangle\exp(-iE_{a'}t/\hbar)$$

$$i\hbar\langle\vec{x}'|a'\rangle\frac{-iE_{a'}}{\hbar}\exp(-iE_{a'}t/\hbar)=\text{"}\qquad\text{"}$$

$$E_{a'}\langle\vec{x}'|a'\rangle=-\left(\frac{\hbar^2}{2m}\right)\nabla'^2\langle\vec{x}'|a'\rangle+V(\vec{x}')\langle\vec{x}'|a'\rangle$$

$-\left(\frac{\hbar^2}{2m}\right)\nabla'^2u_E(\vec{x}')+V(\vec{x}')u_E(\vec{x}')=Eu_E(\vec{x}')$ time-independent wave equation

Particle must be bound or confined within a finite region of space

Subsection *Interpretations of the Wave Function*

- $\rho(\vec{x}', t) = |\psi(\vec{x}', t)|^2 = |\langle \vec{x}' | \alpha, t_0; t \rangle|^2$ *probability density*
- $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$ Continuity equation

  $\vec{j}(\vec{x}, t) = \frac{\hbar}{m} \operatorname{Im}(\psi^* \vec{\nabla} \psi)$ Probability flux

  $\int \vec{j}(\vec{x}, t)\, d^3x = \frac{\langle p \rangle_t}{m}$ where $\langle p \rangle_t$ is expectation value of momentum operator at time t
- $\psi(\vec{x}, t) = \sqrt{\rho(\vec{x}, t)} \exp\left(\frac{iS(\vec{x}, t)}{\hbar}\right)$

  $\psi^* \vec{\nabla} \psi = \sqrt{\rho(\vec{x}, t)} \exp\left(\frac{-iS(\vec{x}, t)}{\hbar}\right) \left[\vec{\nabla} \sqrt{\rho(\vec{x}, t)} \exp\left(\frac{iS(\vec{x}, t)}{\hbar}\right) + \sqrt{\rho(\vec{x}, t)} \cdot \frac{i}{\hbar} \vec{\nabla} S(\vec{x}, t) \exp\left(\frac{iS(\vec{x}, t)}{\hbar}\right) \right]$

  $= \sqrt{\rho}\, \vec{\nabla}(\sqrt{\rho}) + (i/\hbar) \rho \vec{\nabla} S$

  $\vec{j}(\vec{x}, t) = \frac{\hbar}{m} \cdot \frac{1}{\hbar} \rho \vec{\nabla} S = \frac{\rho \vec{\nabla} S}{m}$

Subsection *The Classical Limit*

- Say $\psi(\vec{x}, t) = \sqrt{\rho(\vec{x}, t)} \exp(iS(\vec{x}, t)/\hbar)$

  $i\hbar \frac{\partial}{\partial t} \psi = -\left(\frac{\hbar^2}{2m}\right) \vec{\nabla}^2 \psi + V\psi$

  $i\hbar \frac{\partial}{\partial t} \psi = i\hbar \left[\frac{\partial \sqrt{\rho}}{\partial t} \exp(iS/\hbar) + \sqrt{\rho} \cdot i/\hbar \frac{\partial S}{\partial t} \exp(iS/\hbar) \right.$

  $\vec{\nabla}^2 \psi$

  $\vec{\nabla} \psi = \vec{\nabla} \sqrt{\rho} \exp(iS/\hbar) + \sqrt{\rho} \cdot \frac{i\vec{\nabla} S}{\hbar} \exp(iS/\hbar)$

  $\vec{\nabla}^2 \psi = \vec{\nabla}^2 \sqrt{\rho} \exp(iS/\hbar) + \vec{\nabla} \sqrt{\rho} \cdot \frac{i\vec{\nabla} S}{\hbar} \exp(iS/\hbar) + \vec{\nabla} \sqrt{\rho} \cdot \frac{i\vec{\nabla} S}{\hbar} \exp(iS/\hbar)$
  $+ \sqrt{\rho} \cdot \frac{i\vec{\nabla}^2 S}{\hbar} \exp(iS/\hbar) + \sqrt{\rho} \left(\frac{i\vec{\nabla} S}{\hbar}\right)^2 \exp(iS/\hbar)$

  $i\hbar \left[\frac{\partial \sqrt{\rho}}{\partial t} + \left(\frac{i}{\hbar}\right) \sqrt{\rho} \frac{\partial S}{\partial t} \right] = -\left(\frac{\hbar^2}{2m}\right) \left[\vec{\nabla}^2 \sqrt{\rho} + \left(\frac{2i}{\hbar}\right) (\vec{\nabla} \sqrt{\rho})(\vec{\nabla} S) + \left(\frac{i}{\hbar}\right) \sqrt{\rho} |\vec{\nabla}^2 S| - \left(\frac{1}{\hbar^2}\right) \sqrt{\rho} |\vec{\nabla} S|^2 \right]$
  $+ \sqrt{\rho} V$

  If it is assumed that $\hbar$ is a small quantity, we can then kill all terms with an $\hbar$

  $-\sqrt{\rho} \cdot \frac{\partial S}{\partial t} = \frac{1}{2m} \sqrt{\rho} |\vec{\nabla} S|^2 + \sqrt{\rho} V$

  $0 = \frac{1}{2m} |\vec{\nabla} S(\vec{x}, t)|^2 + V(\vec{x}) + \frac{\partial S(\vec{x}, t)}{\partial t}$ *Hamilton-Jacobi equation*

  $S(x, t) = W(x) - Et$

  - $W(x)$ is Hamilton's characteristic function
  - Hamilton's principal function is separable

  $\vec{p}_{class.} = \vec{\nabla} S = \vec{\nabla} W$

Subsection *Semiclassical (WKB) Approximation*

- Obtain an approximate stationary-state solution to Schrodinger's wave equation in one dimension
- $S(x, t) = W(x) - Et$

  $= \pm \int^x \sqrt{2m[E - V(x')]}\, dx' - Et$
- If we have a stationary state, $\frac{\partial \rho}{\partial t} = 0$ for all x

  Back to the continuity equation: $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \frac{\vec{\nabla} S}{m}\right) = 0$

  $\frac{\partial \rho}{\partial t} + \frac{1}{m} \frac{\partial}{\partial x}\left(\rho \cdot \frac{\partial S}{\partial x}\right) = 0$

  $\frac{1}{m} \cdot \frac{\partial \rho}{\partial x} \cdot \frac{\partial S}{\partial x} + \frac{1}{m} \rho \frac{\partial^2 S}{\partial x^2} = 0$

  $\rho \frac{dW}{dx} = \text{constant}$

  $\pm \rho \sqrt{2m(E - V(x))} = \text{constant}$

  $\pm \rho = \frac{\text{constant}}{\sqrt{2m(E - V(x))}}$

  $\sqrt{\rho} = \text{constant}/[E - V(x)]^{1/4} \propto 1/\sqrt{v_{cl}}$

  $\psi(\vec{x}, t) = \sqrt{\rho(\vec{x}, t)} \exp[iS(\vec{x}, t)/\hbar]$

  $= \{\text{constant}/[E - V(x)]^{1/4}\} \exp\left[\pm \frac{i}{\hbar} \int^x \sqrt{2m[E - V(x')]}\, dx' - iEt/\hbar\right]$

  WKB solution
- When is this solution viable?
  - The short answer is in the short-wavelength limit

  $\hbar |\vec{\nabla}^2 S| \ll |\vec{\nabla} S|^2$

  $\hbar \left|\frac{d^2 W}{dx^2}\right| \ll \left|\frac{dW}{dx}\right|^2$ in one-dimension

  $\lambda = \frac{2\pi\hbar}{p}$

  $\frac{dW}{dx} = \sqrt{2m[E - V(x)]} = (2m)^{1/2}(E - V(x))^{1/2}$

  $\frac{d^2 W}{dx^2} = (2m)^{1/2} \cdot 1/2 (E - V(x))^{-1/2} \cdot -\frac{dV}{dx} \cdot 2m$

  $\hbar \cdot \left(\frac{dV}{dx}\right) \frac{m}{\sqrt{2m(E - V(x))}} \ll 2m(E - V(x))$

  $\lambda = \frac{\hbar}{\sqrt{2m(E - V(x))}} \ll \frac{2[E - V(x)]}{\left|\frac{dV}{dx}\right|}$
- If instead we work in the classically forbidden region where $E - V(x)$ is negative

  $W(x) = \pm \int^x \sqrt{2m(V(x') - E)}\, dx'$

  $\Rightarrow \psi(x, t) = \{\text{constant}/[V(x) - E]^{1/4}\} \exp\left[\pm \left(\frac{1}{\hbar}\right) \int^x \sqrt{2m[V(x') - E]}\, dx' - iEt/\hbar\right]$

Note that at the classical turning point defined by $V(x) = E$, λ becomes infinite. In order to deal with this,

- Make a linear approximation to $V(x)$ near the turning points
- Solve $\frac{d^2 u_E}{dx^2} - \left(\frac{2m}{\hbar^2}\right)\left(\frac{dV}{dx}\right)_{x=x_0}(x - x_0) u_E = 0$ which gives a Bessel function of order $\pm 1/3$
- Match the solutions by choosing appropriately various constants of integration

Sakurai doesn't go into these steps in detail, and neither will we. What we end up finding is that

$$\int_{x_1}^{x_2} \sqrt{2m(E - V(x))}\, dx = (n + \tfrac{1}{2})\pi\hbar$$

Sakurai then goes on to find the energy spectrum of a ball bouncing on a hard surface.

Section 5. Propagators and Feynman Path Integrals

Subsection Propagators in Wave Mechanics

$$|\alpha, t_0; t\rangle = \exp\left[-iH(t-t_0)/\hbar\right]|\alpha, t_0\rangle$$
$$= \sum_{a'} |a'\rangle\langle a'|\alpha, t_0\rangle \exp\left[-iE_{a'}(t-t_0)/\hbar\right]$$
$$\langle \vec{x}'|\alpha, t_0; t\rangle = \sum_{a'} \langle \vec{x}'|a'\rangle\langle a'|\alpha, t_0\rangle \exp\left[-iE_{a'}(t-t_0)/\hbar\right]$$
$$\psi(\vec{x}', t) = \sum_{a'} c_{a'}(t_0)\, u_{a'}(\vec{x}') \exp\left[-iE_{a'}(t-t_0)/\hbar\right]$$
$$c_{a'}(t_0) = \langle a'|\alpha, t_0\rangle$$
$$u_{a'}(\vec{x}') = \langle \vec{x}'|a'\rangle$$
$$c_{a'}(t_0) = \int d^3x' \langle a'|\vec{x}'\rangle\langle \vec{x}'|\alpha, t_0\rangle$$
$$= \int u^*_{a'}(\vec{x}')\, \psi(\vec{x}', t_0)\, d^3x'$$
$$\psi(\vec{x}'', t) = \sum_{a'} \langle \vec{x}''|a'\rangle\langle a'|\alpha, t_0\rangle \exp\left(-iE_{a'}(t-t_0)/\hbar\right)$$
$$= \sum_{a'} \langle \vec{x}''|a'\rangle \int \langle a'|\vec{x}'\rangle\langle \vec{x}'|\alpha, t_0\rangle\, d^3x' \exp\left(-iE_{a'}(t-t_0)/\hbar\right)$$
$$= \int \sum_{a'} \langle \vec{x}''|a'\rangle\langle a'|\vec{x}'\rangle \exp\left(-iE_{a'}(t-t_0)/\hbar\right)\langle \vec{x}'|\alpha, t_0\rangle\, d^3x'$$
$$= \int K(\vec{x}'', t; \vec{x}', t_0)\, \psi(\vec{x}', t_0)\, d^3x'$$
$$K(\vec{x}'', t; \vec{x}', t_0) = \sum_{a'} \langle \vec{x}''|a'\rangle\langle a'|\vec{x}'\rangle \exp\left[-iE_{a'}(t-t_0)/\hbar\right]$$

$K(\vec{x}'', t; \vec{x}', t_0)$ is known as the propagator. It is dependant solely on the potential and is independent of the initial wave function.

Two properties of the propagator

- $K(\vec{x}'', t; \vec{x}', t_0)$ satisfies Schrodinger's time-dependent wave equation
- $\lim_{t \to t_0} K(\vec{x}'', t; \vec{x}', t_0) = \delta^3(\vec{x}'' - \vec{x}')$

$$K(\vec{x}'', t; \vec{x}', t_0) = \langle \vec{x}''|\exp\left(\frac{-iH(t-t_0)}{\hbar}\right)|\vec{x}'\rangle$$

Propagator is Green's function for time-dependent wave equation

$$\left(-\frac{\hbar^2}{2m}\nabla''^2 + V(\vec{x}'') - i\hbar\frac{\partial}{\partial t}\right)K(\vec{x}'', t; \vec{x}', t_0) = -i\hbar\,\delta^3(\vec{x}'' - \vec{x}')\,\delta(t - t_0)$$

with $K(\vec{x}'', t; \vec{x}', t_0) = 0$ for $t < t_0$

An example: a free particle in one dimension

$$p|p'\rangle = p'|p'\rangle \qquad H|p'\rangle = \left(\frac{p'^2}{2m}\right)|p'\rangle$$
$$\langle \vec{x}'|p'\rangle = \frac{1}{(2\pi\hbar)^{1/2}} \exp\left(\frac{i\vec{p}'\cdot\vec{x}'}{\hbar}\right)$$
$$K(x'', t; x', t_0) = \langle \vec{x}''|\vec{p}''\rangle\langle \vec{p}''|\exp\left(\frac{-iH(t-t_0)}{\hbar}\right)|p'\rangle\langle p'|\vec{x}'\rangle$$
$$= \langle \vec{x}''|\vec{p}''\rangle\langle p'|\vec{x}'\rangle\langle \vec{p}''|\exp\left(\frac{-iH(t-t_0)}{\hbar}\right)|p'\rangle$$
$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \exp\left(\frac{ip'(x''-x')}{\hbar} - \frac{ip'^2(t-t_0)}{2m\hbar}\right)dp'$$
$$= \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}} \exp\left(\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right)$$
$$G(t) = \int K(\vec{x}', t; \vec{x}', 0)\, d^3x' = \int \sum_{a'} |\langle \vec{x}'|a'\rangle|^2 \exp\left(\frac{-iE_{a'}t}{\hbar}\right) d^3x'$$
$$= \sum_{a'} \exp\left(\frac{-iE_{a'}t}{\hbar}\right)$$

if we then define $\beta = it/\hbar$, then we get the partition function

$$Z = \sum_{a'} \exp(-\beta E_{a'})$$

Subsection Propagator as a Transition Amplitude

$$K(\vec{x}'', t; \vec{x}', t_0) = \sum_{a'} \langle \vec{x}''|a'\rangle\langle a'|\vec{x}'\rangle \exp\left(\frac{-iE_{a'}(t-t_0)}{\hbar}\right)$$
$$= \sum_{a'} \langle \vec{x}''|\exp\left(\frac{-iHt}{\hbar}\right)|a'\rangle\langle a'|\exp\left(\frac{iHt_0}{\hbar}\right)|\vec{x}'\rangle$$
$$= \langle \vec{x}'', t|\vec{x}', t_0\rangle \quad \text{Transition amplitude}$$

Amplitude for a particle to go from $(\vec{x}', t_0)$ to $(\vec{x}'', t)$

Note that we can divide a time interval into as many smaller subintervals as desired. If we then make these subintervals infinitesimally small, this gives us the path integral formulation

Subsection Path Integrals as the Sum over Paths

- $t_j - t_{j-1} = \Delta t = \frac{(t_N - t_1)}{(N-1)}$
- $\langle x_N, t_N | x_1, t_1 \rangle = \int \langle x_N, t_N | x_{N-1}, t_{N-1} \rangle \ldots \langle x_2, t_2 | x_1, t_1 \rangle \, dx_2 \ldots dx_{N-1}$
- Must sum over all possible paths
- However from classical mechanics, there is a unique path. Specifically one such that

$$\delta \int_{t_1}^{t_2} \mathcal{L}_{cl}(x, \dot{x}) \, dt = 0$$

Subsection Feynman's Formulation

- $S(n, n-1) = \int_{t_{n-1}}^{t_n} \mathcal{L}_{cl}(x, \dot{x}) \, dt$
- $\langle x_N, t_N | x_1, t_1 \rangle = \int_{x_1}^{x_2} \mathcal{D}[x(t)] \exp\left(i \int_{t_1}^{t_N} \frac{\mathcal{L}_{cl}(x, \dot{x})}{\hbar} dt\right)$

Section 6. Potentials and Gauge Transformations

Subsection Constant Potentials

- Gauge transformation: can easily choose a new zero point of the energy scale at each instant of time, but potential differences are of nontrivial physical significance

Subsection Gravity in Quantum Mechanics

Gravity-induced quantum-interference:

This chapter and the previous are not covered, so I will be leaving them for the future.

Problems 2.

1. $\mathcal{H} = -\left(\frac{eB}{mc}\right) S_z = \omega S_z$

$\frac{dS_x}{dt} = \frac{1}{i\hbar}[S_x, \omega S_z] = \omega/i\hbar \, (-i\hbar S_y) = -\omega S_y$

$\frac{dS_y}{dt} = \frac{1}{i\hbar}[S_y, \omega S_z] = \omega/i\hbar \, (i\hbar S_x) = \omega S_x$

$\frac{dS_z}{dt} = \frac{1}{i\hbar}[S_z, \omega S_z] = 0$

$S_z(t) = S_z(0)$

Let's guess some S_x + S_y

$S_x(t) = a\cos(\omega t) + b\sin(\omega t)$

$\frac{dS_x}{dt} = -a\omega\sin(\omega t) + b\omega\cos(\omega t)$

$S_y(t) = c\cos(\omega t) + d\sin(\omega t)$

$\frac{dS_y}{dt} = -c\omega\sin(\omega t) + d\omega\cos(\omega t)$

$-a\omega\sin(\omega t) + b\omega\cos\omega t = -c\omega\cos\omega t - d\omega\sin\omega t$

$b = -c$

$a = d$

$-c\omega\sin\omega t + d\omega\cos\omega t = a\omega\cos\omega t + b\omega\sin\omega t$

Given this,

$S_x(t) = \cos(\omega t) + \sin(\omega t)$

$S_y(t) = -\cos(\omega t) + \sin(\omega t)$

2. $\mathcal{H} = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}$ $\mathcal{H}$ is no longer Hermitian

$\mathcal{H} = \mathcal{H}_{12} |1\rangle\langle 2|$

$\mathcal{U}(t, t_0) = \exp\left(-i\mathcal{H}(t-t_0)/\hbar\right)$

$= 1 - i\mathcal{H}t/\hbar = 1 - it\mathcal{H}_{12}/\hbar \, |1\rangle\langle 2|$

$|\psi\rangle = \mathcal{U} \cdot 1/\sqrt{2}\,(|1\rangle + |2\rangle)$

$= 1/\sqrt{2}\,(|1\rangle + |2\rangle) - i\mathcal{H}_{12}t/\sqrt{2}\,(\overset{\to 0}{|1\rangle\langle 2|1\rangle} + |1\rangle\langle 2|2\rangle)$

$= 1/\sqrt{2}\,(|1\rangle + |2\rangle) - i\mathcal{H}_{12}t/\sqrt{2}\,|1\rangle$

$\langle\psi|\psi\rangle = 1/\sqrt{2}\left[(1 - i\mathcal{H}_{12}t/\sqrt{2})|1\rangle + |2\rangle\right]$

$$\langle\psi|\psi\rangle = \tfrac{1}{2}\left[\left(1+\frac{i\mathcal{H}_{12}t}{\hbar}\right)\langle 1| + \langle 2|\right]\left[\left(1-\frac{i\mathcal{H}_{12}t}{\hbar}\right)|1\rangle + |2\rangle\right]$$

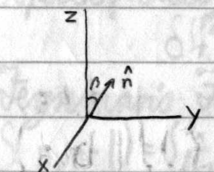
$$= \tfrac{1}{2}\left[\left(1+\frac{\mathcal{H}_{12}^2 t^2}{\hbar^2}\right)\langle 1|1\rangle + \langle 2|2\rangle\right]$$

Note that $\langle 1|2\rangle = \langle 2|1\rangle = 0$

$$\langle\psi|\psi\rangle = \tfrac{1}{2}\left[1+\frac{\mathcal{H}_{12}^2 t^2}{\hbar^2}+1\right] = 1+\frac{\mathcal{H}_{12}^2 t^2}{2\hbar^2}$$

Probability as time-dependent and not conserved

3.



$$\hat{n} = \sin\beta\,\hat{x} + \cos\beta\,\hat{z}$$

$$\vec{S} = \tfrac{\hbar}{2}\vec{\sigma}$$

$$\vec{S}\cdot\hat{n} = \tfrac{\hbar}{2}(\sin\beta\,\sigma_x + \cos\beta\,\sigma_z)$$

$$\vec{S}\cdot\hat{n}|\psi\rangle = \tfrac{\hbar}{2}|\psi\rangle$$

$$\frac{\hbar}{2}\begin{pmatrix}\cos\beta & \sin\beta\\ \sin\beta & -\cos\beta\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}a\\ b\end{pmatrix}$$

$$a\cos\beta + b\sin\beta = a$$

$$a\sin\beta - b\cos\beta = b$$

$$|\psi'\rangle = \begin{pmatrix}1\\ \frac{\sin\beta}{1+\cos\beta}\end{pmatrix}$$

$$|\psi\rangle = \sqrt{\frac{1+\cos\beta}{2}}\begin{pmatrix}1\\ \frac{\sin\beta}{1+\cos\beta}\end{pmatrix}$$

$$\mathcal{H} = -\vec{\mu}_s\cdot\vec{B} = \frac{g_s\mu_B}{2}\cdot\sigma_z B$$

a. $\mathcal{H}|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$

$$-i\omega\begin{pmatrix}A(t)\\ -B(t)\end{pmatrix} = \frac{\partial}{\partial t}\begin{pmatrix}A(t)\\ B(t)\end{pmatrix} \quad ; \omega = \frac{g_s\mu_B B}{2\hbar}$$

$$-i\omega A(t) = \frac{\partial}{\partial t}A(t)$$

$$i\omega B(t) = \frac{\partial}{\partial t}B(t)$$

$$A(t) = A(0)\exp(-i\omega t)$$

$$B(t) = B(0)\exp(i\omega t)$$

$$\psi(t) = \begin{pmatrix}\sqrt{\frac{1+\cos\beta}{2}}\exp(-i\omega t)\\ \frac{\sin\beta}{\sqrt{2(1+\cos\beta)}}\exp(i\omega t)\end{pmatrix}$$

$$|\psi(t)\rangle = \alpha_1|S_x;+\rangle + \alpha_2|S_x;-\rangle$$

$$|S_x;\pm\rangle = \tfrac{1}{\sqrt{2}}|+\rangle \pm \tfrac{1}{\sqrt{2}}|-\rangle$$

$$\alpha_1 = \tfrac{1}{\sqrt{2}}A\exp(-i\omega t) + \tfrac{1}{\sqrt{2}}B\exp(i\omega t)$$

$$\alpha_2 = \tfrac{1}{\sqrt{2}}A\exp(-i\omega t) - \tfrac{1}{\sqrt{2}}B\exp(i\omega t)$$

$$S_x = \tfrac{\hbar}{2}: \alpha_1^*\alpha_1 = \tfrac{1}{2}\left[A^2 + B^2 + 2AB(e^{2i\omega t} + e^{-2i\omega t})\right]$$

$$= \tfrac{1}{2}(1 + \sin\beta\cos 2\omega t)$$

b. $\langle S_x\rangle = \langle\psi(t)|S_x|\psi(t)\rangle = [A^*(t)\;\; B^*(t)]\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{bmatrix}A(t)\\ B(t)\end{bmatrix}$

$$= \tfrac{\hbar}{2}(A^*(t)B(t) + B^*(t)A(t))$$

$$= \tfrac{\hbar}{2}(\sin\beta\cos 2\omega t)$$

c. $\beta\to 0;\ \alpha_1^*\alpha_1 = \tfrac{1}{2}$

$$\langle S_x\rangle = 0$$

$\beta\to\tfrac{\pi}{2}:\ \alpha_1^*\alpha_1 = \tfrac{1}{2}(1+\cos 2\omega t)$

$$\langle S_x\rangle = \tfrac{\hbar}{2}\cos 2\omega t$$

4. $\mathcal{H} = \frac{p^2}{2m}$

$$\frac{dx}{dt} = \frac{1}{i\hbar}[x,\mathcal{H}] = \tfrac{1}{i\hbar}\cdot i\hbar\cdot \tfrac{p}{m} = \tfrac{p}{m}$$

$$\dot{x} = \tfrac{p}{m}$$

$$\frac{dp}{dt} = \frac{1}{i\hbar}[p,\mathcal{H}] = \tfrac{1}{i\hbar}\cdot 0 = 0$$

$$\dot{p} = 0 \Rightarrow p(0) = p(t)$$

$$x(t) = x(0) + \frac{p(0)t}{m}$$

$$[x(t), x(0)] = \cancel{[x(0), x(0)]} + \tfrac{t}{m}[x(0), p(0)]$$

$$= \frac{i\hbar t}{m}$$

5. $[\mathcal{H}, x] = [p^2/2m + V(x), x]$

$= -i\hbar \cdot p/m$

$[[\mathcal{H}, x], x] = [-i\hbar p/m, x] = -i\hbar/m\,[p, x]$

$= -i\hbar/m\,(-i\hbar) = -\hbar^2/m$

$\mathcal{H}|a'\rangle = E_{a'}|a'\rangle$

$\mathcal{H}|a''\rangle = E_{a''}|a''\rangle$

$[[\mathcal{H}, x], x] = [\mathcal{H}x - x\mathcal{H}, x] = \mathcal{H}xx - x\mathcal{H}x - x\mathcal{H}x + xx\mathcal{H}$

$= \mathcal{H}xx - 2x\mathcal{H}x + xx\mathcal{H}$

$\langle a''|[[\mathcal{H}, x], x]|a''\rangle = -\hbar^2/m$

$\langle a''|\mathcal{H}xx|a''\rangle - 2\langle a''|x\mathcal{H}x|a''\rangle + \langle a''|xx\mathcal{H}|a''\rangle = -\hbar^2/m$

$E_{a''}\langle a''|xx|a''\rangle - 2\langle a''|x\mathcal{H}x|a''\rangle + E_{a''}\langle a''|xx|a''\rangle = -\hbar^2/m$

$E_{a''}\langle a''|xx|a''\rangle - \langle a''|x\mathcal{H}x|a''\rangle = -\hbar^2/2m$

$\sum_{a'}(E_{a''}\langle a''|x|a'\rangle\langle a'|x|a''\rangle - \langle a''|x\mathcal{H}|a'\rangle\langle a'|x|a''\rangle) = -\hbar^2/2m$

$\sum_{a'}(E_{a''}|\langle a''|x|a'\rangle|^2 - E_{a'}|\langle a''|x|a'\rangle|^2) = -\hbar^2/2m$

$\sum_{a'}|\langle a''|x|a'\rangle|^2(E_{a'} - E_{a''}) = \hbar^2/2m$

6. $\mathcal{H} = \vec{p}^2/2m + V(\vec{x})$

$[\vec{x}\cdot\vec{p}, \mathcal{H}] = [\vec{x}\cdot\vec{p}, \vec{p}^2/2m + V(\vec{x})]$

$= 1/2m\,[\vec{x}\cdot\vec{p}, \vec{p}^2] + [\vec{x}\cdot\vec{p}, V(\vec{x})]$

$= 1/2m\,\vec{x}[\vec{p}, \vec{p}^2] + 1/2m\,[\vec{x}, \vec{p}^2]\vec{p} + \vec{x}[\vec{p}, V(\vec{x})] + [\vec{x}, V(\vec{x})]\vec{p}$

$= 1/2m\; i\hbar\, 2\vec{p}^2 + \vec{x}(-i\hbar\vec{\nabla}V(\vec{x}))$ (2.2.23)

$\frac{d\langle\vec{x}\cdot\vec{p}\rangle}{dt} = \frac{1}{i\hbar}[\vec{x}\cdot\vec{p}, \mathcal{H}]$ (2.2.21)

$\frac{d\langle\vec{x}\cdot\vec{p}\rangle}{dt} = \left\langle\frac{\vec{p}^2}{m}\right\rangle - \langle\vec{x}\cdot\vec{\nabla}V(\vec{x})\rangle$

Virial theorem $\frac{d\langle\vec{x}\cdot\vec{p}\rangle}{dt} = 0$

$\left\langle\frac{\vec{p}^2}{2m}\right\rangle = \langle\vec{x}\cdot\vec{\nabla}V(\vec{x})\rangle$

or $\langle\vec{x}\cdot\vec{p}\rangle$ is time-independent

7. $\langle\Delta x^2(0)\rangle\langle\Delta p^2(0)\rangle = \hbar^2/4$

$\langle x(0)\rangle = \langle p(0)\rangle = 0$

We know $\langle\Delta x^2(0)\rangle$

$\langle\Delta p^2(0)\rangle = \frac{\hbar^2}{4\langle\Delta x^2(0)\rangle}$

From q1.18b: $\Delta A|\alpha\rangle = \lambda\Delta B|\alpha\rangle$ where λ is purely imaginary

Remember that $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$ (1.4.51)

$\langle\Delta x\rangle = x - \langle x\rangle$ (1.4.50)

From q2.4, $x(t) = x(0) + p(0)t/m$

$\langle x(t)\rangle = \langle x(0)\rangle + t/m\,\langle p(0)\rangle = 0$

$x^2(t) = [x(0) + p(0)t/m]^2$

$= x^2(0) + \frac{t}{m}[x(0)p(0) + p(0)x(0)] + \frac{t^2}{m}p^2(0)$

$\langle x^2(t)\rangle = \langle x^2(0)\rangle + t/m\,\langle x(0)p(0) + p(0)x(0)\rangle + t^2/m\,\langle p^2(0)\rangle$

$\langle x(0)p(0) + p(0)x(0)\rangle = \langle\psi|x(0)p(0)|\psi\rangle + \langle\psi|p(0)x(0)|\psi\rangle$

Forgot to do something that lets us solve the above

$\Delta x(0) = x(0) - \langle x(0)\rangle$

$\Delta x(0) = x(0)$

$\Delta p(0) = p(0)$

$\Delta x(0)|\psi\rangle = \lambda\Delta p(0)|\psi\rangle$

$x(0)|\psi\rangle = \lambda p(0)|\psi\rangle$

$\langle x(0)p(0) + p(0)x(0)\rangle = \langle\psi|x(0)\cdot\frac{1}{\lambda}x(0)|\psi\rangle + \frac{1}{\lambda^*}\langle\psi|x(0)x(0)|\psi\rangle$

$= \frac{1}{\lambda}\langle\psi|x^2(0)|\psi\rangle + \frac{1}{\lambda^*}\langle\psi|x^2(0)|\psi\rangle$

$= \frac{1}{\lambda}\langle\psi|x^2(0)|\psi\rangle - \frac{1}{\lambda}\langle\psi|x^2(0)|\psi\rangle = 0$

Remember, we can do this because λ is purely imaginary

$\langle x^2(t)\rangle = \langle x^2(0)\rangle + t^2/m\,\langle p^2(0)\rangle$

$= \langle\Delta x^2(0)\rangle + t^2/m\,\langle\Delta p^2(0)\rangle$

$\langle x^2(t)\rangle = \langle\Delta x^2(0)\rangle + t^2/m\cdot\frac{\hbar^2}{4\langle\Delta x^2(0)\rangle}$

8.

a. Let's declare $|a'\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|a''\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$\mathcal{H} = \begin{pmatrix}0 & \delta\\ \delta & 0\end{pmatrix}$$

$$\det\begin{pmatrix}-\lambda & \delta\\ \delta & -\lambda\end{pmatrix} = \lambda^2 - \delta^2$$

$\lambda = \pm\delta$ are the energy eigenvalues

$$|\delta\rangle : \begin{pmatrix}-\delta & \delta\\ \delta & -\delta\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$|\delta\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$ ← corresponding eigenstates

$$|-\delta\rangle : \begin{pmatrix}\delta & \delta\\ \delta & \delta\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$|-\delta\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

b. From the Schrodinger equation:

$$i\hbar\tfrac{\partial}{\partial t}|\psi\rangle = \mathcal{H}|\psi\rangle$$

$$|\psi(t)\rangle = \begin{pmatrix}A(t)\\B(t)\end{pmatrix}$$

$$\begin{pmatrix}i\hbar A'(t)\\ i\hbar B'(t)\end{pmatrix} = \begin{pmatrix}0 & \delta\\ \delta & 0\end{pmatrix}\begin{pmatrix}A(t)\\B(t)\end{pmatrix} = \begin{pmatrix}\delta B(t)\\ \delta A(t)\end{pmatrix}$$

$$i\hbar A'(t) = \delta B(t)$$

$$i\hbar B'(t) = \delta A(t)$$

$$i\hbar \cdot \tfrac{i\hbar}{\delta^2} B''(t) = B(t)$$

$$B''(t) = -(\delta/\hbar)^2 B(t)$$

We recognize this as the simple harmonic oscillator, which has the following solution

$$B(t) = B_1 \cos\omega t + B_2 \sin\omega t \qquad \omega = \delta/\hbar$$

$$A(t) = A_1 \cos\omega t + A_2 \sin\omega t$$

$$|\psi(0)\rangle = \begin{pmatrix}A(0)\\B(0)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$A_1 = 1 \qquad A_2 = 0$$

$$B_1 = 0 \qquad B_2 = 1$$

$$|\psi(t)\rangle = \begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}$$

c. $$|\langle a''|\psi(t)\rangle|^2 = \left|\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}\right|^2 = \sin^2\omega t$$

d. $$\mathcal{H} = \delta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \delta\sigma_x = S_x \text{ if } \delta = \hbar/2$$

9. $|R\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|L\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$\mathcal{H} = \begin{pmatrix}0 & \Delta\\ \Delta & 0\end{pmatrix}$$

$$\det\begin{pmatrix}-\lambda & \Delta\\ \Delta & -\lambda\end{pmatrix} = \lambda^2 - \Delta^2$$

$$\lambda = \pm\Delta$$

$$|\Delta\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \tfrac{1}{\sqrt{2}}[|R\rangle + |L\rangle]$$

$$|-\Delta\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \tfrac{1}{\sqrt{2}}[|R\rangle - |L\rangle]$$

b. $$|\alpha, t_0 = 0; t\rangle = \exp(-i\mathcal{H}t/\hbar)|\alpha\rangle$$

$$|\alpha\rangle = \langle R|\alpha\rangle|R\rangle + \langle L|A\rangle|L\rangle$$

$$|\Delta\rangle = \tfrac{1}{\sqrt{2}}[|R\rangle + |L\rangle]$$

$$|-\Delta\rangle = \tfrac{1}{\sqrt{2}}[|R\rangle - |L\rangle]$$

$$exp(^{-iHt}/_\hbar)|\alpha\rangle = exp(^{-iHt}/_\hbar)|\alpha\rangle$$
$$= exp(^{-iHt}/_\hbar)\langle R|\alpha\rangle ^1/_{\sqrt{2}}[|\Delta\rangle + |-\Delta\rangle]$$
$$+ exp(^{-iHt}/_\hbar)\langle L|\alpha\rangle ^1/_{\sqrt{2}}[|\Delta\rangle - |-\Delta\rangle]$$
$$exp(^{-iHt}/_\hbar)|\Delta\rangle = exp(^{-i\Delta t}/_\hbar)|\Delta\rangle$$
$$exp(^{-iHt}/_\hbar)|-\Delta\rangle = exp(^{i\Delta t}/_\hbar)|-\Delta\rangle$$
$$exp(^{-iHt}/_\hbar)|\alpha\rangle = \langle R|\alpha\rangle \cdot ^1/_{\sqrt{2}}[exp(^{-i\Delta t}/_\hbar)|\Delta\rangle + exp(^{i\Delta t}/_\hbar)|-\Delta\rangle]$$
$$+\langle L|\alpha\rangle \cdot ^1/_{\sqrt{2}}[exp(^{-i\Delta t}/_\hbar)|\Delta\rangle - exp(^{i\Delta t}/_\hbar)|-\Delta\rangle]$$
$$= \langle R|\alpha\rangle \cdot ^1/_2[(exp(^{-i\Delta t}/_\hbar) + exp(^{i\Delta t}/_\hbar))|R\rangle + (exp(^{-i\Delta t}/_\hbar) - exp(^{i\Delta t}/_\hbar))|L\rangle]$$
$$+\langle L|\alpha\rangle \cdot ^1/_2[(exp(^{-i\Delta t}/_\hbar) - exp(^{i\Delta t}/_\hbar))|R\rangle + (exp(^{-i\Delta t}/_\hbar) + exp(^{i\Delta t}/_\hbar))|L\rangle]$$
$$= \langle R|\alpha\rangle[\cos(^{\Delta t}/_\hbar)|R\rangle - i\sin(^{\Delta t}/_\hbar)|L\rangle]$$
$$+\langle L|\alpha\rangle[-i\sin(^{\Delta t}/_\hbar)|R\rangle + \cos(^{\Delta t}/_\hbar)|L\rangle]$$
$$= [\langle R|\alpha\rangle\cos(^{\Delta t}/_\hbar) - \langle L|\alpha\rangle i\sin(^{\Delta t}/_\hbar)]|R\rangle$$
$$+[\langle L|\alpha\rangle\cos(^{\Delta t}/_\hbar) - \langle R|\alpha\rangle i\sin(^{\Delta t}/_\hbar)]|L\rangle$$

c. $\langle L|\alpha\rangle = 0$

$\langle R|\alpha\rangle = 1$

$|\alpha, t_0; t=0\rangle = \cos(^{\Delta t}/_\hbar)|R\rangle - i\sin(^{\Delta t}/_\hbar)|L\rangle$

$|\langle L|\alpha, t_0; t=0\rangle|^2 = |-i\sin(^{\Delta t}/_\hbar)|^2 = \sin^2(^{\Delta t}/_\hbar)$

d. Expect $\langle R|\alpha, t_0=0; t\rangle = \langle R|\alpha\rangle\cos(^{\Delta t}/_\hbar) - i\langle L|\alpha\rangle\sin(^{\Delta t}/_\hbar)$

$\langle L|\alpha, t_0=0; t\rangle = \langle L|\alpha\rangle\cos(^{\Delta t}/_\hbar) - i\langle R|\alpha\rangle\sin(^{\Delta t}/_\hbar)$

$i\hbar\frac{d}{dt}a_R(t) = \Delta a_L(t)$

$i\hbar\frac{d}{dt}a_L(t) = \Delta a_R(t)$

$a_L(t) = A\cos(^{\Delta t}/_\hbar) + B\sin(^{\Delta t}/_\hbar)$

$a_R(t) = C\cos(^{\Delta t}/_\hbar) + D\sin(^{\Delta t}/_\hbar)$

$|\alpha\rangle = a_R(0)|R\rangle + a_L(0)|L\rangle$

$= |R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle$

$C = \langle R|\alpha\rangle \quad A = \langle L|\alpha\rangle$

To normalize, $B = -i\langle R|\alpha\rangle$

$D = -i\langle L|\alpha\rangle$

e. $H = \Delta|L\rangle\langle R|$

$U = 1 - ^{it\Delta}/_\hbar|L\rangle\langle R|$

$U|\alpha\rangle = (1 - ^{it\Delta}/_\hbar|L\rangle\langle R|)[|R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle]$

$= |R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle - ^{it\Delta}/_\hbar|L\rangle\langle R|\alpha\rangle$

$|\langle R|\alpha\rangle|^2 + |\langle L|\alpha\rangle|^2 \neq |\langle R|U|\alpha\rangle|^2 + |\langle L|U|\alpha\rangle|^2$

$\langle R|U|\alpha\rangle = \langle R|\alpha\rangle$

$\langle L|U|\alpha\rangle = \langle L|\alpha\rangle - ^{it\Delta}/_\hbar\langle R|\alpha\rangle$

Can see there is a time-dependent component in probability

10.

a. We know from table 2.1 observables are stationary in the Schrodinger picture and moving in the Heisenberg picture. We also know the two pictures should coincide at $t=0$

$x_S = x_H(0) = x_0 \qquad H = ^{p^2}/_{2m} + ^{m\omega^2 x^2}/_2$

$p_S = p_H(0) = p_0$

$x_H(t) = exp(^{iHt}/_\hbar)x_0 exp(^{-iHt}/_\hbar)$ (2.2.10)

$= x_0\cos(\omega t) + ^{p_0}/_{m\omega}\sin(\omega t)$ (2.3.50)

$p_H(t) = p_0\cos(\omega t) - m\omega x_0\sin(\omega t)$

b. Again, from table 2.1, the state ket is moving in the Schrodinger equation and stationary in the Heisenberg picture

$|\alpha, t=0\rangle_H = |\alpha, t=0\rangle_S$

$|\alpha, t\rangle_H = |\alpha, t=0\rangle$

$i\hbar\frac{\partial}{\partial t}|\alpha, t\rangle_S = H|\alpha, t\rangle_S$ (2.1.27)

11. $\mathcal{H} = p^2/2m + m\omega^2 x^2/2$ $\quad x(t) = x_0 \cos(\omega t) + p_0/m\omega \sin(\omega t)$

$\exp(ip_0 a/\hbar)\, x_0 \exp(-ip_0 a/\hbar)$

$= x_0 + \frac{ia}{\hbar}[p_0, x_0] = x_0 + \frac{ia}{\hbar}\cdot(-i\hbar)$ (2.3.47)

$= x_0 + a$

$\exp(ip_0 a/\hbar)\, p_0 \exp(-ip_0 a/\hbar)$

$= p_0 + \frac{ia}{\hbar}[p_0, p_0] = p_0$

$\langle 0 | \exp(ip_0 a/\hbar)[x_0 \cos(\omega t) + p_0/m\omega \sin(\omega t)] \exp(-ip_0 a/\hbar) | 0 \rangle$

$= \langle 0 | (x_0 + a)\cos(\omega t) | 0 \rangle + \langle 0 | \frac{p_0}{m\omega} \sin(\omega t) | 0 \rangle$

$\langle 0 | x_0 | 0 \rangle = 0$

$\langle 0 | p_0 | 0 \rangle = 0$ (2.3.37)

$\langle x(t) \rangle = a\cos(\omega t)$

12.

a. $\langle x' | \alpha \rangle = \langle x' | \exp(-ipa/\hbar) | 0 \rangle$

$\mathcal{U} | x' \rangle = \exp(-ipa/\hbar) | x' \rangle = | x' + a \rangle$

$\langle x' | \exp(-ipa/\hbar) = \langle x' - a |$

$\langle x' | \alpha \rangle = \langle x' - a | 0 \rangle$

$= \pi^{-1/4} x_0^{-1/2} \exp\left(-\frac{1}{2}\left(\frac{x'-a}{x_0}\right)^2\right)$ $\quad x_0 = (\hbar/m\omega)^{1/2}$

b. $P = \int \langle \alpha | x' \rangle \langle x' | 0 \rangle dx'$

$= \int_{-\infty}^{\infty} \frac{1}{\pi^{1/4} x_0^{1/2}} \exp\left(\frac{-1}{2}\frac{(x'-a)^2}{x_0^2}\right) \cdot \frac{1}{\pi^{1/4} x_0^{1/2}} \exp\left(-\frac{1}{2}\cdot\frac{x'^2}{x_0^2}\right) dx'$

$= \frac{1}{\pi^{1/2} x_0} \int_{-\infty}^{\infty} \exp\left(-\frac{(x'^2 - 2ax' + a^2 + x'^2)}{x_0^2}\right) dx'$

$= \frac{1}{\pi^{1/2} x_0} \exp\left(\frac{-a^2}{2x_0^2}\right) \int_{-\infty}^{\infty} \exp\left(\frac{-x'(x'-2a)}{x_0^2}\right) dx'$

$= \frac{1}{\pi^{1/2} x_0} \exp\left(\frac{-a^2}{2x_0^2}\right) \cdot \sqrt{\frac{\pi}{1/x_0^2}} = \exp\left(\frac{-a^2}{2x_0^2}\right)$

$\int_{-\infty}^{\infty} \exp(-a(x+b)^2) dx = \sqrt{\frac{\pi}{a}}$

Probability will not change since it is time-independent

13.

a. $x = \sqrt{\hbar/2m\omega}\,(a + a^\dagger)$

$p = i\sqrt{m\hbar\omega/2}\,(-a + a^\dagger)$ (2.3.24)

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ (2.3.23)

$\langle m|x|n\rangle = \sqrt{\hbar/2m\omega}\,[\langle m|a|n\rangle + \langle m|a^\dagger|n\rangle]$

$= \sqrt{\hbar/2m\omega}\,(\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1})$

$\langle m|p|n\rangle = i\sqrt{m\hbar\omega/2}\,[-\langle m|a|n\rangle + \langle m|a^\dagger|n\rangle]$

$= i\sqrt{m\hbar\omega/2}\,(-\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1})$

$\{x,p\} = xp + px = i(a+a^\dagger)(-a+a^\dagger) + i(-a+a^\dagger)(a+a^\dagger)$

$= i(-a^2 + aa^\dagger - a^\dagger a + a^{\dagger 2} - a^2 - aa^\dagger + a^\dagger a + a^{\dagger 2})$

$= -2i(a^2 - a^{\dagger 2})$

$\langle m|\{x,p\}|n\rangle = -2i(\langle m|a^2|n\rangle - \langle m|a^{\dagger 2}|n\rangle)$

$= -2i(\sqrt{n(n-1)}\,\delta_{m,n-2} - \sqrt{(n+1)(n+2)}\,\delta_{m,n+2})$

$x^2 = \hbar/2m\omega\,(a+a^\dagger)(a+a^\dagger) = \hbar/2m\omega\,(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2})$

$\langle m|x^2|n\rangle = \hbar/2m\omega\,(\sqrt{n(n-1)}\,\delta_{m,n-2} + (n+1)\delta_{m,n} + n\,\delta_{m,n} + \sqrt{(n+1)(n+2)}\,\delta_{m,n+2})$

$p^2 = -m\hbar\omega/2\,(-a+a^\dagger)(-a+a^\dagger) = -m\hbar\omega/2\,(a^2 - aa^\dagger - a^\dagger a + a^{\dagger 2})$

$\langle m|p^2|n\rangle = -m\hbar\omega/2\,(\sqrt{n(n-1)}\,\delta_{m,n-2} - (n+1)\delta_{m,n} - n\delta_{m,n} + \sqrt{(n+1)(n+2)}\,\delta_{m,n+2})$

b. $\langle p^2/m \rangle = \langle x \frac{dV}{dx} \rangle$

$= \langle m\omega^2 x^2 \rangle$

$\langle n | p^2/m | n \rangle = 1/2m \cdot -m\hbar\omega/2\,(-(n+1) - n)$

$= \hbar\omega/2\,(2n+1)$

$\langle n | m\omega^2 x^2 | n \rangle = m\omega^2 \cdot \hbar/2m\omega\,(2n+1)$

$= \hbar\omega/2\,(2n+1)$

14.

a. $\langle p'|x|\alpha\rangle = \langle p'|x'\rangle\langle x'|x|\alpha\rangle$

$= x\langle p'|x'\rangle\langle x'|\alpha\rangle$

$= i\hbar\frac{\partial}{\partial p'}\langle p'|x'\rangle\langle x'|\alpha\rangle$

$= i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$

b. $\mathcal{H}|\alpha\rangle = i\hbar\frac{\partial}{\partial t}|\alpha\rangle$

$(p^2/2m + m\omega^2x^2/2)|\alpha\rangle = i\hbar\frac{\partial}{\partial t}|\alpha\rangle$

$\langle p'|p^2/2m|\alpha\rangle + \langle p'|m\omega^2x^2/2|\alpha\rangle = i\hbar\frac{\partial}{\partial t}\langle p'|\alpha\rangle$

$p'^2/2m\langle p'|\alpha\rangle + \frac{m\omega^2}{2}\cdot(-\hbar^2)\frac{\partial^2}{\partial p'^2}\langle p'|\alpha\rangle = i\hbar\frac{\partial}{\partial t}\langle p'|\alpha\rangle$

$$\frac{p'^2}{2m}\langle p'|\alpha\rangle - \frac{m\omega^2\hbar^2}{2}\cdot\frac{d^2}{dp'^2}\langle p'|\alpha\rangle = i\hbar\frac{d\langle p'|\alpha\rangle}{dt}$$

$$\langle p'|n\rangle = \frac{1}{\pi^{1/4}\sqrt{2^n n!}}\cdot\frac{1}{p_0^{n+1/2}}\left(p' - p_0^2\frac{d}{dp'}\right)^n \exp\left(-\frac{1}{2}\left(\frac{p'}{p_0}\right)^2\right)$$

$p_0 = i\sqrt{m\hbar\omega}$

15. $x(t) = x_0\cos(\omega t) + \frac{p_0}{m\omega}\sin(\omega t)$

$x(t)x(0) = x_0^2\cos(\omega t) + \frac{p_0x_0}{m\omega}\sin(\omega t)$

$\langle x(t)x(0)\rangle = \cos(\omega t)\langle 0|x_0^2|0\rangle + \frac{\sin(\omega t)}{m\omega}\langle 0|p_0x_0|0\rangle$

$p_0x_0 = \frac{1}{2}(\{x_0,p_0\} - [x_0,p_0]\}$

From q. 15, $\langle 0|x_0^2|0\rangle = 2(0)+1 = 1\cdot\hbar/2m\omega$

$\langle 0|\{x_0,p_0\}|0\rangle = 0$

$\langle 0|[x,p]|0\rangle = i\hbar$

$\langle x(t)x(0)\rangle = \hbar/2m\omega(\cos(\omega t) - i\sin(\omega t))$

16

a. $|\alpha\rangle = a|0\rangle + b|1\rangle$

$\langle x\rangle = \langle\alpha|x|\alpha\rangle = (\langle 0|a^* + \langle 1|b^*)x(a|0\rangle + b|1\rangle)$

$= \cancel{a^*a\langle 0|x|0\rangle} + a^*b\langle 0|x|1\rangle + b^*a\langle 1|x|0\rangle + \cancel{b^*b\langle 1|x|1\rangle}$

$= \sqrt{\hbar/2m\omega}\,(a^*b + b^*a)$

If we say a and b are real

$\langle x\rangle = 2ab\sqrt{\hbar/2m\omega}$

$a^2 + b^2 = 1$

$b = \sqrt{1-a^2}$

$\langle x\rangle = 2a\sqrt{1-a^2}\sqrt{\hbar/2m\omega}$

$$\frac{d\langle x\rangle}{da} = 2\sqrt{1-a^2}\sqrt{\hbar/2m\omega} + \frac{a}{\sqrt{1-a^2}}\cdot -2a\sqrt{\hbar/2m\omega} = 0$$

$$2\sqrt{1-a^2} = +\frac{2a^2}{\sqrt{1-a^2}}$$

$2(1-a^2) = +2a^2$

$1-a^2 = a^2$

$a^2 = 1/2$

$a = 1/\sqrt{2}$

$b = 1/\sqrt{2}$

$\langle x\rangle_{max} = \sqrt{\hbar/2m\omega}$

$|\alpha\rangle_{max} = 1/\sqrt{2}|0\rangle + 1/\sqrt{2}|1\rangle$

b. $\mathcal{U}(t) = \exp(-i\mathcal{H}t/\hbar)$

$E_n = (n+1/2)\hbar\omega$

$\mathcal{U}(t)|\alpha\rangle = 1/\sqrt{2}\cdot\exp(-i\mathcal{H}t/\hbar)|0\rangle + 1/\sqrt{2}\cdot\exp(-i\mathcal{H}t/\hbar)|1\rangle$

$$= \frac{\exp(-it\omega/2)|0\rangle}{\sqrt{2}} + \frac{\exp(-3i\omega t/2)|1\rangle}{\sqrt{2}} = |\alpha,t\rangle$$

Schrodinger picture:

$$\langle\alpha,t|x|\alpha,t\rangle = \frac{\langle 0|\exp(i\omega t/2) + \langle 1|\exp(3i\omega t/2)}{\sqrt{2}}\cdot x\cdot\frac{\exp(-i\omega t/2)|0\rangle + \exp(-3i\omega t/2)|1\rangle}{\sqrt{2}}$$

note, only $\langle 0|x|1\rangle$ and $\langle 1|x|0\rangle$ terms survive

$\frac{1}{2}\exp(-i\omega t)\langle 0|x|1\rangle + \frac{1}{2}\exp(i\omega t)\langle 1|x|0\rangle$

$= \frac{1}{2}(\cos\omega t - i\sin\omega t + \cos\omega t + i\sin\omega t)\cdot\sqrt{\hbar/2m\omega}$

$= \sqrt{\hbar/2m\omega}\cos(\omega t)$

Heisenberg: $x(t) = x_0\cos(\omega t) + \frac{p_0}{m\omega}\sin(\omega t)$

$$\tfrac{1}{\sqrt{2}}[\langle 0| + \langle 1|](x_0\cos(\omega t) + {}^{p_0}/_{m\omega}\sin(\omega t))[|0\rangle + |1\rangle]\cdot{}^{1}/_{\sqrt{2}}$$

$$= {}^{\cos(\omega t)}/_{2}[\langle 0| + \langle 1|]x_0[|0\rangle + |1\rangle]$$

$$+ {}^{\sin(\omega t)}/_{2m\omega}[\langle 0| + \langle 1|]p_0[|0\rangle + |1\rangle]$$

$$= {}^{\cos(\omega t)}/_{2}[\langle 0|x_0|1\rangle + \langle 1|x_0|0\rangle]$$

$$+ {}^{\sin(\omega t)}/_{2m\omega}[\langle 0|p_0|1\rangle + \langle 1|p_0|0\rangle]$$

$$= \sqrt{{}^{\hbar}/_{2m\omega}}\cos(\omega t)$$

c. $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$

Schrodinger:

$\langle x^2\rangle = \langle\alpha, t|x^2|\alpha, t\rangle$ Only $\langle 0|x^2|0\rangle$ and $\langle 1|x^2|1\rangle$ terms survive

$$= \tfrac{1}{2}\cdot\exp(0)\langle 0|x^2|0\rangle + \tfrac{1}{2}\cdot\exp(0)\langle 1|x^2|1\rangle$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\cdot 3 = 2\cdot{}^{\hbar}/_{2m\omega} = {}^{\hbar}/_{m\omega}$$

$$\langle x^2\rangle - \langle x\rangle^2 = {}^{\hbar}/_{m\omega} - {}^{\hbar}/_{2m\omega}\cdot\cos^2(\omega t)$$

$$= {}^{\hbar}/_{m\omega}(1 - {}^{\cos^2(\omega t)}/_{2})$$

Heisenberg:

$$x^2 = x_0^2\cos^2(\omega t) + {}^{1}/_{m\omega}\cos(\omega t)\sin(\omega t)\{x, p\} + \frac{p_0^2}{m^2\omega^2}\sin^2(\omega t)$$

$$\langle x^2\rangle = \cos^2(\omega t)\langle\alpha|x_0^2|\alpha\rangle + {}^{\cos(\omega t)\sin(\omega t)}/_{m\omega}\cdot\langle\alpha|\{x, p\}|\alpha\rangle$$

$$+ {}^{\sin^2(\omega t)}/_{m^2\omega^2}\langle\alpha|p_0^2|\alpha\rangle$$

$$= {}^{\cos^2(\omega t)}/_{2}\cdot{}^{\hbar}/_{2m\omega}[\langle 0|x^2|0\rangle + \langle 1|x^2|1\rangle]$$

$$+ 0 + {}^{\sin^2(\omega t)}/_{2m^2\omega^2}\cdot{}^{-m\hbar\omega}/_{2}[\langle 0|p^2|0\rangle + \langle 1|p^2|1\rangle]$$

$$= \frac{\cos^2(\omega t)\,\hbar}{4m\omega}\cdot 4 - \frac{\sin^2(\omega t)\cdot\hbar}{4m\omega}\cdot[-1-3]$$

$$= {}^{\hbar}/_{m\omega}$$

$$\langle x^2\rangle - \langle x\rangle^2 = {}^{\hbar}/_{m\omega}(1 - {}^{\cos^2(\omega t)}/_{2})$$

17. $\exp[-k^2\langle 0|x^2|0\rangle/2] = \exp({}^{-k^2\hbar}/_{4m\omega})$

$\langle 0|\exp(ikx)|0\rangle$

$$|0\rangle = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\exp\left(\frac{-x^2 m\omega}{2\hbar}\right)$$

$$\langle 0|\exp(ikx)|0\rangle = \int\left(\frac{m\omega}{\hbar\pi}\right)^{1/2}\exp\left(ikx - \frac{m\omega x^2}{2\hbar}\right)dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int\exp\left(-\left(\frac{m\omega x^2}{\hbar} - ikx + \alpha\right)\right)\cdot\exp(\alpha)\,dx$$

$$\exp\left(-\frac{m\omega}{\hbar}\left(x^2 - \frac{ik\hbar x}{m\omega} + \frac{\hbar\alpha}{m\omega}\right)\right)$$

$$x^2 - \frac{ik\hbar x}{m\omega} + \frac{\hbar\alpha}{m\omega} = (x-\beta)^2$$

$$2\beta = \frac{ik\hbar}{m\omega}$$

$$\beta^2 = \frac{\hbar\alpha}{m\omega}$$

$$\frac{-k^2\hbar^2}{4m^2\omega^2} = \frac{\hbar\alpha}{m\omega}$$

$$\alpha = \frac{-k^2\hbar}{4m\omega}$$

$$\langle 0|\exp(ikx)|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left(\frac{-k^2\hbar}{4m\omega}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{m\omega}{\hbar}(x-\beta)^2\right)dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left(\frac{-k^2\hbar}{4m\omega}\right)\cdot\left(\frac{\pi\hbar}{m\omega}\right)^{1/2} = \exp\left(\frac{-k^2\hbar}{4m\omega}\right) = \exp\left(\frac{-k^2\langle 0|x^2|0\rangle}{2}\right)$$

18.

a. $a|\lambda\rangle = \exp({}^{-|\lambda|^2}/_{2})\,a\exp(\lambda a^\dagger)|0\rangle$

$$\exp(\lambda a^\dagger) = \sum_{n=0}^{\infty}\frac{\lambda^n a^{\dagger n}}{n!}$$

$$a^{\dagger n}|0\rangle = \sqrt{n!}\,|n\rangle$$

$$a(a^\dagger)^n|0\rangle = \sqrt{n}\sqrt{n!}\,|n-1\rangle$$

$$a|\lambda\rangle = \exp({}^{-|\lambda|^2}/_{2})\cdot\sum_{n=0}^{\infty}\lambda^n\cdot\frac{\sqrt{n}}{\sqrt{n!}}|n-1\rangle$$

$$= \exp({}^{-|\lambda|^2}/_{2})\cdot\sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{(n-1)!}}|n-1\rangle$$

$$= \exp(-|\lambda|^2/2) \sum_{n=0}^{\infty} \frac{\lambda^n \cdot \lambda}{\sqrt{n!}} |n\rangle$$

$$= \lambda \exp(-|\lambda|^2/2) \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle = \lambda |\lambda\rangle$$

$$\langle\lambda|\lambda\rangle = \exp(-|\lambda|^2) \langle 0| \exp(\lambda^* a) \exp(\lambda a^+) |0\rangle$$

$$= \exp(-|\lambda|^2) \langle 0| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\lambda^{*m} a^m}{m!} \frac{\lambda^n}{\sqrt{n!}} |n\rangle$$

$\neq 0$ only when $n = m$

$$\langle\lambda|\lambda\rangle = \exp(-|\lambda|^2) \langle 0| \sum_{n=0}^{\infty} \frac{\lambda^{*n} \lambda^n}{n!} |0\rangle$$

$$= \exp(-|\lambda|^2) \cdot \exp(|\lambda|^2) = 1$$

b. $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \hbar^2/4$ (1.7.41)

$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$ (1.4.51)

$x = \sqrt{\hbar/2m\omega}(a + a^+)$ $\quad p = i\sqrt{m\hbar\omega/2}(-a + a^+)$ (2.3.24)

$a|\lambda\rangle = \lambda|\lambda\rangle$

$\langle\lambda|a^+ = \langle\lambda|\lambda^*$

$$\langle x\rangle = \langle\lambda|x|\lambda\rangle = \sqrt{\hbar/2m\omega}\langle\lambda|(a + a^+)|\lambda\rangle$$

$$= \sqrt{\hbar/2m\omega}\left[\langle\lambda|a|\lambda\rangle + \langle\lambda|a^+|\lambda\rangle\right]$$

$$= \sqrt{\hbar/2m\omega}\left[\lambda\langle\lambda|\lambda\rangle + \lambda^*\langle\lambda|\lambda\rangle\right]$$

$$= \sqrt{\hbar/2m\omega}(\lambda + \lambda^*)$$

$$x^2 = \frac{\hbar}{2m\omega}(a^2 + aa^+ + a^+a + a^{+2})$$

Note that aa^+ doesn't play nice since we don't have an expression for $\langle\lambda|a$ or $a^+|\lambda\rangle$. Thus, we need to use (2.3.3) to get a different form

$$x^2 = \frac{\hbar}{2m\omega}(a^2 + (1 + a^+a) + a^+a + a^{+2})$$

$$= \frac{\hbar}{2m\omega}(a^2 + 2a^+a + a^{+2} + 1)$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}(\lambda^2 + 2\lambda^*\lambda + \lambda^{*2} + 1)$$

$$\langle(\Delta x)^2\rangle = \hbar/2m\omega$$

$$\langle p\rangle = i\sqrt{m\hbar\omega/2}(-\lambda + \lambda^*)$$

$$\langle p\rangle^2 = -m\hbar\omega/2(\lambda^2 - 2\lambda\lambda^* + \lambda^{*2})$$

$$p^2 = -\frac{m\hbar\omega}{2}(a^2 - aa^+ - a^+a + a^{+2})$$

$$= -m\hbar\omega/2(a^2 - 1 - 2a^+a + a^{+2})$$

$$\langle p^2\rangle = -\frac{m\hbar\omega}{2}(\lambda^2 - 2\lambda\lambda^* + \lambda^{*2} - 1)$$

$$\langle(\Delta p)^2\rangle = m\hbar\omega/2$$

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{\hbar}{2m\omega} \cdot \frac{m\hbar\omega}{2} = \hbar^2/4$$

c. $|\lambda\rangle = e^{-|\lambda|^2/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle$

$$= \sum_{n=0} f(n)|n\rangle$$

$$f(n) = \frac{\exp(-|\lambda|^2/2) \cdot \lambda^n}{\sqrt{n!}}$$

Poisson distribution: $P = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$

$$|f(n)|^2 = \frac{\exp(-|\lambda|^2) \cdot \lambda^{2n}}{n!} \quad \text{where } \lambda \to \lambda^2$$

$$= \frac{\exp(-|\lambda|^2)\lambda^{2n}}{n!}$$

$$\ln(|f(n)|^2) = -|\lambda|^2 + n\ln(\lambda^2) - \ln n!$$

$$\frac{\partial \ln(|f(n)|^2)}{\partial n} = \ln(\lambda^2) - \frac{\partial}{\partial n}\ln n! = 0$$

$$\approx \ln(\lambda^2) - \frac{\partial}{\partial n}\left[n\ln n - n\right] \quad \text{Stirling approximation}$$

$$= \ln(\lambda^2) - \ln n - 1 + 1 = 0$$

$$\ln(\lambda^2) = \ln n$$

$$\lambda^2 = n$$

d. $\exp(-ipl/\hbar) = \exp\left(\sqrt{\frac{m\hbar\omega}{2}}\,(-a+a^\dagger)\,l/\hbar\right)$

$= \exp\left(l\sqrt{\frac{m\omega}{2\hbar}}\,a^\dagger\right)\exp\left(-l\sqrt{\frac{m\omega}{2\hbar}}\,a\right)\exp\left(\frac{l^2}{2}\,\frac{m\omega}{2\hbar}\,[a^\dagger, a]\right)$

Looking up the solution $e^{A+B} = e^{A}e^{B}e^{-\frac{1}{2}[A,B]}$

$= \exp\left(\frac{-l^2 m\omega}{4\hbar}\right)\exp\left(-l\sqrt{\frac{m\omega}{2\hbar}}\,a\right)\exp\left(l\sqrt{\frac{m\omega}{2\hbar}}\,a^\dagger\right)$

$\exp\left(l\sqrt{\frac{m\omega}{2\hbar}}\,a\right)|0\rangle = |0\rangle$

$\exp(-ipl/\hbar)|0\rangle = \exp\left(\frac{-l^2 m\omega}{4\hbar}\right)\exp\left(l\sqrt{\frac{m\omega}{2\hbar}}\,a^\dagger\right)|0\rangle$

Say $\lambda = l\sqrt{\frac{m\omega}{2\hbar}}$ and we return

$|\lambda\rangle = e^{-|\lambda|^2/2}\,e^{\lambda a^\dagger}|0\rangle$

19. First, we know different harmonic oscillators don't interact with each other, so we have the following relations

$[a, a^\dagger] = 1 \qquad a a^\dagger = 1 + a^\dagger a$

$[a_+, a_-^\dagger] = 0 \qquad a_+ a_-^\dagger = a_-^\dagger a_+ \qquad a_- a_+^\dagger = a_+^\dagger a_-$

$[a_+, a_-] = 0 \qquad a_+ a_- = a_- a_+ \qquad a_+^\dagger a_-^\dagger = a_-^\dagger a_+^\dagger$

a. $[J_z, J_+] = J_z J_+ - J_+ J_z$

$= \frac{\hbar^2}{2}(a_+^\dagger a_+ - a_-^\dagger a_-)(a_+^\dagger a_-) - \frac{\hbar^2}{2}(a_+^\dagger a_-)(a_+^\dagger a_+ - a_-^\dagger a_-)$

$= \frac{\hbar^2}{2}(a_+^\dagger a_+ a_+^\dagger a_- - a_-^\dagger a_- a_+^\dagger a_- - a_+^\dagger a_- a_+^\dagger a_+ + a_+^\dagger a_- a_-^\dagger a_-)$

$= \frac{\hbar^2}{2}[a_+^\dagger(1 + a_+^\dagger a_+)a_- - a_+^\dagger a_- a_+^\dagger a_+ + a_+^\dagger(1 + a_-^\dagger a_-)a_- - a_-^\dagger a_- a_+^\dagger a_-]$

$= \frac{\hbar^2}{2}[a_+^\dagger a_- + a_+^\dagger a_+^\dagger a_+ a_- - a_+^\dagger a_+^\dagger a_- a_+ + a_+^\dagger a_- + a_+^\dagger a_-^\dagger a_- a_- - a_-^\dagger a_+^\dagger a_- a_-]$

$= \frac{\hbar^2}{2}[2a_+^\dagger a_- + a_+^\dagger a_+^\dagger a_+ a_- - a_+^\dagger a_+^\dagger a_+ a_- + a_+^\dagger a_-^\dagger a_- a_- - a_+^\dagger a_-^\dagger a_- a_-]$

$= \hbar^2 a_+^\dagger a_- = \hbar J_+$ $\qquad$ $[J_z, J_-]$ is left to the reader as an exercise

b. $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$

$J_\pm = J_x \pm i J_y \qquad$ (1.4.19) or (3.5.5)

$[J_+, J_-] = (J_x + iJ_y)(J_x - iJ_y) - (J_x - iJ_y)(J_x + iJ_y)$

$= J_x^2 - iJ_xJ_y + iJ_yJ_x + J_y^2 - J_x^2 - iJ_xJ_y + iJ_yJ_x - J_y^2$

$= 2i[J_x, J_y] = 2i \cdot i\hbar J_z = -2\hbar J_z \qquad$ (1.4.20)

$\vec{J}^2 = J_+J_- - \hbar J_z + J_z^2 \qquad$ (3.5.19)

$[\vec{J}^2, J_z] = (J_+J_- - \hbar J_z + J_z^2)J_z - J_z(J_+J_- - \hbar J_z + J_z^2)$

$= J_+J_-J_z - J_zJ_+J_- - \hbar J_z^2 + J_z^3 + \hbar J_z^2 - J_z^3$

$= J_+J_-J_z - J_zJ_+J_-$

$J_zJ_+ - J_+J_z = \hbar J_+ \qquad J_zJ_+ = J_+J_z + \hbar J_+$

$J_zJ_- - J_-J_z = -\hbar J_- \qquad J_-J_z = J_zJ_- + \hbar J_-$

$[\vec{J}^2, J_z] = J_+(J_zJ_- + \hbar J_-) - (J_+J_z + \hbar J_+)J_-$

$= J_+J_zJ_- + \hbar J_+J_- - J_+J_zJ_- - \hbar J_+J_- = 0$

c. $N = a_+^\dagger a_+ + a_-^\dagger a_-$

$\vec{J}^2 = J_+J_- - \hbar J_z + J_z^2$

$= \hbar^2(a_+^\dagger a_-)(a_-^\dagger a_+) - \frac{\hbar^2}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) + \frac{\hbar^2}{4}(a_+^\dagger a_+ - a_-^\dagger a_-)(a_+^\dagger a_+ - a_-^\dagger a_-)$

$= \hbar^2(a_+^\dagger a_- a_-^\dagger a_+) - \frac{\hbar^2}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) + \frac{\hbar^2}{4}(a_+^\dagger a_+ a_+^\dagger a_+ - a_-^\dagger a_- a_+^\dagger a_+ - a_+^\dagger a_+ a_-^\dagger a_- + a_-^\dagger a_- a_-^\dagger a_-)$

$= \hbar^2(a_+^\dagger a_+ a_- a_-^\dagger) - \frac{\hbar^2}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) + \frac{\hbar^2}{4}(a_+^\dagger a_+ a_+^\dagger a_+ - 2a_+^\dagger a_+ a_-^\dagger a_- + a_-^\dagger a_- a_-^\dagger a_-)$

$= \hbar^2 a_+^\dagger a_+(a_- a_-^\dagger - \frac{1}{2} + \frac{1}{4}a_+^\dagger a_+ - \frac{1}{2}a_-^\dagger a_-)$

$+ \hbar^2 a_-^\dagger a_-(\frac{1}{2} + \frac{1}{4}a_-^\dagger a_-)$

$= \hbar^2 a_+^\dagger a_+(a_- a_-^\dagger - \frac{1}{2}a_- a_-^\dagger + \frac{1}{4}a_+^\dagger a_+) + \frac{\hbar^2}{2}a_-^\dagger a_-(1 + \frac{1}{2}a_-^\dagger a_-)$

$= \frac{\hbar^2}{2}a_+^\dagger a_+(a_- a_-^\dagger + \frac{1}{2}a_+^\dagger a_+) + \frac{\hbar^2}{2}a_-^\dagger a_-(1 + \frac{1}{2}a_-^\dagger a_-)$

$= \frac{\hbar^2}{2}a_+^\dagger a_+(1 + a_-^\dagger a_- + \frac{1}{2}a_+^\dagger a_+) + \frac{\hbar^2}{2}a_-^\dagger a_-(1 + \frac{1}{2}a_-^\dagger a_-)$

$= \frac{\hbar^2}{2}a_+^\dagger a_+(1 + \frac{1}{2}a_-^\dagger a_-) + \frac{\hbar^2}{4}a_+^\dagger a_+(a_-^\dagger a_- + a_+^\dagger a_+) + \frac{\hbar^2}{2}a_-^\dagger a_-(1 + \frac{1}{2}a_-^\dagger a_-)$

$= \frac{\hbar^2}{2}(a_+^\dagger a_+ + a_-^\dagger a_-)(1 + \frac{1}{2}a_-^\dagger a_-) + \frac{\hbar^2}{4}a_+^\dagger a_+(a_-^\dagger a_- + a_+^\dagger a_+)$

$= \frac{\hbar^2}{2}(a_+^\dagger a_+ + a_-^\dagger a_-)(1 + \frac{1}{2}a_-^\dagger a_- + \frac{1}{2}a_+^\dagger a_+)$

$= \frac{\hbar^2}{2}N[(N/2) + 1]$

20.

a. We see that this is solved by the simple harmonic oscillator with the condition that the solution disappears at the origin. Thus, we need odd-parity solutions

$E = (4n + 3)\frac{\hbar\omega}{2}, \quad \omega = \sqrt{k/m}$

$E_0 = \frac{3\hbar\omega}{2}$

b. $\langle x'|1\rangle = \frac{1}{\sqrt{2}\,x_0}\left(x' - x_0^2\frac{d}{dx'}\right)\langle x'|0\rangle \qquad$ (2.3.31)

$\langle x'|0\rangle = \frac{1}{\pi^{1/4}\sqrt{x_0}}\exp\left(-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right) \qquad$ (2.3.30)

$x_0 = \sqrt{\hbar/m\omega}$

$$\psi_1 = \langle x'|1\rangle = \frac{1}{\sqrt{2}\,x_0}\left[\frac{x'}{\pi^{1/4}\sqrt{x_0}}\exp\left(-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right) - \frac{x_0^2}{\pi^{1/4}\sqrt{x_0}}\cdot\frac{-x'}{x_0^2}\exp\left(-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right)\right]$$

$$= \frac{2x'\exp\left(-\frac{1}{2}(x'/x_0)^2\right)}{\sqrt{2}\,x_0^{3/2}\pi^{1/4}}$$

$$\langle x^2\rangle = \int_0^\infty \psi_1 x'^2 \psi_1\, dx'$$

$$= \frac{4}{2x_0^3\pi^{1/2}}\int_0^\infty x'^4 \exp\left(-(x'/x_0)^2\right) dx'$$

$$\left(\frac{x'}{x_0}\right)^2 = r \qquad x' = x_0 r^{1/2}$$

$$\frac{2x'}{x_0}dx' = dr \qquad dx' = \frac{x_0^2}{2x'}dr$$

$$\Gamma(z) = \int_0^\infty r^{z-1}e^{-r}dr$$

$$= (z-1)!$$

$$\langle x^2\rangle = \frac{4}{2x_0^3\pi^{1/2}}\int_0^\infty \frac{x'^4 x_0^2}{2x'}\exp(-r)\,dr$$

$$= \frac{x_0^3 x_0^2}{x_0^3\pi^{1/2}}\int_0^\infty r^{3/2}\exp(-r)\,dr$$

$$= \frac{x_0^2}{\pi^{1/2}}\Gamma(5/2) = \frac{x_0^2}{\sqrt{\pi}}\cdot\frac{3\sqrt{\pi}}{4} = \frac{3x_0^2}{4} = \frac{3\hbar}{4m\omega}$$

21. $\psi_n = A_n \sin\left(\frac{n\pi x}{L}\right)$ (A.2.4)

$$A_n = \sqrt{\frac{2}{L}}$$

$$E_n = \frac{\hbar^2 n^2\pi^2}{2mL^2}$$

$$\psi(x,0) = \delta(x - L/2)$$

$$\psi(x,t) = \sum_n c_n \exp(-iE_n t/\hbar)\psi_n(x)$$

$$c_n = \langle\psi_n(x)|\psi_n(x,0)\rangle$$

$$= \int_{-\infty}^{\infty}\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\delta\left(x-\frac{L}{2}\right)dx$$

$$= \sqrt{2/L}\,\sin\left(\frac{n\pi}{2}\right)$$

$$= (-1)^{\frac{n-1}{2}}\sqrt{\frac{2}{L}} \quad n \text{ odd}$$

$$= 0 \quad n \text{ even}$$

$P = |c_n|^2 = \frac{2}{L}$ for n odd

$$\psi_n = \sum_n (-1)^{\frac{n-1}{2}}\sqrt{\frac{2}{L}}\exp\left(\frac{-iE_n t}{\hbar}\right)\cdot\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_n (-1)^{n-1/2}\cdot\frac{2}{L}\cdot\exp\left(\frac{-i\hbar^2 n^2\pi^2 t}{2mL^2\hbar}\right)\sin\left(\frac{n\pi x}{L}\right)$$

22. $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - v_0\delta(x)\psi = -E\psi$ (2.4.11)

To find the ground state energy, we want to look at a small region $(-\varepsilon,\varepsilon)$ then let $\varepsilon\to 0$

$$\int_{-\varepsilon}^{\varepsilon}\frac{-\hbar^2}{2m}\cdot\frac{d^2\psi}{dx^2}dx - \int_{-\varepsilon}^{\varepsilon}v_0\delta(x)\psi\,dx = \int_{-\varepsilon}^{\varepsilon}-E\psi\,dx$$

$$-\frac{\hbar^2}{2m}\left(\frac{d\psi}{dx}\right)\Bigg|_{-\varepsilon}^{\varepsilon} - v_0\psi(0) = 0$$

For other regions, $(x \neq 0)$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = -E\psi$$

$$\psi = A\exp\left(-\left(\frac{2mE}{\hbar^2}\right)^{1/2}x\right) \qquad x>0$$

$$= A\exp\left(\left(\frac{2mE}{\hbar^2}\right)^{1/2}x\right) \qquad x<0$$

This is a unique solution, so we can plug this into the region around $x=0$

$$-\frac{\hbar^2}{2m}\cdot A\left(\frac{-2mE}{\hbar^2}\right)^{1/2} + \frac{\hbar^2}{2m}A\left(\frac{2mE}{\hbar^2}\right)^{1/2} - v_0 A = 0$$

$$\frac{\hbar^2}{m}\left(\frac{2mE}{\hbar^2}\right)^{1/2} = v_0$$

$$\frac{2mE}{\hbar^2} = \frac{m^2v_0^2}{\hbar^4}$$

$$E = \frac{mv_0^2}{2\hbar^2}$$

Since $E>0$, no bound state

23. We can use the solution from the previous problem

$$\psi(x,0) = A\exp\left(-\left(\frac{2mE}{\hbar^2}\right)^{1/2} x\right)$$

$$E = \frac{m\lambda^2}{2\hbar^2}$$

$$\psi(x,0) = A\exp\left(-\left(\frac{2m}{\hbar^2}\cdot\frac{m\lambda^2}{2\hbar^2}\right)^{1/2} x\right)$$

$$= A\exp\left(-\frac{m\lambda|x|}{\hbar^2}\right)$$

$$1 = \int_{-\infty}^{\infty}\psi^2\,dx = 2A^2\int_0^{\infty}\exp\left(\frac{-2m\lambda x}{\hbar^2}\right)dx$$

$$= 2A^2\cdot\left(\frac{-\hbar^2}{2m\lambda}\right)\exp\left(\frac{-2m\lambda x}{\hbar^2}\right)\Bigg|_0^{\infty}$$

$$= \frac{A^2\hbar^2}{m\lambda} = 1$$

$$A = \left(m\lambda/\hbar^2\right)^{1/2}$$

Apparently to solve this, you need to find the wave function, you need to know how to find the propagator, so I'm going to leave the rest of this problem and problems to future Ben.