Graduate Classical Electrodynamics

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Acknowledgements

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Chapter 1

Introduction to Electrostatics

"Everyone must leave something behind when he dies, my grandfather said...Something your hand touched some way so your soul has somewhere to go when you die, and when people look at that tree or that flower you planted, you're there. It doesn't matter what you do, so long as you change something from the way it was before you touched it into something that's like you after you take your hands away. The difference between the man who just cuts lawns and a real gardener is in the touching. The lawn-cutter might just as well not have been there at all; the gardener will be there a lifetime." - (Ray Bradbury, Fahrenheit 451).

As far as we know, there are four fundamental forces in the universe: gravity, the weak and strong nuclear forces, and electromagnetism. At the large scale (say, between planets), the gravitational force dominates. At the small scale (say, inside the atom), the nuclear force dominates. Everywhere in between, the the electromagnetic force dominates. Of these four, the electromagnetic force is probably the easiest to grasp since it is responsible for (as the name implies) electricity and magnets. The history of how electromagnetism was developed (and of note, is still being developed) is rich and varied and far beyond the scope of this text.

We start our discussion of electromagnetism with electrostatics, the study of charges and fields that have no time dependence. For those of you familiar with electromagnetism, what we mean is that there are no currents and no magnetic fields. Now, as the joke goes, you start with F=ma and the rest is easy to see, so let's begin with the force between two charged particles.

1.1 Coulomb's Law and the Electric Field

1.1.1 Coulomb's Law

Imagine we have two charged particles, q_1 and q_2, placed at positions $\vec{x}_1$ and $\vec{x}_2$ respectively. As an example, figure (1.1) shows two such points. If both of these particles have non-zero charge, there will be some force between these two particles, given by Coulomb's Law (1.1).

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3} \tag{1.1}$$

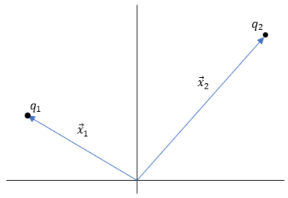

Figure 1.1: Force between two charged particles

There are a couple important takeaways here. First, we note that this is an inverse square law. Looking at the gravitational force law, we also see an inverse square law, which leads us to question if there is some connection between gravitational force and electromagnetic force. The answer is that we don't know. Actually, since I've brought up gravity, it might be worth mentioning here that we can ignore gravitational effects in this course since electromagnetic force is something on the order of 10^{42} times greater in magnitude than the force of gravity. One could then ask why even bother with gravity at all if it's so small. Mass is never negative, so as distances are increased, the amount of mass encompassed also increases. Meanwhile, electric charges can be positive or negative. That is, they will effectively cancel out over these distance scales. If q_1 and q_2 have the same sign, the two charges repel. Conversely, if they have opposite signs, the charges attract.

We also want to make note of ϵ_0, the permittivity of free space 1.2). This can sort of be thought of as the strength of electric interactions in a medium (here specifically, vacuum). As we will see in future chapters, this constant can change if we look at different materials.

$$\epsilon_0 \approx 8.85 \times 10^{-12} \ \frac{C^2}{Nm^2} \tag{1.2}$$

Finally, we want to make note of the 4π constant. We may be inclined to ask if there is some deep physical meaning behind that 4π (perhaps something to do with surface area and force radiating outwards from a single point?). There is not. Rather, it emerges from our definition of the Coulomb (C), which in itself, carries little information about the fundamental nature of the universe. A more physical unit might be the electron charge. In nature, charges are always quantized, they come in some integer multiple of the electron charge (ignoring quarks and fractional quantum hall effect, but there are other factors at play there). However, there is nothing in our construction of electromagnetism that prevents charges from being continuous distributions, and indeed, we can convince ourselves that in the macroscopic limit, it is fine to use continuous charge distributions.

Now, what if we have multiple fixed point charges q_i, each at a position $\vec{x}_i$? How do we fin the force on a particle with charge q at $\vec{x}$? We simply use superposition, which is the principle of summing the force from each particle individually to get the net force of the system. The reason we're allowed to use superposition here is because the interaction between any two charges is completely unaffected by the other charges as per our definition of electrostatics. If all the particles were free to move around, this problem would involve some time-dependent component, and would be more complicated. To answer the question posed at the beginning of this paragraph, the force is given by

$$\vec{F}(\vec{x}) = \frac{q}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3} \tag{1.3}$$

1.1.2 Electric Field

There are two ways to think about electromagnetism. Particles, such as electrons an protons, can be seen as fundamental constituents of nature. In this picture, charged particles create an electric field which dictates how other particles react. On the other hand, we could instead view the field as the fundamental object, creating particles when that field is quantized. In graduate level courses, we want to move from the first picture to the second. This is actually true in general for graduate level courses: fields are fundamentally how the universe works.

Imagine that we have some electric field $\vec{E}$. We can define the electric field as the force per unit charge acting at a given point (1.4). Given this, we can define the electric field due to a single particle in first quantization (1.5) by comparing to Coulomb's Law (1.1). Similarly, if we have a collection of particles, we can find the electric field (1.6) by using superposition (1.3).

$$\vec{F} = q\vec{E} \tag{1.4}$$

$$\vec{E}(\vec{x}) = \frac{q}{4\pi\epsilon_0}\frac{\vec{x}-\vec{x}_1}{|\vec{x}-\vec{x}_1|^3} \tag{1.5}$$

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i\frac{\vec{x}-\vec{x}_i}{|\vec{x}-\vec{x}_i|^3} \tag{1.6}$$

So far, we've been working with discrete charges, but at the macroscopic level, it is often easier to work with charge distributions. We don't really have to worry about quantized particles until we get to the quantum level. Starting from equation (1.6), we make the substitution $q \to \rho(\vec{x}')$ and integrate. This gives us the electric field for any arbitrary charge distribution (1.7). N.B. $\vec{x}'$ denotes the location of the source and $\vec{x}$ denotes the location of the observer.

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}d^3x' \tag{1.7}$$

1.1.3 Dirac Delta Function and Heaviside Function

When we moved from a discrete world (1.6) to a continuous one (1.7), we appear to have thrown out the ability to find the electric field due to discrete charges, but it turns out we can write those discrete charges using the Dirac delta function (1.8).

$$\delta(x-a) = \begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases} \tag{1.8}$$

Even though the Dirac delta function is infinite, it is only infinitely large for an infinitesimally small amount of time. Thus, when integrated, the Dirac delta function returns 1 (1.9). We can then convince ourselves that if we multiply the delta function by some function $f(x)$, the value $f(a)$ is picked out as long as the region of integration includes $x = a$ (1.10).

$$\int \delta(x-a)\ dx = \begin{cases} 0, & if\ region\ of\ integration\ does\ not\ include\ x = a \\ 1, & if\ region\ of\ integration\ includes\ x = a \end{cases} \tag{1.9}$$

$$\int f(x)\delta(x-a)\ dx = f(a) \tag{1.10}$$

Let's now look at some properties of the Dirac delta function. What happens if we take the derivative of the delta function? We can convince ourselves that this is ill-defined at $x = a$; however, we can find the integral of the derivative of the delta function.

$$\int f(x)\delta'(x-a)\ dx$$

Using integration of parts, setting $u = f(x)$ and $dv = \delta'(x-a)$,

$$= f(x)\delta(x-a) - \int f'(x)\delta(x-a)\ dx$$

The first term disappears since we evaluate it at the bounds, which are usually set to $-\infty$ and ∞. Thus, we find that the analog of equation (1.10) with the derivative of a delta function picks out the value of the derivative of the other function at $x = a$ (1.11).

$$\int f(x)\delta'(x-a)\ dx = -f'(a) \tag{1.11}$$

Now, what happens if a function $f(x)$ is the argument for the delta function?

$$\int \delta(f(x))\ dx$$

We can convince ourselves that the delta function should have non-zero values for the zeroes of $f(x)$, which we say occur at $x = x_i$. Using u-substitution setting $u = f(x)$,

$$= \int \delta(u) \frac{1}{\frac{d}{dx}f(x)}\ du$$

We have an integral on both sides, so we can take that out. We then pick out the values where the function is non-zero (1.12).

$$\delta(f(x)) = \sum_i \frac{1}{\left|\frac{d}{dx}f(x_i)\right|}\delta(x-x_i) \tag{1.12}$$

Next, we introduce the Heaviside function (1.13), which is the integral of the delta function. We can think of this as a function that "turns on".

$$\Theta(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \tag{1.13}$$

To connect this all to actual physics, if we want to write the charge density due to a discrete set of point charges, we use the Dirac delta function (1.8). We see that if we integrate this, it gives us a charge distribution with quantized points.

$$\rho(\vec{x}) = \sum_{i=1}^{n} q_i\ \delta(\vec{x} - \vec{x}_i)$$

If we have a constant charge density ρ smeared on a disk of radius b, we can use the Heaviside function (1.13). Here, when integrated, we can ignore $r > b$ since the Heaviside function starts turned on and then turns off at that boundary.

$$\rho(\vec{x}) = \rho\ \Theta(b - r)$$

1.1.4 Example: Electric Field due to an Infinite Line of Charge

Imagine we have a line of infinite length carrying a uniform charge density λ in the $\hat{x}$ direction. We want to know what the electric field $\vec{x} = (0, 0, a)$ is.

For the charge density, we want it to have a magnitude of λ and to be 0 at $y \neq 0$ or $z \neq 0$. Using the Dirac delta function (1.8),

$$\rho(\vec{x}) = \lambda\ \delta(y)\ \delta(z)$$

We can then use equation (1.7) to find the electric field.

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda\delta(y)\delta(z) \frac{a\hat{z} - x\hat{x}}{(a^2 + x^2)^{3/2}}\ dx\ dy\ dz$$

$$= \frac{\lambda}{4\pi\epsilon_0} \left(\int_{-\infty}^{\infty} \frac{a\hat{z}}{(a^2 + x^2)^{3/2}}\ dx - \int_{-\infty}^{\infty} \frac{x\hat{x}}{(a^2 + x^2)^{3/2}}\ dx \right)$$

$$\vec{E}(\vec{x}) = \frac{\lambda}{2\pi\epsilon_0\ a} \hat{z}$$

You should now be able to do Jackson 1.2 and 1.3.

1.2 Gauss's Law

Even though we can equation (1.7) to determine the electric field given any arbitrary configuration of charges, actually working out that integral can sometimes get a little nasty. For example, imagine we have a sphere of uniform of charge density. This is a rather messy problem (see Griffiths 2.7), and I don't particularly want to work it out here. In this section, we will introduce a method for finding the electric field given certain symmetries.

1.2.1 Derivation of Gauss's Law

Figure (1.2) shows a point charge q enclosed by a surface S. We let r denote the distance from the charge to a surface element da. $\hat{n}$ is the unit vector normal to the surface, and θ is the angle between $\hat{n}$ and $\vec{E}$. From these quantities, we can write the solid angle element (1.14). N.B., we assume the charge is inside the surface, but the derivation s the same if the charge is outside the surface sans a negative sign since the electric field is now pointing into the surface.

$$d\Omega = \frac{cos(\theta)}{r^2} da \tag{1.14}$$

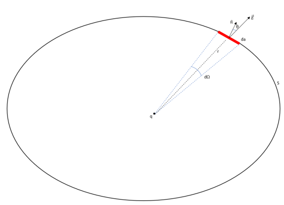

Figure 1.2: Terms used in Gauss's Law

Since we're looking at just a single particle, we can use the electric field due to a single particle (1.5) and dot it with the normal vector,

$$\vec{E} \cdot \hat{n} = \frac{q}{4\pi\epsilon_0 r^2} cos(\theta)\ da$$

Substituting in the solid angle element (1.14),

$$= \frac{q}{4\pi\epsilon_0} d\Omega$$

Integrating over the entire surface,

$$\oint_S \vec{E} \cdot \hat{n}\ da = \int_0^{2\pi} \int_{-1}^{1} \frac{q}{4\pi\epsilon_0} d(cos(\theta))\ d\phi$$

$$\oint_S \vec{E} \cdot \hat{n}\ da = \begin{cases} \frac{q}{\epsilon_0}, & if\ q\ lies\ inside\ S \\ 0, & if\ q\ lies\ outside\ S \end{cases} \tag{1.15}$$

We can imagine this as the flux of the electric field through the surface. We can convince ourselves that only the charges inside the surface will contribute since charges outside of the surface will pass in one side and out the other.

So now, what happens if rather than a single charge, we have a charge density? In this case, we will get the integral form of Gauss's Law (1.16). To make the conversion, we use the map from before, $q \rightarrow \rho(\vec{x})\ d^3x$. Of note, equation (1.16) also works with gravitational forces (up to some constants and with matter density instead of charge density).

$$\oint_S \vec{E}\hat{n}\ da = \frac{1}{\epsilon_0} \int_V \rho(\vec{x})\ d^3x \tag{1.16}$$

A more compact form of Gauss's Law can be obtained by using the divergence theorem (1.17). This gives us the form we are used to from Maxwell's equation (1.18).

$$\oint_S \vec{A} \cdot \hat{n} \; da = \int_V \nabla \cdot \vec{A} \; d^3x \tag{1.17}$$

$$\nabla \cdot \vec{E} = \frac{\rho(\vec{x})}{\epsilon_0} \tag{1.18}$$

N.B., while Gauss's Law is always true, it is not always useful. Specifically, there are three symmetries for which Gauss's Law is useful: spherical, cylindrical, or plane symmetry. For these three symmetries, we want to create a Gaussian surface such that the electric field is always normal to the surface. If we have one of these symmetries, Gauss's Law becomes

$$E \cdot (Area \; of \; surface) = \frac{q_{enclosed}}{\epsilon_0} \tag{1.19}$$

Let's look at a couple of examples.

1.2.2 Symmetric Gauss's Law: Uniform Sphere

If we have spherical symmetry, we want to use a concentric sphere as the Gaussian surface. Imagine we have a sphere of uniform charge density with radius R with total charge Q. We can show that the charge distribution is

$$\rho(\vec{x}) = \frac{3Q}{4\pi R^3}\Theta(R - r)$$

Looking at the charge distribution, we see that the general form of the electric field changes based on whether we are inside the sphere or outside. If we are outside the sphere, all of the charge is inside of our Gaussian surface. Gauss's Law tells us

$$\vec{E}_{out}(\vec{x}) \cdot (4\pi r^2) = \frac{Q}{\epsilon_0}\hat{r}$$

$$\vec{E}_{out}(\vec{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

However, if we are inside the sphere, only part of the charge is enclosed.

$$q_{enclosed} = \int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} \rho(\vec{x}) r^2 \; dr \; d(cos(\theta)) \; d\phi$$

$$= 4\pi \frac{3Q}{4\pi R^3} \int_0^r r^2 \; dr$$

$$q_{enclosed} = \frac{Qr^3}{R^3}$$

Using this, the electric field produced by a sphere of uniform charge,

$$\vec{E}_{in}(\vec{x}) \cdot (4\pi r^2) = \frac{Qr^3}{\epsilon_0 R^3}\hat{r}$$

$$\vec{E}_{in}(\vec{x}) = \frac{Qr}{4\pi\epsilon_0 R^3}\hat{r}$$

We see that if we start at the origin and move outwards, the electric field increases as we encompass more charge until we reach the edge of the sphere. At that point, as we move outwards, we increase the distance, but the charge encompassed remains constant, leading to a decrease in the electric field. Furthermore, we compare E_{out} to the electric field due to a point charge and note that as long as we are outside of the sphere, we can treat it as a point particle with charge Q.

One thing important to remember is that when we are inside the sphere, the charge outside of our Gaussian surface still exists. We ignore it, when calculating the electric field, but when we go to calculate the potential, we can no longer disregard this charge.

1.2.3 Symmetric Gauss's Law: Line Charge

Now imagine that we have an infinitely long wire along the z-axis. We paint a charge density λ along the wire. Since we have cylindrical symmetry, we want our Gaussian surface to be a cylinder of length L and radius r.

$$\vec{E}(\vec{x}) \cdot (2\pi r L) = \frac{L\lambda}{\epsilon_0}\hat{\rho}$$

$$\vec{E}(\vec{x}) = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\rho}$$

N.B., we don't care about the end caps of the cylinder since the normal points in the same direction as the wire. Further, this methodology only works for an infinitely long wire (or if we are sufficiently far from the ends). These sorts of edge effects were not covered in undergraduate courses, but now that we're in graduate school, we can. In a later chapter.

1.2.4 Symmetric Gauss's Law: Parallel Plates

Finally, imagine we have two infinite parallel conducting plates holding equal and opposite charge densities of magnitude σ. We'll say that the positive surface charge density is at $z = 0$ an the negative surface charge density is at $z = a$. Here we have plane symmetry, so our Gaussian surface is going to be a pillbox of height h and surface area A. For a single plate,

$$\vec{E}(\vec{x}) \cdot (2A) = \frac{\sigma A}{\epsilon_0}\hat{z}$$

$$\vec{E}(\vec{x}) = \frac{\sigma}{2\epsilon_0}\hat{z}$$

N.B., we use $2A$ since we count the surface above and below the plate. Interestingly enough, we see that the field is independent of the distance from the plane. This has important implications when we have two plates near each other. We can show that the electric field outside of the parallel plate is 0 while the field between the two plates is

$$\vec{E}(\vec{x}) = \frac{\sigma}{\epsilon_0}\hat{z}$$

This is a fairly common configuration known as the parallel plate capacitor.

You should now be able to do Jackson 1.4.

1.3 Scalar Potential

In the previous section, we saw that we can use Gauss's Law (1.18) to find the electric field given a symmetric charge distribution. Naturally, we ask how to determine the electric field given a non-symmetric charge distribution. A vector field can be specified almost completely if both the divergence and curl are known. Gauss's Law (1.18) gives us the divergence of the electric field, so all we have to do now is find the curl of the electric field.

1.3.1 Faraday's Law

Before finding the curl, it might be useful to show another relation first. I posit that the negative gradient of $1/|\vec{x}-\vec{x}'|$, or $1/r$, is equal to $\hat{r}/r^2$ (1.20). We can show this in one of two ways. The first is through brute force. Alternatively, we can think of the gradient as a derivative, and the result pops out relatively simply.

$$\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) \tag{1.20}$$

Inserting this into the equation for the electric field (1.7),

$$\vec{E}(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\ \nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\ d^3x'$$

Since the gradient operation acts on $\vec{x}$, but the integration variable is $\vec{x}'$, we can take the gradient operator outside of the integral.

$$= -\frac{1}{4\pi\epsilon_0}\nabla\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'$$

Taking the curl, we see that the curl of the electric field is 0 since the curl of the gradient of a function of position always vanishes 1.21). This is Faraday's Law, another one of Maxwell's equations. Well, a reduced form that we will complete once we get to magnetodynamics.

$$\nabla\times\vec{E} = 0 \tag{1.21}$$

We also note that we can write the electric field as the gradient of some scalar function, which we call the scalar potential, Φ (1.22). We can write the scalar potential (1.23) in a format similar to equation (1.7).

$$\vec{E} = -\nabla\Phi \tag{1.22}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x})}{|\vec{x}-\vec{x}'|}d^3x' \tag{1.23}$$

1.3.2 Alternate Derivation of Faraday's Law

In undergraduate EM, we think of the scalar potential as being the work needed to move a charge q from point A to point B in the presence of an electric field $\vec{E}$. We know that the charge should feel a force from equation (1.4), so the work done on the charge is given by

$$W = -\int_A^B \vec{F}\cdot d\vec{l} = -q\int_A^B \vec{E}\cdot d\vec{l}$$

Inserting equation (1.22),

$$W = q\int_A^B \nabla\Phi \cdot d\vec{l}$$

$$= q\int_A^B d\Phi = q(\Phi_B - \Phi_A)$$

If we take the integral over a closed path, it comes out to zero since we end up back where we started. Note that this is true regardless of the actual path taken. This gives us the integral form of Faraday's Law (1.24). By applying Stokes's theorem (1.25), we are able to return Faraday's Law (1.21).

$$\oint \vec{E} \cdot d\vec{l} = 0 \tag{1.24}$$

$$\oint_C \vec{A} \cdot d\vec{l} = \int_S (\nabla \times \vec{A}) \cdot \hat{n}\ da \tag{1.25}$$

1.3.3 A Note on the Scalar Potential

The scalar potential is defined up to the addition of a constant. This constant actually leads to the concept of gauge invariance, which we will touch on in a later chapter.

If we treat the scalar potential as the work done on a charge, we need to have some baseline scalar potential to which we compare. By convention, we set this comparison point infinitely far away, where the potential is zero.

When writing the scalar potential as a function of charge density (1.23), we compared equations (1.22) and (1.7). However, this method doesn't always work. If we have some complicated topology (space is not three-dimensional), we run into some issues. We will not be dealing with these issues, and we will always assume we are in three-dimensional space.

1.3.4 Example: Scalar Potential of a Sphere

In the previous section, we found the electric field due to a sphere of uniform charge density. As a reminder,

$$\vec{E}_{out}(\vec{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

$$\vec{E}_{in}(\vec{x}) = \frac{Qr}{4\pi\epsilon_0 R^3}\hat{r}$$

Outside the sphere, we can use equation (1.22) to determine the potential.

$$\frac{Q}{4\pi\epsilon_0 r^2} = -\frac{\partial\Phi_{out}}{\partial r}$$

$$\Phi_{out} = \frac{Q}{4\pi\epsilon_0 r}$$

We recognize this as the potential due to a point charge Q centered at the origin. Inside the sphere, we cannot use the same methodology. We have to account for the charges that we ignored when we set the Gaussian surface. In this, we need to calculate the work needed to bring a charge from infinity to a point r inside the surface. We split the integral into two parts: going from infinity to the edge of the sphere and from the edge of the sphere to the desired point.

$$\Phi_{in} = -\int_{\infty}^{R} \frac{Q}{4\pi\epsilon_0 r^2} dr - \int_{R}^{r} \frac{Qr}{4\pi\epsilon_0 R^3} dr$$

$$\Phi_{in} = \frac{Q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

You should now be able to do Jackson 1.1, 1.6, 1.7, and 1.9.

1.4 Electric Field Discontinuity

The study of graduate level electrostatics largely focuses on a class of problems known as boundary-value problems. In order to solve these types of problems, we need boundary conditions, which we will lay out here. Most boundary-value problems can be solved with the information from this chapter (with varying degrees of leg work).

1.4.1 Electric Field Behaviour at the Boundary

The electric field is not continuous at the boundary. For simplicity, we'll say that there is only a surface charge density σ on the boundary. Using Gauss's Law (1.18),

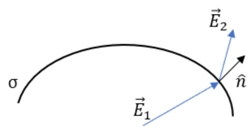

Figure 1.3: Surface with charge density

$$\vec{E}_1 \cdot \hat{n} = 0$$

$$\vec{E}_2 \cdot \hat{n} = \frac{\sigma}{\epsilon_0}$$

From this, it can be seen that there is a normal discontinuity in the electric field (1.26). This holds true for any arbitrary boundary and for any arbitrary charge density configuration. N.B., this argument holds true even if there is some additional charge density since it will contribute to both $\vec{E}_1$ and $\vec{E}_1$ equally.

$$(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = \frac{\sigma}{\epsilon_0} \tag{1.26}$$

1.4.2 Potential Behaviour at the Boundary

Using Faraday's Law (1.21),

$$\vec{E}_1 \times \hat{n} = 0$$

$$\vec{E}_2 \times \hat{n} = 0$$

The electric field tangential to the surface is continuous, which translates to continuity in the potential (1.27).

$$\Phi_2 - \Phi_1 = 0 \tag{1.27}$$

1.4.3 Example: Discontinuity at the Boundary of a Sphere

Let's look at a charged spherical shell of radius R with a total charge Q smeared on the surface. The surface charge density is given by

$$\sigma = \frac{Q}{4\pi R^2}$$

Using Gauss's Law (1.18),

$$E_{in} = 0$$

$$E_{out} = \frac{Q}{4\pi\epsilon_0 r^2}$$

From this, we can find the potential (1.22),

$$\Phi_{in} = \frac{Q}{4\pi\epsilon_0 R}$$

$$\Phi_{out} = \frac{Q}{4\pi\epsilon_0 r}$$

We can clearly see that the electric field is discontinuous, as in agreement with equation (1.26), while the potential is continuous, as in agreement with equation (1.27), at the boundary.

You should now be able to do Jackson 1.11.

1.5 Electrostatic Potential Energy

An electric field stores energy. We can convince ourselves that this is true by constructing the corresponding charge distribution. This requires work, which translates to energy needed to construct the system. Then, using conservation of energy, we can convince ourselves that this is equal to the energy needed to prevent the charges from flying apart once they've been brought together.

1.5.1 Electrostatic Potential Energy of a Discrete Charge Distribution

In order to construct a charge distribution made up of point charges, we bring charges in sequentially from infinity until we have the desired configuration. Say that we have some charge distribution already, or more importantly, an electric field and associated potential which vanishes at infinity. The work required to bring a charge q from infinity to a point $\vec{r}$ is given by

$$W = -\int_{\infty}^{\vec{r}} \vec{F} \cdot d\vec{l}$$

Using equation (1.4) to convert the force into an electric field, and then using equation (1.22) to convert from the electric field to the scalar potential,

$$W = q \int_{\infty}^{\vec{r}} \nabla\Phi \cdot d\vec{l}$$

$$W = q\Phi(\vec{x}) \tag{1.28}$$

Now, let's take our result and work backwards to construct our charge distribution from void. Bringing in the first charge is free (since we say there is no underlying vacuum electric field). To place the second charge, the work required is

$$W_2 = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_1 - \vec{r}_2|}$$

Note that if the two charges have the same sign, $W_2 > 0$, which tells us we need to put work into the system to push the two charges together. If they have different signs, $W_2 < 0$, we need to do work to keep them apart. If we then want to bring a third charge in,

$$W_3 = \frac{q_3}{4\pi\epsilon_0} \left(\frac{q_2}{|\vec{r}_2 - \vec{r}_3|} + \frac{q_1}{|\vec{r}_1 - \vec{r}_3|} \right)$$

If we continue on in this manner, we can convince ourselves that the total work needed is (1.29). We can actually rewrite this more symmetrically if we intentionally over count and then divide by 2 (1.30). Note that we have to avoid $i = j$ terms since that would count the energy due to two particles occupying the same space, effectively resulting in infinite energy.

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \tag{1.29}$$

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i} \sum_{j\neq i} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \tag{1.30}$$

1.5.2 Electrostatic Potential Energy of a Continuous Charge Distribution

In moving from a discrete world to a continuous one, we again utilize the same tricks (i.e. $q \to \rho(\vec{x})\ d^3x$). Equation (1.30) turns into (1.31). Using equation (1.23), we can write the electrostatic potential energy due to some scalar potential (1.32).

$$W = \frac{1}{8\pi\epsilon_0}\int\int \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\ d^3x\ d^3x' \tag{1.31}$$

$$W = \frac{1}{2}\int \rho(\vec{x})\Phi(\vec{x})\ d^3x \tag{1.32}$$

If we want to write the energy due to an electric field (1.33), we start by applying Poisson's equation (1.35).

$$W = -\frac{\epsilon_0}{2}\int \Phi(\vec{x})\nabla^2\Phi(\vec{x})\ d^3x$$

Integrating by parts, setting $u = \Phi(\vec{x})$ and $dv = \nabla^2\Phi(\vec{x})$,

$$W = -\frac{\epsilon_0}{2}\left(\Phi\nabla\Phi - \int \nabla\Phi\cdot\nabla\Phi\ d^3x\right)$$

The first term disappears due to the condition that the potential should vanish at infinity. Using equation (1.23) on the second term,

$$W = \frac{\epsilon_0}{2}\int |\vec{E}|^2\ d^3x \tag{1.33}$$

If we integrate over a unit volume, we will then get the energy density

$$w = \frac{\epsilon_0}{2}|\vec{E}|^2 \tag{1.34}$$

1.5.3 Example: Electrostatic Energy due to Two Point Charges

Imagine we have two charges, q_1 and q_2, located at $\vec{x}_1$ and $\vec{x}_2$ respectively. The electric field at $\vec{x}$ is given by

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(q_1\frac{\vec{x}-\vec{x}_1}{|\vec{x}-\vec{x}_1|^3} + q_2\frac{\vec{x}-\vec{x}_2}{|\vec{x}-\vec{x}_2|^3}\right)$$

Using equation (1.33),

$$W = \frac{1}{32\pi^2\epsilon_0}\int \frac{q_1^2}{(x-x_1)^3} + \frac{q_2^2}{(x-x_2)^3} + \frac{2q_1q_2(\vec{x}-\vec{x}_1)\cdot(\vec{x}-\vec{x}_2)}{|\vec{x}-\vec{x}_1|^3\ |\vec{x}-\vec{x}_2|^3}\ d^3x$$

$$= -\frac{1}{32\pi^2\epsilon_0}\frac{q_1^2}{3|\vec{x}-\vec{x}_1|^3} - \frac{1}{32\pi^2\epsilon_0}\frac{q_2^2}{3|\vec{x}-\vec{x}_2|^3} + \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{|\vec{x}_1-\vec{x}_2|}$$

If we instead use the discrete formulas (1.30),

$$W = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|}$$

These don't match, and this is because we actually have some self-energy terms in the continuous case. The first two terms are those self-energy terms, and the third term gives us the proper interaction energy. When working out problems, this is something we need to keep an eye out for.

1.5.4 Example: Energy Stored in a Sphere

Going back to our previous example of a sphere of uniform charge density. As a reminder,

$$\rho(\vec{x}) = \frac{3Q}{4\pi R^3}\Theta(R - r)$$

$$\Phi_{in} = \frac{Q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

We expect that our charge configuration should only have energy inside of the sphere, so we will limit our integration to that region. Using equation (1.32),

$$W = \frac{1}{2}\int_0^{2\pi}\int_{-1}^{1}\int_0^{R}\frac{3Q}{4\pi R^3}\frac{Q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) r^2 \ dr \ d(cos(\theta)) \ d\phi$$

$$W = \frac{3Q^2}{20\pi\epsilon_0 R}$$

You should now be able to do Jackson 1.8.

1.6 Poisson and Laplace Equations

Gauss's law (1.18) gives us the relation between the electric field and the charge density. Using equation (1.22), we can relate the electric field and the scalar potential. If we combine the two, we should be able to relate the scalar potential and charge density. This equation is particularly important to electrostatics, going so far as to garner its own name: Poisson's equation an Laplace's equation.

1.6.1 Statement of Poisson and Laplace Equations

Poisson's equation is given in (1.35). If there is no charge density, this simplifies to Laplace's equation (1.36). Now, near as I can tell, there is no reason not to use Poisson's equation in every case, sort of like how a Maclaurin series is a Taylor series about 0.

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \tag{1.35}$$

$$\nabla^2 \Phi = 0 \tag{1.36}$$

1.6.2 Example: Poisson's Equation and Scalar Potential

Let's make sure that the explicit form of the scalar potential (1.23) follows Poisson's equation. First, we note the identity (1.37).

$$\nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}'|^2} \right) = -4\pi \ \delta(\vec{x} - \vec{x}') \tag{1.37}$$

Given this,

$$\nabla^2 \Phi = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}') \nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \ d^3x'$$

$$= \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}')(-4\pi\delta(\vec{x} - \vec{x}')) \ d^3x'$$

$$= -\frac{1}{\epsilon_0} \int \rho(\vec{x}')\delta(\vec{x} - \vec{x}') \ d^3x'$$

$$\nabla^2 \Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0}$$

Thus we see that in general, the scalar potential satisfies Poisson's equation, which is what we expected. At the moment, the Poisson and Laplace equations may seem a little silly, but they turn out to be rather fundamental to the study of electrostatics. In fact, the next two chapters are just different applications of Poisson equation.

You should now be able to do Jackson 1.5.

1.7 Green's Theorem

In order to deal with charges in finite regions of space, we will need to develop new methods, starting with Green's Theorem. You have likely seen an elementary boundary condition problem already. Imagine we have a conductor that we connect to the Earth, which for this example, we say is a reservoir containing infinite charge. Charge will then flow in and out of the conductor to maintain a constant potential (which we will declare to be 0 by convention). This constant potential is an example of a boundary condition, also referred to as grounding. In practice, we do not need to connect every electronic device to the planet. We can, for example, ground the conductors inside of cell phones by attaching them by a small metal wire to the metal casing.

1.7.1 Derivation of Green's Theorem

We'll start by defining some vector field

$$\vec{A} = \phi\nabla\psi$$

where ϕ and ψ are arbitrary scalar fields. Inserting this into the divergence theorem (1.17), we get Green's first identity (1.38). Interchanging ϕ and ψ and subtracting the two results in Green's second identity, or Green's theorem.

$$\int_V \nabla\cdot(\phi\nabla\psi)\ d^3x = \oint(\phi\nabla\psi)\cdot\hat{n}\ da$$

$$\int_V \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\ d^3x = \oint_S \phi\frac{\partial\psi}{\partial n}\ da \tag{1.38}$$

$$\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi) - (\psi\nabla^2\phi + \nabla\phi\cdot\nabla\psi)\ d^3x = \oint_S \phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\ da$$

$$\int_V \phi\nabla^2\psi - \psi\nabla^2\phi\ d^3x = \oint_S \phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\ da \tag{1.39}$$

1.7.2 Example: Scalar Potential and Green's Theorem

Let's apply Green's theorem to the following fields:

$$\phi = \Phi$$

$$\psi = \frac{1}{R}$$

Substituting these fields in,

$$\int_V \Phi\nabla'^2\frac{1}{R} - \frac{1}{R}\nabla'^2\Phi\ d^3x' = \oint_S \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial\Phi}{\partial n'}\ da'$$

Using Poisson's equation (1.35) and equation (1.37),

$$\int_V \Phi(\vec{x}')(-4\pi\delta(\vec{x}-\vec{x}') - \frac{1}{R}\frac{-\rho(\vec{x}')}{\epsilon_0} = \oint_S \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial \Phi}{\partial n'}\ da'$$

$$-4\pi\Phi(\vec{x}) + \frac{1}{\epsilon_0}\int_V \frac{\rho(\vec{x}')}{R}\ d^3x' = \oint_S \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial \Phi}{\partial n'}\ da'$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')}{R} d^3x' + \frac{1}{4\pi}\oint_S \frac{1}{R}\frac{\partial \Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\ da'$$

What we notice is that if we let the surface S go to infinity, the surface integral dies and we are left with just the equation for scalar potential (1.23). This is to be expected since by setting the potential to 0 at infinity, we are setting a boundary condition, albeit a slightly less tangible one than expected. If we are working in charge free space, the potential becomes a function of the potential on the surface and the normal derivative of the potential on the surface. It turns out, we have two types of boundary conditions which correspond to these two cases.

1.7.3 Dirichlet and Neumann Boundary Conditions

We can categorize boundary conditions in one of two ways. If we fix the value of the potential on the surface, we have Dirichlet boundary conditions. If we fix the value of $\nabla\Phi\cdot\hat{n}$, or electric field (1.22), we have Neumann boundary conditions. As it turns out, we need only determine either Dirichlet or Neumann boundary conditions; if we have both, we end up over-specifying the problem.

Let us now prove that given a boundary condition, we have a unique solution. In order to do this, we suppose there exist two potentials, Φ_1 and Φ_2, both of which satisfy the same boundary condition. We define

$$U = \Phi_2 - \Phi_1$$

For both boundary conditions, Poisson's equation (1.35) should hold true.

$$\nabla^2\Phi_2 = \nabla^2\Phi_1$$

$$\nabla^2 U = 0$$

Furthermore, $U = 0$ or $\frac{\partial U}{\partial n} = 0$ on the surface for Dirichlet or Neumann boundary conditions respectively. Using Green's first identity (1.38), with $\phi = \psi = U$,

$$\int_V U\nabla^2 U + \nabla U\cdot\nabla U\ d^3x = \oint_S U\frac{\partial U}{\partial n}\ da$$

$$\int_V |\nabla U|^2\ d^3x = 0$$

$$\nabla U = 0$$

Since U is a constant, and since $U = 0$, this implies that $\Phi_1 = \Phi_2$ everywhere.

1.7.4 Green Functions

The solutions to the Poisson equations are given by a class of functions known as Green functions (1.40), which in general follow the condition given in equation (1.42).

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}') \tag{1.40}$$

$$\nabla'^2 F(\vec{x},\vec{x}') = 0 \tag{1.41}$$

$$\nabla'^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}') \tag{1.42}$$

We can quickly see that $1/r$ is a Green function with $F(\vec{x},\vec{x}') = 0$. To see how we use Green functions to determine the potential given specific boundary conditions, we turn to Green's theorem (1.39) using $\phi = \Phi$ and $\psi = G(\vec{x},\vec{x}')$.

$$\int_V \Phi(\vec{x}')\ \nabla^2 G(\vec{x},\vec{x}') - G(\vec{x},\vec{x}')\ \nabla^2\Phi(\vec{x}')\ d^3x' = \oint_S \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} - G(\vec{x},\vec{x}')\frac{\partial \Phi(\vec{x}')}{\partial n'}\ da'$$

Using Poisson's equation (1.35) and equation (1.42),

$$\int_V -4\pi\Phi(\vec{x}')\delta(\vec{x}-\vec{x}') + G(\vec{x},\vec{x}')\frac{\rho(\vec{x}')}{\epsilon_0}\ d^3x' = \oint_S \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} - G(\vec{x},\vec{x}')\frac{\partial \Phi(\vec{x}')}{\partial n'}\ da'$$

$$-4\pi\Phi(\vec{x}) + \int_V G(\vec{x},\vec{x}')\frac{\rho(\vec{x}')}{\epsilon_0}\ d^3x' = \oint_S \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} - G(\vec{x},\vec{x'})\frac{\partial \Phi(\vec{x}')}{\partial n'}\ da'$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G(\vec{x},\vec{x}')\ d^3x' + \frac{1}{4\pi}\oint_S G(\vec{x},\vec{x}')\frac{\partial \Phi(\vec{x}')}{\partial n'} - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\ da' \tag{1.43}$$

This is the general solution to Poisson's equation (1.35). Note that this includes both Dirichlet and Neumann boundary conditions, but as we said before, we only need one. For Dirichlet boundary conditions, if we insert the Green's Dirichlet function (1.44), equation (1.43) simplifies to (1.45).

$$G_D(\vec{x},\vec{x}') = 0, \quad for\ \vec{x}'\ on\ S \tag{1.44}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G_D(\vec{x},\vec{x}')\ d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}\ da' \tag{1.45}$$

For Neumann boundary conditions, our condition is given by

$$\frac{\partial G_N(\vec{x},\vec{x}')}{\partial n'} = -\frac{4\pi}{S}, \quad for\ \vec{x}'\ on\ S \tag{1.46}$$

Plugging this in, equation (1.43) simplifies to

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G_N(\vec{x},\vec{x}')\ d^3x' + \frac{1}{4\pi}\oint_S G_N(\vec{x},\vec{x}')\frac{\partial \Phi(\vec{x}')}{\partial n'} + \Phi(\vec{x}')\frac{4\pi}{S}\ da'$$

$$\Phi(\vec{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G_N(\vec{x},\vec{x}')\ d^3x' + \frac{1}{4\pi}\oint_S G_N(\vec{x},\vec{x'})\frac{\partial \Phi(\vec{x}')}{\partial n'}da' \tag{1.47}$$

where $\langle\Phi\rangle_S$ is the average value of the potential over the surface. Now we theoretically know how to find the potential given any boundary condition. However, in practice, things are a little bit more complicated because the Green's function will depend on the given boundaries, a problem which shall be addressed in the next chapter.

1.7.5 Example: Green Function with Dirichlet Boundary Conditions

Imagine we have a particle with charge q at the point (x', y', z'). The boundary condition is that the potential must be 0 at infinity. I posit that the Green function is given by

$$G_D(x',y',z') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = \frac{1}{|\vec{x}-\vec{x}'|}$$

This fits the general form of the Green function (1.40), setting $F(\vec{x},\vec{x}')=0$. We also see that it satisfies equation (1.42) if we use equation (1.37). Plugging this into equation (1.45),

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x})}{|\vec{x}-\vec{x}'|}\, d^3x'$$

The second term in equation (1.45) drops out since the potential is 0 at the boundary, and we are left with the expected scalar potential. This is a simple example, but it serves to provide us with a relatively easy method of finding the Green function for a given charge distribution and Dirichlet boundary condition of the potential being 0 at infinity. We find the potential everywhere in space, then we remove the charge density term and the $4\pi\epsilon_0$ term.

You should now be able to do Jackson 1.10, 1.12, 1.13, and 1.14.

Chapter 2

Boundary-Value Problems in Electrostatics

"There is a place where time stands still. Raindrops hang motionless in air. Pendulums of clocks float mid-swings. Dogs raise their muzzles in silent howls. Pedestrians are frozen in the dusty streets, their legs cocked as if held by strings. The aromas of dates, mangoes, coriander, cumin are suspended in space"- (Alan Lightman, Einstein's Dreams)

In this chapter, we will deal with boundary-value problems. Specifically, we'll be looking at the method of images, and then we'll look at some simple separation of variables. I personally think that this chapter and the next could be combined into one (or maybe move the separation of variable stuff to the next chapter), but we want to stick with how Jackson does things since we only have one source here.

2.1 Method of Images

In undergraduate, your professor most likely method of images, proved that they had a unique solution, and then promptly moved on. Here, we're going to revisit the method of images and look at some more complicated geometries.

Imagine we have a conductor and some charge distribution, and we want to find the resultant potential. As a reminder, a conductor is an object that maintains a constant potential across its body. In general, the trick is to erase the conductor and draw some equivalent image charge distribution which satisfies the same boundary conditions. N.B. if you see a problem that mentions conductors, the first thing you should think is method of images.

2.1.1 Infinite Plane Conductor

This is the method of images problem that you most likely saw in undergraduate EM. Imagine we hold a point charge q some distance d above an infinite grounded conducting plane as shown in figure (2.1).

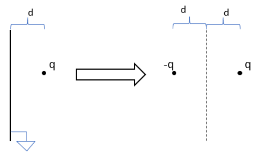

Figure 2.1: Method of images, plane conductor

The potential will not be just $\frac{q}{4\pi\epsilon_0 r}$ since the charge will induce some surface charge on the conductor. If you remember your undergraduate EM, the solution is to pretend the conductor isn't there and draw an image charge $-q$ at $-d$. We know we want to satisfy the condition that the potential is zero at $x = 0$ and vanishes at infinity. The potential of this configuration is fairly straightforward, it's just the potential due to two point charges (2.1). We can convince ourselves that this potential satisfies both of our boundary conditions by looking at it. One thing to note is that the thickness of the plate does not matter as long as we are looking at the region outside of the conductor. If we are inside the conductor, then our potential is zero. This doens't show up in the math; it's just something we have to remember and insert manually.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{q}{\sqrt{(x+d)^2+y^2+z^2}}\right) \tag{2.1}$$

The Green function of this system (2.2) is fairly straightforward if we remember that we only need to remove the charges and the $1/4\pi\epsilon_0$ factor.

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} + \vec{x}'|} \tag{2.2}$$

We mentioned that some surface charge is induced on the conductor, which begs the question: what is that induced surface charge? In order to answer this question, let's look at the system where our conductor is infinitely thick in the -x direction. This way, we can find the induced charge by looking at the discontinuity in the electric field (1.26). We take the negative of the normal derivative of the potential, multiply by ϵ_0, and we get our induced surface charge density (2.3).

$$\sigma(\vec{x}) = -\epsilon_0 \frac{\partial \Phi}{\partial x}\bigg|_{x=0} = \frac{q}{4\pi}\left(\frac{x-d}{\left((x-d)^2+y^2+z^2\right)^{3/2}} - \frac{x+d}{\left((x+d)^2+y^2+z^2\right)^{3/2}}\right)\bigg|_{x=0}$$

$$\sigma(\vec{x}) = -\frac{qd}{2\pi(d^2+y^2+z^2)^{3/2}} \tag{2.3}$$

If we integrate this induced charge over the entire surface,

$$q_{induced} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\frac{qd}{2\pi(d^2+y^2+z^2)^{3/2}}\ dy\ dz = -q$$

The charge induced on the conductor is the same as the image charge, and as it turns out, this will always be true when using method of images.

Because there is an induced charge on the conductor, we can reason that our charge will feel some force from the conductor. Since we can treat our conductor like the image charge, the real charge will fell the same attractive force towards the conductor as it would towards the image charge (2.4).

$$\vec{F}(\vec{x}) = -\frac{q^2}{16\pi\epsilon_0 d^2}\hat{x} \tag{2.4}$$

2.1.2 Grounded, Conducting Sphere

The other method of images problem we can solve easily is that of a grounded, conducting sphere of radius a. We place a point charge q at $\vec{y}$ and again ask, what is the resulting potential?

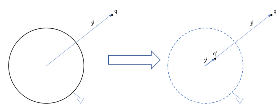

Figure 2.2: Method of images, conducting sphere

The boundary conditions here are that the potential must be 0 at $r = a$ and again must vanish as we go to infinity. We can convince ourselves that due to symmetry reasons, the image charge q' will lie on the line from the origin to q. For the purposes of this derivation, we will say that our charge lies outside the sphere. As can be shown, we actually get the same relation if our charge is inside the sphere.

If we have an image charge q' at $\vec{y}'$, the potential of this system is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|x\hat{n} - y\hat{n}'|} + \frac{q'}{|x\hat{n} - y'\hat{n}'|}\right)$$

We define $\hat{n}$ to be the unit vector in the direction $\vec{x}$ and $\hat{n}'$ to be the unit vector in the direction $\vec{y}$. We now want to find y' and q' such that the potential is 0 at $x = a$.

$$\Phi(x = a) = 0 = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a\left|\hat{n} - \frac{y}{a}\hat{n}'\right|} + \frac{q'}{y'\left|\hat{n}' - \frac{a}{y'}\hat{n}\right|}\right)$$

$$\frac{q}{a\left|\hat{n} - \frac{y}{a}\hat{n}'\right|} = -\frac{q'}{y'\left|\hat{n}' - \frac{a}{y'}\hat{n}\right|}$$

$$\frac{q}{a}\left(\frac{a}{y'} - 1\right) = -\frac{q'}{y'}\left(\frac{y}{a} - 1\right)$$

By setting the parts outside of the parenthesis equal to each other and doing likewise with the pieces inside the parenthesis,

$$q' = -\frac{a}{y}q \tag{2.5}$$

$$y' = \frac{a^2}{y} \tag{2.6}$$

As the charge is brought closer to the sphere, the image charge will grow larger and will move closer towards the surface to meet it. When the real charge is just outside the boundary, the image will have equal and opposite charge and lie just inside the boundary. Again, because this system reduces to two point charges, the potential is fairly straightforward (2.7). From this, we can get the Green function (2.8), which might be more useful in spherical coordinates (2.9) where γ is the angle between $\vec{x}$ and $\vec{x}'$.

$$\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{x} - \vec{x}'|} - \frac{a}{x'\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|}\right) \tag{2.7}$$

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'\left|\vec{x}' - \frac{a^2}{x'^2}\vec{x}'\right|} \tag{2.8}$$

$$G(\vec{x},\vec{x}') = \frac{1}{\sqrt{x^2 + x'^2 - 2xx'\ cos(\gamma)}} - \frac{1}{\sqrt{\frac{x^2x'^2}{a^2} + a^2 - 2xx'\ cos(\gamma)}} \tag{2.9}$$

Just as with the plane conductor, when we bring the charge close, it induces some surface charge density on the sphere, which in turn creates a force on the real charge.

$$\sigma(\vec{x}) = -\epsilon_0 \frac{\partial \Phi}{\partial a}\bigg|_{x=a} = \frac{q}{4\pi}\left(\frac{x - x'\ cos(\gamma)}{(x^2 + x'^2 - 2xx'\ cos(\gamma))^{3/2}} - \frac{\frac{xx'^2}{a^2} - x'\ cos(\gamma)}{\left(\frac{x^2x'^2}{a^2} + a^2 - 2xx'\ cos(\gamma)\right)^{3/2}}\right)\Bigg|_{x=a}$$

$$\sigma(\vec{x}) = -\frac{q}{4\pi a^2}\left(\frac{a}{x'}\right)\frac{1 - \frac{a^2}{x^2}}{\left(1 + \frac{a^2}{x'^2} - 2\frac{a}{x'}\ cos(\gamma)\right)^{3/2}} \tag{2.10}$$

$$\vec{F}(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\frac{q^2 a}{x'^3}\left(1 - \frac{a^2}{x'^2}\right)^{-2}\hat{x}' \tag{2.11}$$

These are really the two main method of image problems we can solve (I guess we could do the two-dimensional circle, but you can figure that out for yourself). Jackson does provide some more examples and applications of spherical method of images, so nothing doing.

2.1.3 Charged, Insulated, Conducting Sphere

Imagine we take the sphere from the previous section and unground it. We then place a charge Q on it (reminder that this charge will spread out since the sphere is a conductor), bring a point charge q near it, and ask what the resulting potential is. We can get to the solution through superposition. We draw two spheres: a grounded, conducting sphere in the presence of a charge q and a conducting sphere with charge $Q - q'$ painted on it. The reason we have a charge $Q - q'$ is because the first sphere will have a surface charge induced on it, which has a total charge q'.

The first sphere is the same configuration we solved in the previous section. Furthermore, we know that for the potential outside of the sphere, we can treat the charged conductor as a point charge at the origin. Thus, the total potential is given by equation (2.12). To find the force on the real charge, we look at the system as follows: we have the real charge q at $\vec{x}'$, a point charge $Q - q'$ at the origin, and an image charge q' at $\frac{a^2}{x'}\vec{x}'$. We then use Coulomb's law (1.1) to find the total force (2.13). We note that as we move the charge away from the sphere ($\vec{x}' \to \infty$), the force on the point charge reduces to Coulomb's law (1.1) for a point charge q at $\vec{x}'$ and a point charge Q at the origin.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{x}-\vec{x}'|} - \frac{aq}{x'\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|} + \frac{Q + \frac{a}{x'}q}{|\vec{x}|}\right) \tag{2.12}$$

$$\vec{F}(\vec{x}') = \frac{q}{4\pi\epsilon_0}\left(\frac{Q-q}{x'^2} - \frac{q'}{\left(x' - \frac{a^2}{x'}\right)^2}\right)\hat{x}'$$

$$\vec{F}(\vec{x}') = \frac{1}{4\pi\epsilon_0}\frac{q}{x'^2}\left(Q - \frac{qa^3(2x'^2 - a^2)}{x'(x'^2 - a^2)^2}\right)\hat{x}' \tag{2.13}$$

2.1.4 Conducting Sphere at Fixed Potential

Imagine that instead of specifying the charge on the sphere, we specify the potential. We can convince ourselves that the logic will be similar to that of the previous section. We have a superposition of a grounded, conducting sphere and a second conducting sphere with some charge on it. The first sphere, again, we can solve using equation (2.7). For the potential to be V at the boundary, we look at the potential of the charged sphere (2.12). The first two terms come from the grounded sphere, so at the boundary, they go to zero. This means that we simply have to replace the last term with something that is equal to V at the boundary to get the full potential (2.14). The force on the charged particle can be found in the same manner as the previous section (2.15).

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{x} - \vec{x}'|} - \frac{aq}{x'\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|}\right) + \frac{Va}{|\vec{x}|} \tag{2.14}$$

$$\vec{F}(\vec{x}) = \frac{q}{4\pi\epsilon_0}\left(\frac{Va}{x'^2} + \frac{q'}{\left(x' - \frac{a^2}{x'}\right)^2}\right)\hat{x}'$$

$$\vec{F}(\vec{x}) = \frac{q}{x'^2}\left(Va - \frac{1}{4\pi\epsilon_0}\frac{qax'^3}{(x'^2 - a^2)^2}\right)\hat{x}' \tag{2.15}$$

2.1.5 Conducting Sphere in a Uniform Electric Field

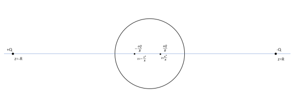

Figure 2.3: Method of images, uniform electric field

Imagine that we have a conducting sphere in a uniform electric field, E_0. Even though there aren't any charges immediately apparent with which to use method of images, not to worry. We can approximate our field as two charges $\pm Q$ placed at $z = \mp R$ respectively as shown in figure (2.3). If R is sufficiently large, we can create a uniform field parallel to the z-axis. To find the magnitude, we simply use equation (1.5). $E_0 = \frac{Q}{2\pi\epsilon_0 R^2}$.

We now have two charges, and we can use method of images. If our conducting sphere is located at the origin, we create image charges $\pm\frac{Qa}{R}$

at $z = \pm\frac{a^2}{R}$, an image dipole. The potential is simply the potential due to four point charges.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{|\vec{x}+R\hat{z}|} - \frac{Q}{|\vec{x}-R\hat{z}|} - \frac{aQ}{R\left|\vec{x}+\frac{a^2}{R}\hat{z}\right|} + \frac{aQ}{R\left|\vec{x}-\frac{a^2}{R}\hat{z}\right|}\right) \tag{2.16}$$

We can write this in spherical coordinates,

$$\Phi = \frac{Q}{4\pi\epsilon_0}\frac{1}{(r^2+R^2+2rR\ cos(\theta))^{1/2}} - \frac{Q}{4\pi\epsilon_0}\frac{1}{(r^2+R^2-2rR\ cos(\theta))^{1/2}}$$

$$-\frac{aQ}{4\pi\epsilon_0 R}\frac{1}{\left(r^2+\frac{a^4}{R^2}+\frac{2a^2r}{R}cos(\theta)\right)^{1/2}} + \frac{aQ}{4\pi\epsilon_0 R}\frac{1}{\left(r^2+\frac{a^4}{R^2}-\frac{2a^2r}{R}cos(\theta)\right)^{1/2}}$$

$$= \frac{Q}{4\pi\epsilon_0 R}\left(1+\left(\frac{r^2}{R^2}+\frac{2r}{R}cos(\theta)\right)\right)^{-1/2} - \frac{Q}{4\pi\epsilon_0 R}\left(1+\left(\frac{r^2}{R^2}-\frac{2r}{R}cos(\theta)\right)\right)^{-1/2}$$

$$-\frac{aQ}{4\pi\epsilon_0 Rr}\left(1+\left(\frac{a^4}{R^2r^2}+\frac{2a^2}{Rr}cos(\theta)\right)\right)^{-1/2} + \frac{aQ}{4\pi\epsilon_0 Rr}\left(1+\left(\frac{a^4}{R^2r^2}-\frac{2a^2}{Rr}cos(\theta)\right)\right)^{-1/2}$$

Working under the assumption that $R >> r$, we can use the binomial approximation to get an approximate potential (2.17). This allows us to easily calculate the surface charge density induced on the sphere (2.18).

$$\Phi \approx \frac{Q}{4\pi\epsilon_0}\left(-\frac{2r}{R^2}cos(\theta) + \frac{2a^3}{r^2R^2}cos(\theta)\right)$$

$$\Phi \approx -E_0\left(r - \frac{a^3}{r^2}\right)\ cos(\theta) \tag{2.17}$$

$$\sigma = -\epsilon_0\frac{\partial\Phi}{\partial r}\bigg|_{r=a} = 3\epsilon_0 E_0\ cos(\theta) \tag{2.18}$$

One thing to note is that the induced surface charge is positive on the right hemisphere and negative on the left, so the total induced charge is zero. Further, we note that the conductor feels no net force since we can treat it like a dipole pointing parallel to the electric field.

You should now be able to do Jackson 2.1, 2.2, 2.3, 2.4, 2.5, 2.8, 2.9, 2.10, and 2.11.

2.2 Conducting Sphere with Hemispheres at Different Potentials

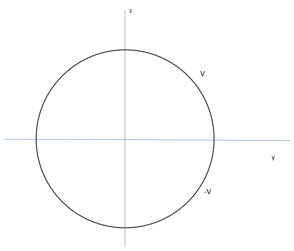

Figure 2.4: Cross-section view of sphere with hemispheres at opposite potential

There's one more conductor problem we want to look at. Imagine we have a sphere of radius a made up of two hemispherical shells separated by an insulating ring lying in the z=0 plane as shown in figure (2.4). The hemispheres are then kept at different potentials. Let's say the top hemisphere at $+V$ and the bottom hemisphere at $-V$.

We see the word "conductor" which leads us to try using method of images. However, we don't have any charges, so we can't create any image charges. In this section, we'll show both the general solution (well, we'll get close to a general solution, but we can't actually get a closed-form solution) as well as an approximate closed-form solution.

2.2.1 General Solution

We need to use Dirichlet's solution to the Poisson equation (1.45). Since we are in charge-free space, the potential simplifies to

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D}{\partial n'}\ da'$$

We'll take the derivative of the spherical Green function (2.9),

$$\left.\frac{\partial G}{\partial(-x')}\right|_{x'=a} = -\frac{(x^2-a^2)}{a(x^2+a^2-2ax\ cos(\gamma))^{3/2}}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\Phi\frac{a(x^2-a^2)}{(a^2+x^2-2ax\ cos(\gamma))^{3/2}}\ d(cos(\theta'))d\phi'$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{1}V\frac{a(x^2-a^2)}{(a^2+x^2-2ax\ cos(\gamma))^{3/2}}\ d(cos(\theta'))d\phi' - \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{0}V\frac{a(x^2-a^2)}{(a^2+x^2-2ax\ cos(\gamma))^{3/2}}\ d(cos(\theta'))d\phi'$$

$cos(\gamma)$ is the angle between $\vec{x}$ and $\vec{x}'$ given by $cos(\theta)\ cos(\theta') + sin(\theta)\ sin(\theta')\ cos(\phi-\phi')$. If we then make the substitution, $\theta' = \pi - \theta'$, $\phi' = \phi' + \pi$ in the second integral,

$$= V\frac{a(x^2-a^2)}{4\pi}\int_0^{2\pi}\int_0^{1}(a^2+x^2-2ax\ cos(\gamma))^{-3/2} - (a^2+x^2+2ax\ cos(\gamma))^{-3/2}\ d(cos(\theta'))d\phi' \quad (2.19)$$

We could try to integrate this, but it turns out, we can't solve to a closed form. Instead, let's look at a couple approximations.

2.2.2 Solution on the z-axis

First, let's look at the solution along the positive z-axis. In this case, we have no ϕ dependence and $\theta = 0$, so $cos(\gamma) = cos(\theta')$. Equation (2.19)

$$\Phi(z) = V\frac{a(z^2-a^2)}{4\pi}\int_0^{2\pi}\int_0^{1}(a^2+z^2-2az\ cos(\theta'))^{-3/2} - (a^2+z^2+2az\ cos(\theta'))^{-3/2}\ d(cos(\theta'))d\phi'$$

$$\Phi(z) = V\left(1 - \frac{z^2 - a^2}{z\sqrt{z^2+a^2}}\right) \tag{2.20}$$

As expected, the potential is V at $z = a$.

2.2.3 Binomial Approximation

One approximation we can do is to expand the denominator as a power series. We'll define $\alpha = \frac{ax}{a^2+x^2}$ and assume $2\alpha\ cos(\gamma) << 1$. We can either expand assuming α is small (2.21) or expand assuming $a << x$ (2.22).

$$\Phi(\vec{x}) = \frac{Va(x^2-a^2)}{4\pi(x^2+a^2)^{3/2}}\int_0^{2\pi}\int_0^1 (1-2\alpha\ cos(\gamma))^{-3/2} - (1+2\alpha\ cos(\gamma))^{-3/2}\ d(cos(\theta'))d\phi'$$

$$= \frac{Va(x^2-a^2)}{4\pi(x^2+a^2)^{3/2}}\int_0^{2\pi}\int_0^1 6\alpha\ cos(\gamma) + 35\alpha^3\ cos^3(\gamma)\ d(cos(\theta'))d\phi'$$

$$\Phi(\vec{x}) = \frac{3Va^2x(x^2-a^2)\ cos(\theta)}{2(x^2+a^2)^{5/2}}\left(1 + \frac{35a^2x^2}{24(a^2+x^2)^2}(3 - cos^2(\theta)) + ...\right) \tag{2.21}$$

$$\Phi = \frac{Va}{4\pi x}\int_0^{2\pi}\int_0^1 \left(1 - \frac{2a}{x}\ cos(\gamma)\right)^{-3/2} - \left(1 + \frac{2a}{x}\ cos(\gamma)\right)^{-3/2}\ d(cos(\theta'))d\phi'$$

$$= \frac{Va}{4\pi x}\int_0^{2\pi}\int_0^1 \frac{6a}{x}cos(\gamma) + \frac{35a^3}{x^3}cos^3(\gamma)\ d(cos(\theta'))d\phi'$$

$$\Phi(\vec{x}) = \frac{3Va^2}{2x^2}\left[cos(\theta) - \frac{7a^2}{12x^2}\left(\frac{5}{2}cos^3(\theta) - \frac{3}{2}cos(\theta)\right) + ...\right] \tag{2.22}$$

You should now be able to do Jackson 2.7 and 2.22.

2.3 Orthogonal Functions

Let us now turn our attention to a topic that doesn't directly involve physics. I once heart that graduate EM is really just a course in remedial maths, and this section is an example of that. We introduce orthogonal functions, which provide the solution to a large number of problems that we will encounter in future sections.

2.3.1 Orthogonal Functions

Imagine we have some complex function $U_n(\xi)$ integrable on the range (a, b). If our function satisfies,

$$\int_a^b U_n^*(\xi) U_m(\xi)\ d\xi$$

for different n and m. If our functions are normalized, then they are said to be orthonormal.

$$\int_a^b U_n^*(\xi) U_m(\xi)\ d\xi = \delta_{nm} \tag{2.23}$$

The most common example of orthogonal functions are $sin(nx)$ and $cos(nx)$ or in complex space, e^{ix}. A lot of the math behind orthonormal functions can probably be better explained in quantum using bra-ket notation, so let's skip ahead to some examples.

2.3.2 Fourier Series

You probably encountered Fourier series in undergraduate. To find the coefficients in the expansion, we use orthogonal functions. We'll start with some function $f(x)$ bound to the region $(-a/2,\ a/2)$. We remember (or look up) that the Fourier expansion is given by equation (2.24).

$$f(x) = \frac{1}{2}A_0 + \sum_{m=1}^{\infty}\left[A_m cos\left(\frac{2\pi mx}{a}\right) + B_m sin\left(\frac{2\pi mx}{a}\right)\right] \tag{2.24}$$

We know that in this range, if we multiply $cos(mx)$ and $sin(nx)$, we'll get 0, so to isolate A_m, we multiply both sides by $cos\left(\frac{2\pi nx}{a}\right)$ and integrate over the region of interest, we'll get the coefficient A_m (2.25). We can convince ourselves that if we repeat this process with sin instead, we'll get B_m (2.26).

$$\int_{-a/2}^{a/2} f(x)\ cos\left(\frac{2\pi mx}{a}\right)\ dx = \sum_{m=1}^{\infty}\int_{-a/2}^{a/2} A_m cos\left(\frac{2\pi mx}{a}\right) cos\left(\frac{2\pi nx}{a}\right)\ dx$$

$$= \frac{a}{2}A_m\delta_{nm}$$

$$A_m = \frac{2}{a}\int_{-a/2}^{a/2} f(x) cos\left(\frac{2\pi mx}{a}\right)\ dx \tag{2.25}$$

$$B_m = \frac{2}{a}\int_{-a/2}^{a/2} f(x) sin\left(\frac{2\pi mx}{a}\right)\ dx \tag{2.26}$$

You should now be able to do Jackson 2.24.

2.4 Solution to Laplace Equation, Rectangular Coordinates

Laplace's equation (1.36) looks rather simple at first glance (I have to be careful to not say on the surface, haha), but it actually takes a little bit of work to get to a solution. There are eleven different variations of Laplace's equation that can be solved using separation of variables, but we'll only look at three of them. Nothing doing, we'll start with rectangular coordinates.

2.4.1 Separation of Variables

The gist of separation of variables is to assume the solution can be written as a product of three functions, each of which depends on only one coordinate,

$$\Phi = X(x)Y(y)Z(z)$$

We can write Laplace's equation in rectangular coordinates (2.27) and substitute this in.

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{2.27}$$

$$\frac{\partial^2(XYZ)}{\partial x^2} + \frac{\partial^2(XYZ)}{\partial y^2} + \frac{\partial^2(XYZ)}{\partial z^2} = 0$$

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0$$

Dividing both sides by XYZ,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

In order for this statement to hold for arbitrary values, they must either all be equal to zero everywhere (trivial solution), or they must be separately constant (less trivial solution).

$$\begin{cases} \dfrac{1}{X}\dfrac{d^2X}{dx^2} = -\alpha^2 \\[2ex] \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = -\beta^2 \\[2ex] \dfrac{1}{Z}\dfrac{d^2Z}{dz^2} = \gamma^2 \end{cases}$$

Note that

$$\alpha^2 + \beta^2 = \gamma^2$$

We recognize that each of these is simply a simple harmonic oscillator equation, i.e., $x'' = -\omega^2 x$, the solutions to which are

$$\begin{cases} X(x) = e^{\pm i\alpha x} \\ \\ Y(y) = e^{\pm i\beta y} \\ \\ Z(z) = e^{\pm\gamma z} = e^{\pm\sqrt{\alpha^2+\beta^2}\ z} \end{cases}$$

Combining all of these, the general solution to equation (2.27) is given by equation (2.28) where the values for α and β will depend on the specific boundary conditions of the system.

$$\Phi = e^{\pm i\alpha x}\ e^{\pm i\beta y}\ e^{\pm\sqrt{\alpha^2+\beta^2}\ z} \tag{2.28}$$

2.4.2 Example: Box with One Side at a Set Potential

Imagine we have a rectangular box with dimension (a, b, c) in the (x, y, z) directions. The box is kept at zero potential except the surface $z = c$, which is kept at $V(x, y)$. We can write the individual components of the potential as

$$\begin{cases} X(x) = cos(\alpha x) \pm i\ sin(\alpha x) \\ \\ Y(y) = cos(\beta y) \pm i\ cos(\beta y) \\ \\ Z(z) = cosh\left(\sqrt{\alpha^2+\beta^2}\ z\right) \pm sinh\left(\sqrt{\alpha^2+\beta^2}\ z\right) \end{cases}$$

If we look at the boundary, $\Phi = 0$ for $x = 0$, $y = 0$, and $z = 0$, we can kill the cos terms in each solution,

$$\begin{cases} X(x) = sin(\alpha x) \\ \\ Y(y) = sin(\beta y) \\ \\ Z(z) = sinh\left(\sqrt{\alpha^2+\beta^2}\ z\right) \end{cases}$$

Since we have the boundary, $\Phi = 0$ for $x = a$ and $y = b$, the potential must be periodic,

$$\begin{cases} X(x) = sin\left(\frac{n\pi x}{a}\right) \\ \\ Y(y) = sin\left(\frac{m\pi y}{b}\right) \\ \\ Z(z) = sinh\left(\pi\sqrt{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}}\ z\right) \end{cases}$$

For simplicity, we will set

$$\begin{cases} \alpha_n = \dfrac{n\pi}{\alpha} \\ \beta_m = \dfrac{m\pi}{b} \\ \gamma_{nm} = \pi\sqrt{\dfrac{n^2}{a^2} + \dfrac{m^2}{b^2}} \end{cases}$$

Now, we want to use the boundary, $\Phi = V(x, y)$ at $z = c$.

$$V(x,y) = \sum_{n,m} A_{nm} sin(\alpha_n x)\ sin(\beta_m y)\ sinh(\gamma nmc)$$

We recognize this as a Fourier series, so in order to sovle for A_{nm}, we multiply both sides by $\sum sin(\alpha_{n'} x)\ sin(\beta_{m'} y)$ and integrate.

$$\sum_{n',m'} \int_0^a \int_0^b V(x,y)\ sin(\alpha_{n'} x)\ sin(\beta_{m'} y)\ dydx$$

$$= \sum_{n,m} \sum_{n',m'} \int_0^a \int_0^b A_{nm} sin(\alpha_n x)\ sin(\alpha_{n'} x)\ sin(\beta_m y)\ sin(\beta_{m'} y)\ sinh(\gamma_{nm} c)\ dydx$$

Using orthonormal functions,

$$= A_{nm} sinh(\gamma_{nm} c) \int_0^a \int_0^b sin^2(\alpha_n x)\ sin^2(\beta_m y)\ delta_{nn'} \delta_{mm'}\ dydx$$

$$A_{nm} = \frac{4}{ab\ sinh(\gamma_{nm} c)} \int_0^a \int_0^b V(x,y)\ sin(\alpha_n x)\ sin(\beta_m y)\ dydx$$

We can write the full potential,

$$\Phi(x,y,z) = \sum_{n,m} A_{nm} sin(\alpha_n x)\ sin(\beta_m y)\ sinh(\gamma_{nm} z)$$

You should now be able to do Jackson 2.23 (a and c).

2.5 Two-Dimensional Potential

It might seem a little backwards to go from three dimensions to only two, but we can put alot of the infrastructure from the previous chapter to good use here. We'll start with Laplace's equation in rectangular coordinates (2.27). Because we are working in two dimensions, our equation should be independent of z.

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{2.29}$$

Our separation of variables potential is

$$\Phi = X(x)Y(y)$$

Substituting this in,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

The solution to this is

$$\begin{cases} X(x) = e^{\pm i\alpha x} \\ \\ Y(y) = e^{\pm\alpha y} \end{cases}$$

Which gives a general potential,

$$\Phi = e^{\pm i\alpha x}\ e^{\pm\alpha y} \tag{2.30}$$

Again, if we want the value of α, we'll need to look at the specific boundary conditions.

2.5.1 Example: Square Well Potential

Imagine we have a two-dimensional box with $\Phi = 0$ at $x = 0$ and $x = a$. We impose $\Phi = V$ at $y = 0$. Furthermore, we want our potential to vanish for large y. We are interested in the region, $0 \leq x \leq a$, $y \geq 0$.

We know the potential must disappear at $x = 0$ and must be periodic in x. In addition, we know that the potential should fall off as y goes to infinity. Our preliminary potential is

$$\Phi = \sum_n A_n\ exp\left(-\frac{n\pi y}{a}\right)\ sin\left(\frac{n\pi x}{a}\right)$$

We'll start with the condition that $\Phi = V$ at $y = 0$. We multiply both sides by $\sin\left(\frac{n'\pi x}{a}\right)$ and integrate.

$$\int_0^a V\ \sin\left(\frac{n\pi x}{a}\right)\ dx = A_n \int_0^a \sin^2\left(\frac{n\pi x}{a}\right)\ dx$$

$$A_n = \frac{4V}{n\pi}, \quad for\ odd\ n$$

The full potential is given by

$$\Phi = \frac{4V}{\pi} \sum_{n\ odd} \frac{1}{n} \exp\left(-\frac{n\pi y}{a}\right)\ \sin\left(\frac{n\pi x}{a}\right)$$

We can write $\sin(\theta) = Im(e^{i\theta})$,

$$= \frac{4V}{\pi} Im\left(\sum_{n\ odd} \frac{1}{n} \exp\left(\frac{in\pi}{a}(x+iy)\right)\right)$$

If we define $Z = \exp\left(\frac{i\pi}{a}(x+iy)\right)$,

$$= \frac{4V}{\pi} Im\left(\sum_{n\ odd} \frac{Z^n}{n}\right)$$

Using the expansion of $\ln(1+Z)$,

$$= \frac{2V}{\pi} Im\left[\ln\left(\frac{1+Z}{1-Z}\right)\right]$$

$$\Phi(x,y) = \frac{2V}{\pi} \tan^{-1}\left(\frac{\sin\left(\frac{\pi x}{a}\right)}{\sinh\left(\frac{\pi y}{a}\right)}\right)$$

You should now be able to do Jackson 2.15 and 2.16.

2.6 Two Dimensional Potential: Polar Coordinates

In this section, we'll look at another geometry for potentials in two dimensions. We can also look at what happens to the potential near corners.

2.6.1 Separation of Variables: Polar Coordinates

We could use Cartesian coordinates for any two-dimensional system (2.30), but sometimes the math is a little easier if we use polar coordinates instead. We start with the Laplacian for cylindrical coordinates, setting z=0 (2.31). We then use separation of variables to break up the potential into a radial and an angular component.

$$\nabla^2 \Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0 \tag{2.31}$$

$$\Phi(\rho, \phi) = R(\rho)\Psi(\phi)$$

$$\frac{\Psi}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{R}{\rho^2}\frac{d^2\Psi}{d\phi^2} = 0$$

We want each term to be in terms of a single variable. In order to accomplish this, we multiply by $\frac{\rho^2}{\Phi}$.

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} = 0$$

Again, each term must be equal to a constant,

$$\begin{cases} \frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = \nu^2 \\ \\ \frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} = -\nu^2 \end{cases}$$

For arbitrary, non-zero integer values of ν, the first equation returns a polynomial solution, and the second is the simple harmonic oscillator equation.

$$R(\rho) = a\rho^{\nu} + b\rho^{-\nu}$$

$$\Psi(\phi) = A\ \cos(\nu\phi) + B\ \sin(\nu\phi)$$

If we have $\nu = 0$, we could either get the trivial solution (R and Ψ don't depend on any variables) or the less trivial solutions,

$$R(\rho) = a_0 + b_0 \ln(\rho)$$

$$\Psi(\phi) = A_0 + B_0\phi$$

We note that we can write combinations of $\cos(\theta)$ and $\sin(\theta)$ as sin (or cos) with some phase shift. Combining the angular and radial components, we get the general potential in polar coordinates (2.32).

$$\Phi = a_0 + b_0\ln(\rho) + \sum_{n=1}^{\infty}\left(a_n\rho^n + b_n\rho^{-n}\right)\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)$$

$$= a_0 + b_0\ln(\rho) + \sum_{n=1}^{\infty}\left(a_nA_n\rho^n\cos(n\phi) + a_nB_n\rho^n\sin(n\phi) + b_nA_n\rho^{-n}\cos(n\phi) + b_nB_n\rho^{-n}\sin(n\phi)\right)$$

$$\Phi(\rho,\ \phi) = a_0 + b_0\ln(\rho) + \sum_{n=1}^{\infty}a_n\rho^n\sin(n\phi+\alpha_n) + \sum_{n=1}^{\infty}b_n\rho^{-n}\sin(n\phi+\beta_n) \qquad (2.32)$$

2.6.2 Example: Cylinder with a Potential on the Surface

Imagine we have an infinitely long cylinder of radius a centered along the z-axis which holds a potential $V(\phi) = V_0\cos^2(\phi)$ on its surface. We want to know what the potential everywhere is due to this cylinder. We can use equation (2.32) to determine the potential since we can ignore the z-direction due to the infinite length. If the cylinder had finite length, we would not be able to use the method established in this section (we cover this in the next chapter).

Let's look inside the cylinder first. Immediately, we can say that the b terms should all be equal to zero since our region of interest includes the origin. $\ln(\rho)$ and ρ^{-n} both both explode at the origin, so in order to keep our potential from going to infinity at the origin, we need to suppress these terms. Using the angle addition formula, we can break the $\sin(n\phi + \alpha_n)$ into cos and sin terms,

$$\Phi(\rho,\ \phi) = a_0 + \sum_{n=1}^{\infty}a_n\rho^n\sin(n\phi) + \sum_{n=1}^{\infty}c_n\rho^n\cos(n\phi)$$

Since $\cos^2(\phi) = \frac{1}{2}[1+\cos(2\phi)]$, if we match terms, we see that only the $n = 0$ and $n = 2$ terms survive. Matching terms at the boundary $\rho = a$,

$$\frac{V_0}{2}[1+\cos(2\phi)] = a_0 + c_2a^2\cos(2\phi)$$

$$a_0 = \frac{V_0}{2}$$

$$c_2 = \frac{V_0}{2a^2}$$

Similarly, outside the cylinder, we say that b_0 and a_n terms should all be zero since our region of interest includes infinity.

$$\Phi(\rho,\ \phi) = a_0 + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi) + \sum_{n=1}^{\infty} d^n \rho^{-n} \cos(n\phi)$$

Again, looking at the boundary $\rho = a$, only the $n = 0$ and $n = 2$ terms survive.

$$\frac{V_0}{2}[1 + \cos(2\phi)] = a_0 + d_2 a^{-2} \cos(2\phi)$$

$$a_0 = \frac{V_0}{2}$$

$$d_2 = \frac{V_0\ a^2}{2}$$

Combining these,

$$\Phi_{in}(\rho,\ \phi) = \frac{V_0}{2} + \frac{V_0}{2}\frac{\rho^2}{a^2}\cos(2\phi)$$

$$\Phi_{out}(\rho,\ \phi) = \frac{V_0}{2} + \frac{V_0}{2}\frac{a^2}{\rho^2}\cos(2\phi)$$

You should now be able to do Jackson 2.12, 2.13, 2.14, 2.17, 2.18, 2.19, 2.20, and 2.21.

2.6.3 Separation of Variables: Corner

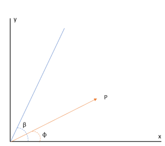

Figure 2.5:

Now we have the question of what happens to the potential near a corner. We can look at this system as two planes with a constant potential V intersecting and forming an angle β between them as shown in figure (2.5). Going back to the derivation in the previous section, we now have boundary conditions for $\phi = 0$ and $\phi = \beta$.

Using the condition that our region of interest includes the origin, b and b_0 must be zero.

$$\Phi(\rho, \phi) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi)$$

If we want our potential to be V at $\phi = 0$ and $\phi = \beta$, we need to set $a_0 = V$ and $n = m\pi/\beta$,

$$\Phi(\rho, \phi) = V + \sum_{m=1}^{\infty} a_m \rho^{m\pi/\beta} \sin\left(\frac{m\pi\phi}{\beta}\right) \tag{2.33}$$

We can find the electric field by taking the derivative (1.22). Further, if we are close to the origin, we only need to keep the $m = 1$ term since ρ will drop off fast enough that we can neglect higher order terms.

$$E_\rho(\rho, \phi) = -\frac{\partial \Phi}{\partial \rho} = -\frac{\pi a_1}{\beta} \rho^{(\pi/\beta)-1} \sin\left(\frac{\pi \phi}{\beta}\right)$$

$$E_\phi(\rho, \phi) = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi} = -\frac{\pi a_1}{\beta} \rho^{(\pi/\beta)-1} \cos\left(\frac{\pi \phi}{\beta}\right)$$

Using equation (1.26), we can find the charge density,

$$\sigma(\rho) = \epsilon_0 E_\phi(\rho, 0) = -\frac{\epsilon_0 \pi a_1}{\beta} \rho^{(\pi/\beta)-1}$$

As the corner becomes sharper, β becomes smaller and the charge density and electric field start to increase by larger and larger powers. For example, a right angle corner $\left(\beta = \frac{\pi}{2}\right)$ has linear dependence in ρ, but if we cut that in half $\left(\beta = \frac{\pi}{4}\right)$, the induced charge density goes by ρ^3. If you've ever put crumpled aluminium foil in the microwave, it starts to spark, and this is the reason why. A smooth ball of aluminium would be perfectly fine in a microwave since it doesn't have any sharp corners (please do not try this at home). This is also why when soldering, you want to make rounded ends rather than pointed ones.

You should now be able to do Jackson 2.25, 2.26, and 2.27.

Chapter 3

More Boundary-Value Problems in Electrostatics

"He couldn't see her in the darkness, but there were plenty of faces he could remember from the old days which fitted the voice. When you visualized a man or woman carefully, you could always begin to feel pity-that was a quality God's image carried with it. When you saw the lines at the corner of the eyes, the shape of the mouth, how the hair grew, it was impossible to hate. Hate was just a failure of imagination." - (Graham Greene, The Power and the Glory)

In the last chapter, we introduced boundary value problems, specifically, those in Cartesian and polar coordinates. In this chapter, we will deal with boundary-value problems in spherical and cylindrical coordinates. These problems are relatively straightforward (other than a bit of algebra) if we have symmetry, but if we don't have those symmetries, these problems get noticeably more difficult to the point where I don't think it is reasonable to try memorizing the solutions. Nothing doing, let's start with some mathematical background.

3.1 Legendre Polynomials

As we will show, the solution to Laplace's equation in spherical coordinates requires the use of Legendre Polynomials and Spherical Harmonics, both of which are solutions to the Legendre equation (3.1). In this section, we'll look at the specific case where $m = 0$ (3.2).

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0 \tag{3.1}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + l(l+1)P = 0 \tag{3.2}$$

3.1.1 Legendre Polynomials

I posit that the solution to equation (3.2) takes the form of an infinite sum of polynomials.

$$P(x) = x^{\alpha} \sum_{j=0}^{\infty} a_j x^j \tag{3.3}$$

Substituting this into equation (3.2),

$$\frac{d}{dx}\left[(1-x^2)\sum_{j=0}^{\infty} a_j (j+\alpha) x^{\alpha+j-1}\right] + l(l+1)\sum_{j=0}^{\infty} a_j x^{\alpha+j} = 0$$

$$\sum_{j=0}^{\infty} a_j (j+\alpha)(j+\alpha-1) x^{\alpha+j-2} - [(\alpha+j)(\alpha+j+1) - l(l+1)] a_j x^{\alpha+j} = 0$$

We want the coefficient of each power in x to vanish separately. Since we're taking a sum over j, let's look at the $j = 0$ case,

$$\alpha(\alpha-1)a_0 x^{\alpha-2} - [\alpha(\alpha+1) - l(l+1)]a_0 x^{\alpha} = 0$$

Since, as we stated previously, we want each power in x to disappear, if $a_0 \neq 0$, then the first term implies,

$$\alpha(\alpha-1) = 0$$

Note that this implies $\alpha = 0$ *or* 1. Similarly, ifwe look at the $j = 1$ case,

$$\alpha(\alpha+1)a_1 x^{\alpha-1} - [(\alpha+1)(\alpha+2) - l(l+1)]a_1 x^{\alpha+1} = 0$$

Again, looking at the first term, if $a_1 \neq 0$,

$$\alpha(\alpha+1) = 0$$

In general, if we want to look at the j term in the sum, we use $j = j + 2$ for the first term,

$$(\alpha+j+2)(\alpha+j+1)a_{j+2} x^{\alpha+j} - [(\alpha+j)(\alpha+j+1) - l(l+1)]a_j x^{\alpha+j} = 0$$

Solving for a_j, we end up getting a recursive relation,

$$a_{j+2} = \left[\frac{(\alpha+j)(\alpha+j+1) - l(l+1)}{(\alpha+j+1)(\alpha+j+2)}\right] a_j \tag{3.4}$$

In order to solve for these coefficients, we need to specify either a_0 or a_1. As it turns out, if we have both a_0 and a_1 not zero, we over-specify the problem. Furthermore, we see that if a_0 is set to zero, we only get even powers of x while if set a_1 to zero, we only get odd powers of x. In addition, l must be either 0 or a positive integer, which allows us another index. This index determines the highest power of each Legendre polynomial, so $P_0(x)$ goes up to x^0 while $P_3(x)$ has an x^3 and an x dependence. A compact representation of the Legendre polynomials is given by Rodrigues' formula (3.5). I will not be deriving this here as I believe the proof is a bit non-trivial (it's been a little while since I looked at it, so I could be wrong).

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \tag{3.5}$$

3.1.2 Orthogonality of Legendre Polynomials

Finding the coefficients in boundary value problems can be made a bit easier by using the orthogonality of the Legendre polynomials. To prove this, we'll start by multiplying $P_{l'}(x)$ by the Legendre equation (3.2) and integrating over the interval $[-1, 1]$. Since the Legendre equation is equal to zero, we expect this to return 0.

$$\int_{-1}^{1} P_{l'} \left[\frac{d}{dx} \left((1-x^2) \frac{dP_l}{dx} \right) + l(l+1)P \right] \, dx = 0$$

$$\int_{-1}^{1} P_{l'} \left[\frac{d}{dx} \left((1-x^2) \frac{dP_l}{dx} \right) + l(l+1)P_{l'}P_l \right] \, dx = 0$$

We can integrate by parts on the first term with $u = P_{l'}(x)$ and $v' = \dfrac{d}{dx}\left((1-x^2)\dfrac{dP_l}{dx}\right)$.

$$(1-x^2)P_{l'}\frac{dP_l}{dx}\bigg|_{-1}^{1} - \int_{-1}^{1} (1-x^2)\frac{dP_l}{dx}\frac{dP_{l'}}{dx} \, dx + \int_{-1}^{1} l(l+1)P_{l'}P_l \, dx = 0$$

$$\int_{-1}^{1} (x^2-1)\frac{dP_l}{dx}\frac{dP_{l'}}{dx} + \int_{-1}^{1} l(l+1)P_{l'}P_l \, dx = 0$$

If we had started by multiplying P_l by the Legendre equation,

$$\int_{-1}^{1} (x^2-1)\frac{dP_{l'}}{dx}\frac{dP_l}{dx} \, dx + \int_{-1}^{1} l'(l'+1)P_l P_{l'} \, dx = 0$$

Subtracting these two, we get

$$[l(l+1) - l'(l'+1)] \int_{-1}^{1} P_{l'}P_l \, dx = 0$$

Because of the term out front, this implies that $l \neq l'$, the integral must vanish, which means the Legendre polynomials are orthogonal. Now, let's find the normalization constant so that we can show orthonormality.

$$N_l = \int_{-1}^{1} P_l^2(x) \, dx$$

Writing the Legendre polynomials using Rodrigues' formula (3.5),

$$= \frac{1}{2^{2l}(l!)^2} \int_{-1}^{1} \frac{d^l}{dx^l}(x^2-1)^l \frac{d^l}{dx^l}(x^2-1)^l \, dx$$

Integrating by parts with $u = \dfrac{d^l}{dx^l}(x^2-1)^l$ and $v' = \dfrac{d^l}{dx^l}(x^2-1)^l$,

$$= \frac{1}{2^{2l}(l!)^2} \left[\frac{d^l}{dx^l}(x^2-1)^l \frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l \bigg|_{-1}^{1} - \int_{-1}^{1} \frac{d^{l+1}}{dx^{l+1}}(x^2-1)^l \frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l \, dx \right]$$

$$= -\frac{1}{2^{2l}(l!)^2}\int_{-1}^{1}\frac{d^{l+1}}{dx^{l+1}}(x^2-1)^l\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l\ dx$$

Repeating this process l times,

$$= \frac{(-1)^l}{2^{2l}(l!)^2}\int_{-1}^{1}(x^2-1)^l\frac{d^{2l}}{dx^{2l}}(x^2-1)^l\ dx$$

We can convince ourselves that $\frac{d^{2l}}{dx^{2l}}(x^2-1)^l$ gives $(2l)!$ since it is equivalent to taking derivatives of x^{2l} until we return a constant.

$$= \frac{(2l)!}{2^{2l}(l!)^2}\int_{-1}^{1}(1-x^2)^l\ dx$$

$$N_l = \frac{2}{2l+1}$$

The complete orthonormality condition is thus given by equation (3.6).

$$\int_{-1}^{1}P_{l'}(x)P_l(x)\ dx = \frac{2}{2l+1}\delta_{ll'} \tag{3.6}$$

3.1.3 Alternate Definitions of Legendre Polynomials

By using Rodrigues' formula (3.5), we can derive some recurrence relations involving Legendre polynomials. For example, if we take the derivative of $P_{l+1}(x)$,

$$\frac{dP_{l+1}(x)}{dx} = \frac{1}{2^{l+1}(l+1)!}\frac{d^{l+2}}{dx^{l+2}}(x^2-1)^{l+1}$$

$$= \frac{1}{2^l l!}\frac{d^l}{dx^l}[(x^2-1)^l + 2lx^2(x^2-1)^{l-1}]$$

We notice that the second term looks like it should be related to $P_{l-1}(x)$, so let's try subtracting out the derivative of $P_{l-1}(x)$.

$$\frac{dP_{l+1}(x)}{dx} - \frac{dP_{l-1}(x)}{dx} = \frac{d^l}{dx^l}\left[\frac{(x^2-1)^l}{2^l l!} + \frac{x^2(x^2-1)^{l-1}}{2^{l-1}(l-1)!} - \frac{(x^2-1)^{l-1}}{2^{l-1}(l-1)!}\right]$$

$$= \frac{d^l}{dx^l}\left[\frac{(x^2-1)^l}{2^l l!} + \frac{(x^2-1)^{l-1}(x^2-1)}{2^{l-1}(l-1)!}\right]$$

$$= (2l+1)\frac{d^l}{dx^l}\frac{(x^2-1)^l}{2^l l!}$$

We recognize this as Rodrigues' formula (3.5), so we get the first of our recurrence relations (3.7).

$$\frac{dP_{l+1}(x)}{dx} - \frac{dP_{l-1}(x)}{dx} - (2l+1)P_l(x) = 0 \tag{3.7}$$

Jackson provides further recurrence relations (3.8-3.10), which we shall prove (or rather, show to be internally consistent since I can't seem to figure out how to prove them outright. Sorry about that).

$$(l+1)P_{l+1}(x) - (2l+1)xP_l(x) + lP_{l-1}(x) = 0 \tag{3.8}$$

$$\frac{dP_{l+1}(x)}{dx} - x\frac{dP_l(x)}{dx} - (l+1)P_l(x) = 0 \tag{3.9}$$

$$(x^2-1)\frac{dP_l(x)}{dx} - lxP_l(x) + lP_{l-1}(x) = 0 \tag{3.10}$$

We'll start with equation (3.8). To prove this, we'll start by writing $P_{l+1}(x)$ using the Legendre equation (3.2),

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_{l+1}}{dx}\right] + (l+1)(l+2)P_{l+1} = 0$$

Substituting in equation (3.7),

$$\frac{d}{dx}\left[(1-x^2)\left(\frac{dP_{l-1}}{x} + (2l+1)P_l\right)\right] + (l+1)(l+2)P_{l+1} = 0$$

Using the Legendre equation (3.2) with $P_{l-1}(x)$,

$$-l(l-1)P_{l-1} + (2l+1)\left[\frac{dP_l}{dx} - \frac{d(x^2P_l)}{dx}\right] + (l+1)(l+2)P_{l+1} = 0$$

Using equation (3.10),

$$-(l-1)lP_{l-1} + (2l+1)(-lxP_l + lP_{l-1} - 2xP_l) + (l+1)(l+2)P_{l+1} = 0$$

Simplifying, this returns equation (3.8).

We want to prove the next recurrence relation (3.9) using induction. The $l=0$ case is easy to show using the explicit form of $P_l(x)$. For the $l+1$ case,

$$\frac{dP_{l+2}}{dx} - x\frac{dP_{l+1}}{dx} - (l+2)P_{l+1} = 0$$

Substituting in equation (3.7) for $P_{l+2}(x)$ and $P_{l+1}(x)$,

$$\frac{dP_l}{dx} + (2l+3)P_{l+1} - x\frac{dP_{l-1}}{dx} - x(2l+1)P_l - (l+2)P_{l+1} = 0$$

$$\frac{dP_l}{dx} - x\frac{dP_{l-1}}{dx} + (l+1)P_{l+1} - x(2l+1)P_l = 0$$

Using equation (3.9) with $P_l(x)$,

$$lP_{l-1} + (l+1)P_{l+1} - x(2l+1)P_l = 0$$

We recognize this as equation (3.8), which we showed to be true above. Since that is true, equation (3.9) must also be true.

For the last recurrence relation, we again want to use induction. The $l+1$ case,

$$(x^2-1)\frac{dP_{l+1}}{dx} - (l+1)xP_{l+1} + (l+1)P_l = 0$$

We can use equation (3.7) for the first term,

$$(x^2-1)\left[x\frac{dP_l}{dx} + (l+1)P_l\right] - (l+1)xP_{l+1} + (l+1)P_l = 0$$

Rearranging and gathering like terms,

$$x(x^2-1)\frac{dP_l}{dx} + x^2(l+1)P_l - (l+1)xP_{l+1} = 0$$

The last term can be rewritten using equation (3.8),

$$x(x^2-1)\frac{dP_l}{dx} + x^2(l+1)P_l - x[(2l+1)xP_l - lP_{l+1}] = 0$$

$$(x^2-1)\frac{dP_l}{dx} - xlP_l + lP_{l-1} = 0$$

Which we recognize as equation (3.10).

3.2 Separation of Variables: Spherical Coordinates, Azimuthal Symmetry

Now that we have some of the mathematical background out of the way, let's look at the physics. In the previous chapter, we solved boundary value problems in polar coordinates. Let's now take that problem and generalize it to three dimensions.

3.2.1 General Solution

In spherical coordinates, Laplace's equation (1.36) is given by equation (3.11). Note that we can rewrite the first term using equation (3.12).

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \tag{3.11}$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) \tag{3.12}$$

Just like we did in the previous chapter, we'll use separation of variables to solve for the potential, which takes the form,

$$\Phi = \frac{U(r)}{r}P(\theta)Q(\phi)$$

Substituting this in,

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(UPQ) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\frac{\sin(\theta)}{r}\frac{\partial\, UPQ}{\partial\theta}\right) + \frac{1}{r^3\sin^2\theta}\frac{\partial^2 UPQ}{\partial\phi^2} = 0$$

$$\frac{PQ}{r}\frac{d^2U}{dr^2} + \frac{UQ}{r^3\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{dP}{d\theta}\right) + \frac{UP}{r^3\sin^2(\theta)}\frac{d^2Q}{d\phi^2} = 0$$

Multiplying through by $\dfrac{r^3\sin^2(\theta)}{UPQ}$,

$$r^2\sin^2(\theta)\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{Pr^2\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{dP}{d\theta}\right)\right] + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0 \tag{3.13}$$

We notice that only the final term has a ϕ dependence, and it is the only term which depends on a single variable. We set it equal to a constant (3.14), which we then recognize as the simple harmonic oscillator (3.15).

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2 \tag{3.14}$$

$$Q = e^{\pm im\phi} \tag{3.15}$$

In order for Q to be single-valued, we must let m be an integer. To find expressions for $P(\theta)$ and $U(r)$, we have to go back to the original laplacian (3.11). Multiplying through by $\frac{r^3}{UPQ}$,

$$\frac{r^2}{U}\frac{d^2U}{dr^2} + \frac{1}{P\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{dP}{d\theta}\right) + \frac{1}{Q\sin^2(\theta)}\frac{d^2Q}{d\phi^2} = 0$$

Using equation (3.15),

$$\frac{r^2}{U}\frac{d^2U}{dr^2} + \frac{1}{P\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2(\theta)} = 0$$

We can separate this into radial and angular components. The angular component is solved by assuming the radial component is a constant which, for reasons that will be shown, we set to $l(l+1)$. Rearranging, we get equation (3.16), which we recognize as the Legendre equation (3.1) with $x = \cos(\theta)$.

$$\frac{1}{P\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2(\theta)} = -l(l+1)$$

$$\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{dP}{d\theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2(\theta)}\right)P = 0 \tag{3.16}$$

For the radial component, we get equation (3.17) which is solved by polynomials (3.18).

$$\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0 \tag{3.17}$$

$$U = Ar^{l+1} + Br^{-l} \tag{3.18}$$

3.2.2 Boundary-Value Problems with Azimuthal Symmetry

If we have azimuthal symmetry, $m = 0$ in equation (3.16). What this means is that we have no ϕ component, and the θ component reduces to equation (3.2). Thus, the solution (3.19) is going to be a product of the radial solution to the laplacian (3.18) and the Legendre polynomials (3.5). The coefficients A_l and B_l are determined by the boundary coefficients.

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}[A_l r^l + B_l r^{-(l+1)}P_l(\cos(\theta)) \tag{3.19}$$

3.2.3 Green Function: Spherical Coordinates with Azimuthal Symmetry

Imagine we have some point charge q at $\vec{x}'$, and we want to find the Green function (1.40). For this case, we already have the solution form section 1.7.

$$G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}$$

If we set $\vec{x}$ and $\vec{x}'$ both on the z-axis, this simplifies to

$$= \frac{1}{|r - r'|}$$

We are faced with two possibilities. First, let's look at the case where $r > r'$. We can use Taylor expansion to show,

$$= \frac{1}{r}\left(1 - \frac{r'}{r} + \left(\frac{r'}{r}\right)^2 + ...\right)$$

If we write this in the same vein as the general potential (3.19) with $\theta = 0$.

$$\frac{1}{r}\left(1 - \frac{r'}{r} + \left(\frac{r'}{r}\right)^2 + ...\right) = \sum_{l=0}^{\infty} A_l r^l + B_l r^{-(l+1)}$$

We notice that A_l must be zero for all l and $B_l = r'^l$. Similarly, if we had expanded in the case where $r' > r$, we would see that $A_l = r'^{-(l+1)}$ and $B_l = 0$. If we are then looking at a point offset from the axis, we need to reintroduce the $P_l(\cos(\gamma))$ term. Combining all of this, we get an expansion for $1/R$ (3.20).

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos(\gamma)) \tag{3.20}$$

3.2.4 Example: Sphere with a Potential on the Surface

Imagine we have a sphere of radius a which holds a potential $V(\theta) = V_0 \cos^2(\theta)$ on its surface. We want to know what the potential inside the sphere is.

We start with the general potential (2.7). Since we are looking at the interior, we set B_l to 0 since we want to suppress the $r^{-(l+1)}$ terms. At the boundary, we have

$$V_0 \cos^2(\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos(\theta))$$

The highest power of $\cos(\theta)$ in our potential is 2, so the only terms in our expansion are $l = 2$ and $l = 0$.

$$V_0 \cos^2(\theta) = A_0 P_0(\cos(\theta)) + A_2 a^2 \; P_2(\cos(\theta))$$

$$V_0 \cos^2(\theta) = A_0 + A_2 a^2 \left(\frac{3}{2}\cos^2(\theta) - \frac{1}{2}\right)$$

Solving for the coefficients by matching powers,

$$A_0 = \frac{V_0}{3}$$

$$A_2 = \frac{2V_0}{3a^2}$$

Thus, the general potential inside the sphere is given by

$$\Phi(r,\theta) = \frac{V_0}{3} P_0(\cos(\theta)) + \frac{2V_0}{3a^2} r^2 \; P_2(\cos(\theta))$$

You should now be able to do Jackson 3.1, 3.2, 3.7, and 3.22.

3.3 Spherical Harmonics

We once again introduce some mathematical concepts before diving into the physics. When we solved the Legendre equation (3.1), we set $m = 0$, so now we may ask what the solution is if $m \neq 0$. We get the associated Legendre function $P_l^m(x)$ as our solution, which when normalized and put into spherical coordinates, are the spherical harmonics, $Y_{lm}(\theta, \phi)$.

3.3.1 Associated Legendre Function

We start by having a dummy solution,

$$\Pi_l^m(x) = \frac{d^m}{dx^m} P_l(x)$$

One thing to note is that

$$\Pi_l^m(x) = P_l(x)$$

If we look at the azimuthal form of Legendre's equation (3.2) and take m derivatives,

$$(1-x^2)\frac{d^2\Pi_l^m(x)}{dx^2} - 2(m+1)x\frac{d\Pi_l^m(x)}{dx} + [l(l+1) - m(m+1)]\Pi_l^m(x) = 0$$

If we make the substitution, $\Pi_l^m(x) = (1-x^2)^{-m/2}P_l^m(x)$, we get

$$-m^2x^2(1-x^2)^{-(m/2)-1}P_l^m(x) + (1-x^2)^{-(m/2)+1}\frac{d^2P_l^m(x)}{dx^2} - 2x(1-x^2)^{-m/2}\frac{dP_l^m(x)}{dx}$$
$$+l(l+1)(1-x^2)^{-m/2}P_l^m(x) - m^2(1-x^2)^{-m/2}P_l^m(x) = 0$$

We then multiply by $(1-x^2)^{m/2}$ and combine terms,

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l^m(x)}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m(x) = 0$$

We recognize this as the Legendre equation (3.1). Thus, the associated Legendre function is given by equation (3.21). The $(-1)^m$ is referred to as the Condon-Shortley phase, and is sometimes left out. We'll follow Jackson's example here and leave it in. Combining with Rodrigues' formula, we get a more concise definition, (3.22), that allows for negative values of m. Note that m can only take values between $-l$ and l.

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x) \tag{3.21}$$

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \tag{3.22}$$

Looking at values of $-m$, we see that $P_l^{-m}(x)$ and $P_l^m(x)$ are proportional (3.23).

$$P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x) \tag{3.23}$$

Since the Legendre Polynomials are orthonormal, we expect the associated Legendre functions should also be orthonormal over the interval [-1,1] (3.24).

$$\int_{-1}^{1} P_{l'}^m(x)P_l^m(x)\ dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'} \tag{3.24}$$

3.3.2 Spherical Harmonics

It is often useful (as we shall see in the next section) to write Legendre's equation in spherical coordinates. One tool we use is to combine the angular components into one function (3.25). We do this by combining the associated Legendre function and $e^{im\phi}$.

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos(\theta))e^{im\phi} \tag{3.25}$$

Using equation (3.23), we get the spherical harmonic for negative values (3.26).

$$Y_{l,-m}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}}(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)e^{-im\phi}$$

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y_{lm}^*(\theta,\phi) \tag{3.26}$$

Using equation (3.24) and a Gaussian integral, we can show the orthonormality of the spherical harmonics (3.27).

$$\int_0^{2\pi}\int_0^{\pi} Y_{l'm'}^*(\theta,\phi)Y_{lm}(\theta,\phi)\sin(\theta)\ d\theta d\phi = \delta_{ll'}\delta_{mm'} \tag{3.27}$$

One relation that tends to come up is the $m=0$ case for spherical harmonics,

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos(\theta))$$

3.3.3 Addition Theorem

Imagine we have two vectors $\vec{x}$ and $\vec{x}'$ separated by an angle γ. We can expand the Legendre polynomial of that angle using the addition theorem (3.28).

$$P_l(\cos(\gamma)) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{3.28}$$

This is most useful when we look at the the expansion of $1/R$ using the Legendre polynomial (3.20). We can use the angle addition theorem to expand $1/R$ in the spherical harmonics (3.29).

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \tag{3.29}$$

3.4 Separation of Variables: Spherical Coordinates

Now that we've established the mathematical background, we turn to the physics. We saw the solution to spherical coordinates with azimuthal symmetry (3.19), so now we wonder what happens if we don't have azimuthal symmetry.

3.4.1 Potential

Going back to Laplace's equation (3.11), the radial solution should be the same. Now, we look at the ϕ component, (3.14), which is solved by an exponential. Similarly, the θ component, (3.16) is solved by Legendre's equation. Thus, our potential (3.30) is the same as before with spherical harmonics (3.25) instead,

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}[A_{lm}r^l + B_{lm}r^{-(l+1)}]Y_{lm}(\theta,\phi) \tag{3.30}$$

One way to tell if something has azimuthal symmetry is to do a full rotation about the z-axis and see if your system is the same for all angles.

You should now be able to do Jackson 3.3, 3.4, 3.5, 3.6, 3.8, 3.13, 3.14, 3.15, 3.26, and 3.27

3.5 Bessel Functions

Once again, we do some mathematical background. Just as the spherical harmonics (3.25) were the solution to the radial component of Laplace's equation in spherical coordinates (3.30, the Bessel functions are the radial component of Laplace's equation in cylindrical components. I'm not terribly comfortable with Bessel functions outside of their use in the solution to Laplace's equation in cylindrical coordinates, so this section might be a bit spotty.

3.5.1 Bessel's Differential Equation

Just like the Legendre polynomials were the solution to the differential equation (3.2), Bessel functions are the solution to Bessel's differential equation (3.31) where ν is an integer.

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0 \tag{3.31}$$

Our solution takes the from of a polynomial.

$$R(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j$$

Substituting this into equation (3.31),

$$a_j(\alpha+j)(\alpha+j-1)x^{\alpha+j-2} + a_j(\alpha+j)x^{\alpha+j-2} + a_j x^{\alpha+j} - \nu^2 a_j x^{\alpha+j-2} = 0$$

Making the substitution $j \to 2j$ and matching coefficients,

$$a_{2j} = -\frac{1}{(a+2j)^2 - \nu^2} a_{2j-2}$$

Apparently we then want to set $\alpha = \pm\nu$, but I'm not entirely sure what motivates this other than that it gives us the coefficient in a nice form (3.32). Iterating, this gives the coefficients (3.33) with $a_0 = [2\alpha\Gamma(\alpha+1)]^{-1}$.

$$a_{2j} - \frac{1}{4j(j+\alpha)} a_{2j-2} \tag{3.32}$$

$$a_{2j} = \frac{(-1)^j \Gamma(\alpha+1)}{2^{2j} j! \Gamma(j+\alpha+1)} a_0 \tag{3.33}$$

Substituting these back in, we get the Bessel functions of the first kind (3.34). I'm not entirely sure why Jackson writes them using the gamma function (probably because of Bessel functions of the second and third kind) since the argument will always be an integer if ν is an integer. The gamma function is defined as $\Gamma(n) = (n-1)!$, so using this, we can write Bessel functions of the first kind without the gamma function.

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j} \tag{3.34}$$

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+\nu)!} \left(\frac{x}{2}\right)^{2j}$$

We have some recursion relations (3.35) and (3.36) that are apparently a little difficult to prove from the ground up. Instead, we'll just verify that they hold true.

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) \tag{3.35}$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2\frac{dJ_\nu(x)}{dx} \tag{3.36}$$

To prove the first recursion, we write the Bessel functions using equation (3.34). Note that there should be a sum over j, but I leave it out since I don't want to write it in every line.

$$\frac{(-1)^j}{j!(j+\nu-1)!} \left(\frac{x}{2}\right)^{2j+\nu-1} + \frac{(-1)^j}{j!(j+\nu+1)!} \left(\frac{x}{2}\right)^{2j+\nu+1} - \frac{2\nu}{x} \frac{(-1)^j}{j!(j+\nu)!} \left(\frac{x}{2}\right)^{2j+\nu} = 0$$

In the second term, we make the substitution $2j+1 \to 2j-1$, or $j \to j-1$ so as to match coefficients,

$$\frac{(-1)^j}{j!(j+\nu-1)!} \left(\frac{x}{2}\right)^{2j+\nu-1} + \frac{(-1)^{j-1}}{(j-1)!(j+\nu)!} \left(\frac{x}{2}\right)^{2j+\nu-1} - \frac{(-1)^j \nu}{j!(j+\nu)!} \left(\frac{x}{2}\right)^{2j+\nu-1} = 0$$

$$\frac{j+\nu}{j!(j+\nu)!} - \frac{j}{j!(j+\nu)!} - \frac{\nu}{j!(j+\nu)!} = 0$$

And we're done. To prove the second recursion (3.36), we follow the same steps including the same substitution in the second step,

$$\frac{(-1)^j}{j!(j+\nu-1)!} \left(\frac{x}{2}\right)^{2j+\nu-1} - \frac{(-1)^j}{j!(j+\nu+1)!} \left(\frac{x}{2}\right)^{2j+\nu+1} - 2\frac{(-1)^j}{j!(j+\nu)!} \left(\frac{x}{2}\right)^{2j+\nu-1} \frac{2j+\nu}{2} = 0$$

$$\frac{(-1)^j}{j!(j+\nu-1)!} - \frac{(-1)^{j-1}}{(j-1)!(j+\nu)!} - \frac{(2j+\nu)(-1)^j}{j!(j+\nu)!} = 0$$

$$\frac{j+\nu}{j!(j+\nu)!} + \frac{j}{j!(j+\nu)!} - \frac{2j+\nu}{j!(j+\nu)!} = 0$$

3.5.2 Neumann Functions and Hankel Functions

I don't think these get used as much. For most purposes, it seems we use Bessel functions of the first kind. We have Neumann functions, or Bessel functions of the second kind (3.37), and Hankel functions, or Bessel functions of the third kind (3.38).

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \tag{3.37}$$

$$\begin{cases} H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x) \\ H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x) \end{cases} \tag{3.38}$$

We can convince ourselves that the recursion relations (3.35) and (3.36) also hold true for the Neumann and Hankel functions.

3.5.3 Orthogonality of Bessel Functions

To prove the orthogonality of the Bessel functions, we go back to equation (3.31). If we set $x = k\rho$, it can be shown by substitution,

$$\rho^2 \frac{d^2}{d\rho^2} J_\nu(k\rho) + \rho \frac{d}{d\rho} J_\nu(k\rho) + (k^2\rho^2 - \nu^2) J_\nu(k\rho) = 0$$

We set $k = \alpha_{\nu m}/a$,

$$\rho \frac{d^2}{d\rho^2} J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right) + \frac{d}{d\rho} J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right) + \left(\frac{\alpha_{\nu m}^2 \rho}{a^2} - \frac{\nu^2}{\rho}\right) J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right) = 0$$

Multiplying by $J_\nu(\alpha_{\nu n}\rho/a)$,

$$\rho J_\nu \left(\alpha_{\nu n} \frac{\rho}{a}\right) \frac{d^2}{d\rho^2} J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right) + J_\nu \left(\alpha_{\nu n} \frac{\rho}{a}\right) \frac{d}{d\rho} J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right) + \left(\frac{\alpha_{\nu m}^2 \rho}{a^2} - \frac{\nu^2}{\rho}\right) J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right) J_\nu \left(\alpha_{\nu n} \frac{\rho}{a}\right) = 0$$

If we had started with $J_\nu(\alpha_{\nu n}\rho/a)$ and multiplied by $J_\nu(\alpha_{\nu m}\rho/a)$ and then subtract the two,

$$J_\nu \left(\alpha_{\nu n} \frac{\rho}{a}\right) \frac{d}{d\rho} \left[\rho \frac{d}{d\rho} J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right)\right] - J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right) \frac{d}{d\rho} \left[\rho \frac{d}{d\rho} J_\nu \left(\alpha_{\nu n} \frac{\rho}{a}\right)\right] = \frac{\alpha_{\nu n}^2 - \alpha_{\nu m}^2}{a^2} \rho J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right) J_\nu \left(\alpha_{\nu n} \frac{\rho}{a}\right)$$

Integrating from $\rho = 0$ to $\rho = a$, we want to use integration by parts on the first two terms with $u = J_\nu(\alpha_{\nu n}\rho/a)$ and $dv = \frac{d}{d\rho}\left[\rho \frac{d}{d\rho} J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right)\right]$. The left side becomes

$$\rho J_\nu \left(\alpha_{\nu n} \frac{\rho}{a}\right) \frac{d}{d\rho} J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right) \Bigg|_0^a - \int_0^a \rho \frac{d}{d\rho} J_\nu \left(\alpha_{\nu n} \frac{\rho}{a}\right) \frac{d}{d\rho} J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right)\, d\rho$$
$$-\rho J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right) \frac{d}{d\rho} J_\nu \left(\alpha_{\nu n} \frac{\rho}{a}\right) \Bigg|_0^a + \int_0^a \rho \frac{d}{d\rho} J_\nu \left(\alpha_{\nu n} \frac{\rho}{a}\right) \frac{d}{d\rho} J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right)\, d\rho$$

If we choose α_{nun} such that it returns the roots of the Bessel functions at $\rho = a$, we see that the first and third terms die. The second and fourth terms cancel. We are left with

$$\int_0^a J_\nu \left(\alpha_{\nu m} \frac{\rho}{a}\right) J_\nu \left(\alpha_{\nu n} \frac{\rho}{a}\right) \rho\, d\rho = 0$$

To normalize this, we need to combine the recursion relations. Subtracting them from each other,

$$J_{\nu+1} = \frac{\nu}{x} J_\nu - \frac{dJ_\nu}{dx}$$

Making the substitution, $x = \alpha_\nu \rho/a$,

$$J_{\nu+1} = \frac{\nu a}{\alpha_\nu \rho} J_\nu - \frac{a}{\alpha_\nu} J_\nu'$$

We also want to make the substitution $\alpha_{\nu n} = \alpha_{\nu m} + \epsilon$ and Taylor expand,

$$J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right) = J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right) + \epsilon\frac{a}{\alpha_{\nu m}}J'_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)$$

Note that we have to add a factor of a/α since we go go from taking a derivative in x to taking a derivative according to ρ. For the following, I am going to make the substitutions, $J_{\nu n} = J_\nu(\alpha_{\nu n}\rho/a)$, $\alpha_{\nu m} = \alpha$. We have the equation,

$$\rho J_{\nu n}J'_\nu - \rho J_\nu J'_{\nu n} = \frac{2\alpha\epsilon}{a^2}\int_0^a J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right) J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\rho\, d\rho$$

On the left hand side of the equation, we can get rid of terms involving J_ν since we are evaluating at $\rho = a$, and as we showed previously, those go to zero.

$$\rho J_{\nu n}J'_\nu = \rho(J_\nu + \epsilon_0 J'_\nu)J'_\nu$$

$$= \rho\frac{\alpha^2}{a^2}J^2_{\nu+1}\frac{a}{\alpha}$$

From this, we get the orthonormality of the Bessel functions of the first kind, given by equation (3.39).

$$\int_0^a \rho J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)\ d\rho = \frac{a^2}{2}[J_{\nu+1}(x_{\nu n})]^2\delta_{n'n} \tag{3.39}$$

3.5.4 Modified Bessel Functions

If we modify the Bessel differential equation (3.31) so that we get equation (3.40), the solutions are given by the modified Bessel functions, (3.41) and (3.42).

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)R = 0 \tag{3.40}$$

$$I_\nu(x) = i^{-\nu}J_\nu(ix) \tag{3.41}$$

$$K_\nu(x) = \frac{\pi}{2}i^{\nu+1}H^{(1)}_\nu(ix) \tag{3.42}$$

3.6 Separation of Variables: Cylindrical Coordinates

3.6.1 Laplace's Equation

In cylindrical coordinates, Laplace's equation (1.36) can be written as equation (3.43). We want to use the method of separation of variables, so $\Phi = R(\rho)Q(\phi)Z(z)$.

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{3.43}$$

Substituting in the potential,

$$QZ\frac{d^2R}{d\rho^2} + \frac{QZ}{\rho}\frac{dR}{d\rho} + \frac{RZ}{\rho^2}\frac{d^2Q}{d\phi^2} + RQ\frac{d^2Z}{dz^2} = 0$$

We want to start by isolating the z-component, which we do by dividing by RQZ,

$$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{R\rho}\frac{dR}{d\rho} + \frac{1}{Q\rho^2}\frac{d^2Q}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

Since we've isolated the z-component, let's set it equal to a constant, k^2. This gives us a simple harmonic oscillator equation (3.44), which we know the solution to (3.45).

$$\frac{d^2Z}{dz^2} - k^2Z = 0 \tag{3.44}$$

$$Z = e^{\pm kz} \tag{3.45}$$

We can do the same thing for the angular component. We multiply by $\rho^2/(RQZ)$ and set the resulting component equal to $-\nu^2$ (to give us the Bessel differential equation later). This gives us equation (3.46), which is solved by equation (3.47).

$$\frac{d^2Q}{d\phi^2} + \nu^2Q = 0 \tag{3.46}$$

$$Q = e^{\pm i\nu\phi} \tag{3.47}$$

Substituting this back into equation (3.43), we get equation (3.48). Making the substitution $x = k\rho$ and absorbing all the factors of k^2 into R, we get the Bessel differential equation (3.31).

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0 \tag{3.48}$$

Combining all of the separated variables, we get the general potential for a boundary-value problem with cylindrical symmetry (3.49) where Ω is one of the Bessel functions.

$$\Phi(\rho,\phi,z) = \sum_{k=0}^{\infty}\sum_{\nu=-\infty}^{\infty} e^{\pm kz}e^{\pm i\nu\phi}\Omega(k\rho) \tag{3.49}$$

3.6.2 Green Function

The expansion of $1/R$ in cylindrical coordinates is given by equation (3.50).

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \int_0^{infty} e^{im(\phi-\phi')} \cos[k(z-z')] I_m(k\rho_<) K_m(k\rho_>) \; dk \tag{3.50}$$

You should now be able to do Jackson 3.9, 3.10, 3.11, and 3.16.

Chapter 4

Multipoles and Dielectrics

"Thus did this poor soul struggle in its anguish. Eighteen hundred years before this ill-fated man, the mysterious being in whom are concentrated all the saintliness and all the sufferings of humanity had also refused for a long time the terrible chalice, streaming with darkness and brimming with shadows, that appeared to him in the star-filled depths while the olive trees shook in the fierce blast of the infinite." - (Victor Hugo, The Wretched)

In this chapter, we finish our study of electrostatics with a look at multipole expansion and dielectrics. We've already seen hints at multipoles in previous chapters with brief looks at dipoles. So far we've been looking at electric fields in a vacuum, but the question arises of what happens if we look at the electric field in some material. Jackson does provide some models for what causes this change in electric field, but I will not be going over that here as that is a little too much condensed-matter for me.

4.1 Multipole Expansion

In undergraduate physics, we saw dipoles as two point charges of equal and opposite charges Q and $-Q$, a distance d apart. If we remember back to undergraduate, if we observe the dipole from far away, to first order, the two charges cancel each other, and we have to look at higher order terms to get a non-zero potential. These higher order terms are referred to as multipole terms.

4.1.1 Multipole Moment

Imagine we have some charge density $\rho(\vec{x}')$ constrained to inside of a sphere of radius R. Using equation (1.23), the potential of this system is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3x'$$

Expanding in terms of the spherical harmonics (3.29) with $r_< = r'$ and $r_> = r$,

$$\Phi(\vec{x}) = \frac{1}{\epsilon_0} \int \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r'^l}{r^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \rho(\vec{x}') \, d^3x'$$

$$= \frac{1}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left[\int Y_{lm}^*(\theta', \phi') r'^l \rho(\vec{x}')\ d^3x' \right] \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$$

From this, we can see that the potential can be written as equation (4.1) where q_{lm} are the multipole moments (4.2). We see that if we look at the $l = 0$ term, we get the potential due to a point charge. The $l = 1$ terms are the dipole terms, and $l = 2$ are quadrupole terms. Equation (4.1) is also referred to as the multipole expansion. In general, we don't really look beyond the quadrupole term.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \tag{4.1}$$

$$q_{lm} = \int Y_{lm}^*(\theta', \phi') r'^l \rho(\vec{x}')\ d^3x' \tag{4.2}$$

If we solve for the first couple multipole moments explicitly, we can see that q is the monopole moment. Similarly, we have a dipole moment (4.3) and quadrupole moment tensor (4.4). We can check that the dipole moment follows expectation by thinking about two charges a distance d apart. Placing the negative charge at the origin and the positive charge at $x = d$, we return the familiar $p = Qd$. Further, we see that the multipole expansion can be written as (4.5).

$$\vec{p} = \int \vec{x}' \rho(\vec{x}')\ d^3x' \tag{4.3}$$

$$Q_{ij} = \int (3x_i' x_j' - r'^2 \delta_{ij}) \rho(\vec{x}')\ d^3x' \tag{4.4}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{r} + \frac{\vec{p} \cdot \vec{x}}{r^3} + \frac{1}{2} \sum_{ij} Q_{ij} \frac{x_i x_j}{r^5} + ... \right] \tag{4.5}$$

4.1.2 Example: Multipole Moments of a Dipole

Imagine we have a dipole configuration that we are used to from undergraduate. We have two point charges with equal, opposite charges. We place the particle with charge Q at the origin and the particle with charge $-Q$ at $(0, 0, -d)$. Note that we keep the two particles on the same axis. This is allowed because we can rotate any arbitrary coordinate system such that this holds true. Using equation (1.23), the potential of this system can be found by superposition,

$$\Phi = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{1}{|\vec{r} + \vec{d}|} \right)$$

We can expand the second term using Taylor expansion,

$$\frac{1}{|\vec{r} + \vec{d}|} = \frac{1}{r} + \vec{d} \cdot \nabla \frac{1}{r} + \frac{1}{2} (\vec{d} \cdot \nabla)^2 \frac{1}{r}$$

$$= \frac{1}{r} - \frac{\vec{d} \cdot \vec{r}}{r^3} - \frac{1}{2} \left(\frac{\vec{d} \cdot \vec{d}}{r^3} - \frac{3(\vec{d} \cdot \vec{r})^2}{r^5} \right)$$

Substituting this back into our potential,

$$\Phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{\vec{d}\cdot\vec{r}}{r^3} - \frac{3}{2}\frac{(\vec{d}\cdot\vec{r})^2}{r^5} + \frac{1}{2}\frac{\vec{d}\cdot\vec{d}}{r^3}\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{dz}{r^3} - \frac{3}{2}\frac{(dz)^2}{r^5} + \frac{1}{2}\frac{d^2}{r^3}\right)$$

Let's compare this to equation (4.5). First, we see no monopole term since the total charge of this system is 0. The charge density is

$$\rho(\vec{x}') = Q\delta(\vec{x}') - Q\delta(x')\delta(y')\delta(z'+d)$$

Substituting this into equation (4.3) to find the electric dipole moment,

$$\vec{p} = (0, 0, Qd)$$

Looking at the quadrupole moment tensor (4.4), let's for example look at Q_{xx},

$$Q_{xx} = \int (3x'x' - r'^2)\rho(\vec{x}')\ d^3x'$$

The first term dies from delta functions, and we are left with

$$Q_{xx} = Qd^2$$

We can convince ourselves that $Q_{yy} = Q_{xx}$. Furthermore, all other terms other than Q_{zz} go to zero by delta functions.

$$Q_{zz} = -3Qd^2 + Qd^2$$

Substituting all of this in,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p}\cdot\vec{r}}{r^3} + \frac{1}{2}\frac{1}{r^5}\left(Q_{xx}x^2 + Q_{yy}y^2 + Q_{zz}z^2\right)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{Qdz}{r^3} + \frac{1}{2}\left(\frac{-3Qd^2z^2}{r^5} + \frac{Qd^2r^2}{r^5}\right)\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{dz}{r^3} - \frac{3}{2}\frac{(dz)^2}{r^5} + \frac{1}{2}\frac{d^2}{r^3}\right)$$

And we see that we get the expected result.

4.1.3 Electric Field Multipole

To find the electric field due to a dipole, we use equation (1.22). Ignoring the monopole and quadrupole terms in equation (4.5), and taking the derivative, we get the electric field due to a dipole (4.6). $\hat{n}$ is the unit vector pointing from the dipole to the observation point. Also note that the field is not pointing always towards or away from the dipole as is the case with the monopole. That is, in some parts, the force on a test particle due to a dipole will be positive, but in other parts, it will be negative.

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{x}}{|\vec{x}-\vec{x}_0|^3}$$

$$\vec{E}(\vec{x}) = \frac{3\hat{n}(\vec{p}\cdot\hat{n})-\vec{p}}{4\pi\epsilon_0|\vec{x}-\vec{x}_0|^3} \tag{4.6}$$

4.1.4 Energy due to a Multipole

Imagine we have a charge distribution $\rho(\vec{x})$ in an external potential $\Phi(\vec{x})$. Looking at equation (1.28), we can convince ourselves that the energy of the system is given by

$$W = \int \rho(\vec{x})\Phi(\vec{x})\ d^3x$$

Performing a Taylor expansion on the potential,

$$\Phi(\vec{x}) = \Phi(0) + \vec{x}\cdot\nabla\Phi(0) + \frac{1}{2}\sum_i\sum_j x_i x_j \frac{\partial^2\Phi}{\partial x_i \partial x_j}(0) + ...$$

Using the dipole moment (4.3) and quadrupole moment (4.4), we find the energy of a multipole in an external field (4.7).

$$W = q\Phi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\sum_i\sum_j Q_{ij}\frac{\partial E_j(0)}{\partial x_i} + ... \tag{4.7}$$

You should now be able to do Jackson 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, and 4.7.

4.2 Dielectrics

The reason why we include dielectrics in the same section as multipole expansions is because the behaviour of the electric field in a dielectric is explained using multipole expansion. I won't be going too deep into the mechanisms of how this works. If an electric field is applied to an atom, it perturbs the nucleus and the electron cloud. This in turn creates a dipole and a corresponding dipole moment. If there are multiple atoms, there is an induced electric polarization.

4.2.1 Electric Displacement

Faraday's Law (1.21) stays the same in a dipole, but Gauss's Law becomes,

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0}[\rho - \nabla \cdot \vec{P}]$$

We can rearrange this to pop out the electric displacement,

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

Using the electric displacement, we can rewrite Gauss's law (1.18,

$$\nabla \cdot \vec{D} = \rho$$

Further, if we assume the induced polarization is parallel to the electric field, we can write the polarization (4.8) as a re-scaling of the electric field where χ_e is the electric susceptibility of the medium. Substituting this into the electric displacement, we see that the displacement (4.9) is parallel to the electric field where ϵ is the electric permittivity (4.10). Gauss's law then can be written as equation (4.11).

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \tag{4.8}$$

$$\vec{D} = \epsilon \vec{E} \tag{4.9}$$

$$\epsilon = \epsilon_0 (1 - \chi_e) \tag{4.10}$$

$$\nabla \cdot \vec{E} = \rho / \epsilon \tag{4.11}$$

What happens if we have two materials with different electric permittivity bordering each other? The boundary conditions (1.26) and (1.27) can then be written as

$$(\vec{D}_2 - \vec{D}_1) \cdot \hat{n}_{21} = \sigma \tag{4.12}$$

$$(\vec{E}_2 - \vec{E}_1) \times \hat{n}_{21} = 0 \tag{4.13}$$

4.2.2 Method of Images in Dielectrics

Imagine we have two semi-infinite dielectrics with electric permittivity ϵ_1 and ϵ_2 with a charge q embedded in the first dielectric a distance d from the second. We want to know the potential everywhere. We can actually use the method of images here. We can convince ourselves that since we are using method of images, we can put an image charge in the second medium on the same axis. This gives a potential in the first region of

$$\Phi_1 = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{\sqrt{\rho^2+(d-z)^2}} + \frac{q'}{\sqrt{\rho^2+(d+z)^2}}\right)$$

In the second region,

$$\Phi_2 = \frac{1}{4\pi\epsilon_2}\frac{q''}{\sqrt{\rho^2+(d-z)^2}}$$

We want the tangential components of the electric field to be continuous at the boundary (4.13) while the normal component of the electric field has a discontinuity (4.12).

$$\epsilon_2\frac{\partial\Phi_2}{\partial z}\Big|_{z=0} = \epsilon_1\frac{\partial\Phi_1}{\partial z}\Big|_{z=0}$$

$$\frac{\partial\Phi_1}{\partial\rho}\Big|_{z=0} = \frac{\partial\Phi_2}{\partial\rho}\Big|_{z=0}$$

This gives the requirements

$$\begin{cases} q'' = q - q' \\ \frac{1}{\epsilon_1}(q+q') = \frac{1}{\epsilon_2}q'' \end{cases}$$

Solving, we find,

$$\begin{cases} q' = -\left(\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\right)q \\ q'' = \left(\frac{2\epsilon_2}{\epsilon_2+\epsilon_1}\right)q \end{cases}$$

4.2.3 Boundary-Value Problems in Dielectrics

Imagine we have a dielectric sphere with electric permittivity ϵ in a uniform electric field. Using equation (3.19),

$$\begin{cases} \Phi_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta)) \\ \Phi_{out} = \sum_{l=0}^{\infty}[B_l r^l + C_l r^{-(l+1)}]P_l(\cos(\theta)) \end{cases}$$

We know that very far away, $\Phi \to -E_0 r\cos(\theta)$, so $B_1 = -E_0$ and all other B terms go to zero. Using the continuity in the electric field (4.13),

$$-\frac{1}{a}\frac{\partial\Phi_{in}}{\partial\theta}\Big|_{r=a} = -\frac{1}{a}\frac{\partial\Phi_{out}}{\partial\theta}\Big|_{r=a}$$

Using the discontinuity in the normal displacement (4.12),

$$-\epsilon \frac{\partial \Phi_{in}}{\partial r}\bigg|_{r=a} = -\epsilon_0 \frac{\partial \Phi_{out}}{\partial r}\bigg|_{r=a}$$

Solving,

$$\Phi_{in} = -\left(\frac{3}{\epsilon/\epsilon_0 + 2}\right) E_0 r \cos(\theta)$$

$$\Phi_{out} = -E_0 r \cos(\theta) + \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) E_0 \frac{a^3}{r^2} \cos(\theta)$$

You should now be able to do Jackson 4.8, 4.9, and 4.10.