Solutions to Modern Quantum Mechanics by Sakurai

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Contents

Chapter 1

Fundamental Concepts

1.1 Commutation Relations

Prove

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB$$

I think it's easier if we start with the right hand side. Expanding out the anti-commutation relations (1.10),

$$-AC(DB + BD) + A(CB + BC)D - C(DA + AD)B + (CA + AC)DB$$

Distributing and grouping terms,

$$= -ACDB - ACBD + ACBD + ABCD - CDAB_C ADB + CADB + ACDB$$

$$= ABCD - CDAB = (AB)(CD) - (CD)(AB) = [AB, CD]$$

1.2 Pauli Matrices

Suppose a 2×2 matrix X (not necessarily Hermitian, nor unitary) is written as

$$X = a_0 + \vec{\sigma} \cdot \vec{a}$$

where a_0 and $a_{1,2,3}$ are numbers.

1.2.a How are a_0 and a_k $(k = 1, 2, 3)$ related to $\mathrm{Tr}(X)$ and $\mathrm{Tr}(\sigma_k X)$?

We can take the trace of X,

$$\mathrm{Tr}\, X = \mathrm{Tr}(a_0 I) + \mathrm{Tr}(\vec{\sigma} \cdot \vec{a})$$

By definition, the Pauli matrices (σ) are traceless, so re-scaling them by a scalar factor does nothing to the trace.

$$\mathrm{Tr}(X) = \mathrm{Tr}(a_0 I) = 2a_0$$

If we multiply X by one of the Pauli matrices, we should write this out explicitly,

$$\sigma_k X = a_0 \sigma_k + a_1 \sigma_k \sigma_1 + a_2 \sigma_k \sigma_2 + a_3 \sigma_k \sigma_3$$

When we take the trace, the first term will die since that is just one of the Pauli matrices. Remembering the following relation,

$$\sigma_a \sigma_b = \delta_{ab} I + i\epsilon_{abc}\sigma_c$$

we see that for $a \neq b$, we get just another Pauli matrix. Thus, only the k term survives,

$$\mathrm{Tr}(\sigma_k) = 2a_k$$

Rewriting for convenience,

$$\begin{cases} a_0 = {}^1\!/_2\ \mathrm{Tr}(X) \\ a_k = {}^1\!/_2\ \mathrm{Tr}(\sigma_k X) \end{cases}$$

1.2.b Obtain a_0 and a_k in terms of the matrix elements X_{ij}

As a reminder, the Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$X = \begin{bmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{bmatrix}$$

Multiplying each Pauli matrix by X, i.e., $\sigma_k X$,

$$\begin{cases} \sigma_1 X = \begin{bmatrix} X_{21} & X_{22} \\ X_{11} & X_{12} \end{bmatrix} \\ \\ \sigma_2 X = \begin{bmatrix} -iX_{21} & -iX_{22} \\ iX_{11} & iX_{12} \end{bmatrix} \\ \\ \sigma_3 X = \begin{bmatrix} X_{11} & X_{12} \\ -X_{21} & -X_{22} \end{bmatrix} \end{cases}$$

Using the results from the previous part,

$$\begin{cases} a_0 = {}^1\!/\!_2 \ \mathrm{Tr}(X) = {}^1\!/\!_2 \ (X_{11} + X_{22}) \\ a_1 = {}^1\!/\!_2 \ \mathrm{Tr}(\sigma_1 X) = {}^1\!/\!_2 \ (X_{21} + X_{12}) \\ a_2 = {}^1\!/\!_2 \ \mathrm{Tr}(\sigma_2 X) = {}^1\!/\!_2 \ (-iX_{21} + iX_{12}) \\ a_3 = {}^1\!/\!_2 \ \mathrm{Tr}(\sigma_3 X) = {}^1\!/\!_2 \ (X_{11} - X_{22}) \end{cases}$$

1.3 Invariant Determinant

Show that the determinant of a 2×2 matrix $\vec{\sigma} \cdot \vec{a}$ is invariant under

$$\vec{\sigma} \cdot \vec{a} \to \vec{\sigma} \cdot \vec{a}' = \exp\left(\frac{i\vec{\sigma} \cdot \hat{n}\phi}{2}\right) \vec{\sigma} \cdot \vec{a} \exp\left(\frac{-i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)$$

Find a'_k in terms of a_k when $\hat{n}$ is in the positive z-direction and interpret your result.

Let's go ahead and take the determinant of both sides. What really matters is the right side, so let's look at that one. We know we can break up the determinant,

$$\det(\vec{\sigma} \cdot \vec{a}') = \det\left(\exp\left(\frac{i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right) \det(\vec{\sigma} \cdot \vec{a}) \det\left(\exp\left(\frac{-i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right)$$

Each determinant is just a scalar, so we can rearrange them for free,

$$= \det\left(\exp\left(\frac{i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right) \det\left(\exp\left(\frac{-i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right) \det(\vec{\sigma} \cdot \vec{a})$$

$$= \det\left(\exp\left(\frac{i\vec{\sigma} \cdot \hat{n}\phi}{2}\right) \exp\left(\frac{-i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right) \det(\vec{\sigma} \cdot \vec{a})$$

$$\det(\vec{\sigma} \cdot \vec{a}') = \det(\vec{\sigma} \cdot \vec{a})$$

We now want to find a'_k. We set,

$$\hat{n} = \hat{z} = (0, 0, 1)$$

Substituting this in and writing out explicitly,

$$\vec{\sigma} \cdot \vec{a}' = \exp\left(\frac{i\vec{\sigma}_z\phi}{2}\right) \vec{\sigma} \cdot \vec{a} \exp\left(\frac{-i\vec{\sigma}_z\phi}{2}\right)$$

$$= \begin{pmatrix} \exp(i\phi/2) & 0 \\ 0 & \exp(-i\phi/2) \end{pmatrix} \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} \begin{pmatrix} \exp(-i\phi/2) & 0 \\ 0 & \exp(i\phi/2) \end{pmatrix}$$

$$= \begin{pmatrix} a_3 & (a_1 - ia_2)\exp(i\phi) \\ (a_1 + ia_2)\exp(-i\phi) & -a_3 \end{pmatrix}$$

Using the results from question 1.2, with $X = a_0 + \vec{\sigma} \cdot \vec{a}$

$$\begin{cases} a'_0 = \dfrac{a_3 - a_3}{2} = 0 \\ a'_1 = 1/2\,[(a_1 + ia_2)\exp(-i\phi) + (a_1 - ia_2)\exp(i\phi)] = a_1\cos(\phi) + a_2\sin(\phi) \\ a'_2 = 1/2\,[-i(a_1 + ia_2)\exp(-i\phi) + i(a_1 - ia_2)\exp(i\phi)] = -a_1\sin(\phi) + a_2\cos(\phi) \\ a'_3 = 1/2\,(a_3 + a_3) = a_3 \end{cases}$$

This is rotation about the z-axis.

1.4 Bra-Ket Algebra

Using the rules of bra-ket algebra, prove or evaluate the following:

1.4.a $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$, where X and Y are operators

See Sakurai 1.7.1.

1.4.b $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators

Let's act XY on some ket, $|\alpha\rangle$,

$$(XY)\,|\alpha\rangle$$

In bra-space (1.1),

$$\langle\alpha|\,(XY)^\dagger$$

Alternatively,

$$XY\,|\alpha\rangle = X(Y\,|\alpha\rangle)$$

In dual correspondence (1.1),

$$\langle\alpha|\,Y^\dagger X^\dagger$$

Comparing these two cases,

$$(XY)^\dagger = Y^\dagger X^\dagger$$

1.4.c $\exp[if(A)]$ =? in ket-bra form, where A is a Hermitian operator whose eigenvalues are known

Let's act the function on a vector,

$$\exp(if(A))\,|\alpha\rangle = [\cos(f(A)) + i\sin(f(A))]\,|\alpha\rangle$$

Since we know the eigenvalues,

$$= [\cos(f(\alpha)) + i\sin(f(\alpha))]\,|\alpha\rangle$$

Matching solutions,

$$\exp(if(A)) = \cos(f(A)) + i\sin(f(A))$$

1.4.d $\sum_{a'} \psi^*_{a'}(\vec{x}')\psi_{a'}(\vec{x}'')$, where $\psi_{a'}(\vec{x}') = \langle \vec{x}' | a' \rangle$

Writing it out,

$$\sum_{a'} \psi^*_{a'}(\vec{x}')\psi_{a'}(\vec{x}'') = \sum_{a'} \langle a' | \vec{x}' \rangle \langle \vec{x}'' | a' \rangle$$

The middle two terms must be equal,

$$\sum_{a'} \psi^*_{a'}(\vec{x}')\psi_{a'}(\vec{x}'') = \delta_{\vec{x}',\vec{x}''}$$

1.5 Matrix Representation

1.5.a Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Suppose $\langle a'|\alpha\rangle$, $\langle a''|\alpha\rangle$,... and $\langle a'|\beta\rangle$, $\langle a''|\beta\rangle$,... are all known, where $|a'\rangle$, $|a''\rangle$,... form a complete set of base kets. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$ in that basis

The answer is given in the text,

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} \langle a'|\alpha\rangle\langle a'|\beta\rangle^* & \langle a'|\alpha\rangle\langle a''|\beta\rangle^* & \dots \\ \langle a''|\alpha\rangle\langle a'|\beta\rangle^* & \langle a''|\alpha\rangle\langle a''|\beta\rangle^* & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

1.5.b We now consider a spin-1/2 system and let $|\alpha\rangle$ and $|\beta\rangle$ be $|s_z = \hbar/2\rangle$ and $|s_x = \hbar/2\rangle$, respectively. Write down explicitly the square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the usual (s_z diagonal) basis.

We expect to get a 2×2 matrix,

$$|s_z = \hbar/2\rangle\langle s_x = \hbar/2| = |+\rangle \, {}^1\!/\!\sqrt{2}(\langle +| + \langle -|)$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

1.6 Adding Eigenkets

Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator A. Under what condition can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of A? Justify your answer.

If we act A on our eigenkets,

$$\begin{cases} A\,|i\rangle = a\,|i\rangle \\ A\,|j\rangle = a'\,|j\rangle \end{cases}$$

In order for $|i\rangle + |j\rangle$ to be an eigenket,

$$A(|i\rangle + |j\rangle) = a''(|i\rangle + |j\rangle)$$

Alternatively,

$$A(|i\rangle + |j\rangle) = A\,|i\rangle + A\,|j\rangle = a\,|i\rangle + a'\,|j\rangle$$

Comparing these two results, they are only equal if either $|i\rangle = |j\rangle$ or $a = a'$, i.e., the eigenvalues are degenerate.

1.7 Ket Space

Consider a ket space spanned by the eigenkets of $\{|a'\rangle\}$ of a Hermitian operator A. There is no degeneracy.

1.7.a Prove that

$$\prod_{a'}(A - a')$$

is the null operator Let's act A on some unsuspecting eigenvector,

$$A\,|\Psi\rangle = a'\,|\Psi\rangle$$

$$A\,|\Psi\rangle - a'\,|\Psi\rangle = |0\rangle$$

$$(A - a'I)\,|\Psi\rangle = 0$$

$A - a' = 0$ for at least one case. Since we product over all a', if $A - a' = 0$ for one case, then the product over all of those is 0.

1.7.b What is the significance of

$$\prod_{a''\neq a'} \frac{(A - a'')}{(a' - a'')}$$

If we act the given on $|a'\rangle$,

$$\prod_{a''\neq a'} \frac{(A - a'')}{(a' - a'')}\,|a'\rangle = \prod_{a''\neq a'} \frac{(a' - a'')}{(a' - a'')}\,|a'\rangle$$

The product cancels out,

$$\prod_{a''\neq a'} \frac{(A - a'')}{(a' - a'')}\,|a'\rangle = |a'\rangle$$

If we now act it on an eigenvector,

$$\prod_{a''\neq a'} \frac{(A - a'')}{(a' - a'')}\,|\Psi\rangle$$

We can insert identity and use the above relation,

$$= \prod_{a''\neq a'} \frac{(A - a'')}{(a' - a'')}\,|a'\rangle\,\langle a'|\Psi\rangle$$

$$\prod_{a''\neq a'} \frac{(A - a'')}{(a' - a'')} |\Psi\rangle = |a'\rangle \langle a'|\Psi\rangle$$

This is the projection operator (1.16) of $|a'\rangle$.

1.7.c Illustrate (a) and (b) using A set equal to S_z of a spin $1/2$ system.

As a reminder,

$$S_z = \begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix}$$

with eigenvalues $\omega = \pm\hbar/2$. Showing part (a), we substitute in $A = S_z$ and a' as the eigenvalues,

$$\prod_{a'} (A - a') = (S_z - \hbar/2)(S_z + \hbar/2)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & -\hbar \end{bmatrix} \begin{bmatrix} \hbar & 0 \\ 0 & 0 \end{bmatrix}$$

$$\prod_{a'} (A - a') = 0$$

For part (b), we have $a' = \hbar/2$ and $a'' = -\hbar/2$,

$$\prod_{a''\neq a'} \frac{(A - a'')}{(a' - a'')} = \frac{S_z + \hbar/2}{\hbar} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Acting this on some general vector,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

We pick out the spin-up component.

1.8 Orthonormality

Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove

$$[S_i, S_j] = i\epsilon_{ijk}\hbar S_k, \quad \{S_i, S_j\} = \left(\frac{\hbar^2}{2}\right)\delta_{ij}$$

where

$$\begin{cases} S_x = \dfrac{\hbar}{2}(|+\rangle\langle-| + |-\rangle\langle+|) \\ \\ S_y = \dfrac{i\hbar}{2}(-|+\rangle\langle-| + |-\rangle\langle+|) \\ \\ S_z = \dfrac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|) \end{cases}$$

As an example, let's set $i = x$ and $j = y$ and brute force,

$$[S_x, S_y] = S_xS_y - S_yS_x = \frac{i\hbar^2}{4}[(|+\rangle\langle-|) + (|-\rangle\langle+|)][-(|+\rangle\langle-|) + (|-\rangle\langle+|)]$$
$$-\frac{i\hbar^2}{4}[-(|+\rangle\langle-|) + (|-\rangle\langle+|)][(|+\rangle\langle-|) + (|-\rangle\langle+|)]$$

$$= \frac{i\hbar^2}{4}[-(|+\rangle\langle-|)(|+\rangle\langle-|) + (|+\rangle\langle-|-\rangle\langle+|) - (|-\rangle\langle+|+\rangle\langle-|) + (|-\rangle\langle+|-\rangle\langle+|)$$
$$+(|+\rangle\langle-|+\rangle\langle-|) + (|+\rangle\langle-|-\rangle\langle+|) - (|-\rangle\langle+|+\rangle\langle-|) - (|-\rangle\langle+|-\rangle\langle+|)]$$

Using the orthonormality relationships,

$$\begin{cases} \langle+|+\rangle = \langle-|-\rangle = 1 \\ \langle+|-\rangle = \langle-|+\rangle = 0 \end{cases}$$

$$= \frac{i\hbar^2}{2}[(|+\rangle\langle+|) - (|-\rangle\langle-|)] = i\hbar S_z$$

We do the same thing with the anti-commutation relation,

$$\{S_x, S_y\} = S_xS_y + S_yS_x = \frac{i\hbar^2}{4}[(|+\rangle\langle-|) + (|-\rangle\langle+|)][-(|+\rangle\langle-|) + (|-\rangle\langle+|)]$$
$$+\frac{i\hbar^2}{4}[-(|+\rangle\langle-|) + (|-\rangle\langle+|)][(|+\rangle\langle-|) + (|-\rangle\langle+|)]$$

$$= \frac{i\hbar^2}{4}[-(|+\rangle\langle-|+\rangle\langle-| + (|+\rangle\langle-|-\rangle\langle+|) - (|-\rangle\langle+|+\rangle\langle-|) + (|-\rangle\langle+|-\rangle\langle+|)$$
$$-(|+\rangle\langle-|+\rangle\langle-|) - (|+\rangle\langle-|-\rangle\langle+|) + (|-\rangle\langle+|+\rangle\langle-|) + (|-\rangle\langle+|-\rangle\langle+|)]$$

$$= 0$$

We can repeat this for all other combinations to prove the desired relations.

1.9 Rotation Operators

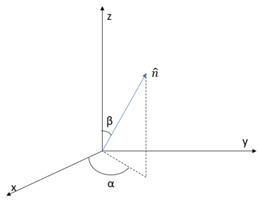

Figure 1.1: Angles

Construct $|\vec{S}\cdot\hat{n};+\rangle$ such that

$$\vec{S}\cdot\hat{n}\,|\vec{S}\cdot\hat{n};+\rangle = \left(\frac{\hbar}{2}\right)|\vec{S}\cdot\hat{n};+\rangle$$

where $\hat{n}$ is characterized by the angles shown in the figure. Express your answer as a linear combination of $|+\rangle$ and $|-\rangle$. [Note: The answer is

$$\cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)\exp(i\alpha)\,|-\rangle$$

But do not just verify that this answer satisfies the above eigenvalue equation. Rather, treat the problem as a straightforward eigenvalue problem. Also do not use rotation operators, which we will introduce later in this book.]

The first thing we do is figure out $\vec{S}\cdot\hat{n}$,

$$\begin{cases}\vec{S} = \hbar/2\;(\sigma_x, \sigma_y, \sigma_z)\\ \hat{n} = (\cos(\alpha)\sin(\beta), \sin(\alpha)\sin(\beta), \cos(\beta))\end{cases}$$

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\left[\begin{pmatrix}0 & \cos(\alpha)\sin(\beta)\\ \cos(\alpha)\sin(\beta) & 0\end{pmatrix} + \begin{pmatrix}0 & -i\sin(\alpha)\sin(\beta)\\ i\sin(\alpha)\sin(\beta) & 0\end{pmatrix} + \begin{pmatrix}\cos(\beta) & 0\\ 0 & \cos(\beta)\end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\begin{bmatrix}\cos(\beta) & \sin(\beta)(\cos(\alpha) - i\sin(\alpha))\\ \sin(\beta)(\cos(\alpha) + i\sin(\alpha)) & \cos(\beta)\end{bmatrix}$$

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\begin{bmatrix}\cos(\beta) & \sin(\beta)\exp(-i\alpha)\\ \sin(\beta)\exp(i\alpha) & \cos(\beta)\end{bmatrix}$$

If we now say that $|\vec{S}\cdot\hat{n};+\rangle$ is some arbitrary vector, we can solve the eigenvalue problem,

$$\vec{S}\cdot\hat{n}\,|\vec{S}\cdot\hat{n};+\rangle = \hbar/2\,|\vec{S}\cdot\hat{n};+\rangle$$

$$\begin{bmatrix}\cos(\beta) & \sin(\beta)\exp(-i\alpha)\\ \sin(\beta)\exp(i\alpha) & -\cos(\beta)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x\\ y\end{bmatrix}$$

$$\begin{bmatrix}x\cos(\beta) + y\sin(\beta)\exp(-i\alpha)\\ x\sin(\beta)\exp(i\alpha) - y\cos(\beta)\end{bmatrix} = \begin{bmatrix}x\\ y\end{bmatrix}$$

Our eigenket should be normalized,

$$|x|^2 + |y|^2 = 1$$

Looking at the first line of the matrix,

$$x\cos(\beta) + y\sin(\beta)\exp(-i\alpha) = x$$

$$y = \frac{(1-\cos(\beta))x}{\sin(\beta)\exp(-i\alpha)}$$

$$|y|^2 = \frac{(1-\cos(\beta))^2|x|^2}{\sin^2(\beta)}$$

Inserting this into the normalization condition,

$$|x|^2 + \frac{|x|^2 - 2|x|^2\cos(\beta) + |x|^2\cos^2(\beta)}{\sin^2(\beta)} = 1$$

$$\frac{2|x|^2 - 2|x|^2\cos(\beta)}{\sin^2(\beta)} = 1$$

$$|x|^2 = \frac{\sin^2(\beta)}{2(1-\cos(\beta))}$$

$$= \frac{1-\cos^2(\beta)}{2(1-\cos(\beta))}$$

$$|x|^2 = \frac{1+\cos(\beta)}{2}$$

Looking up half-angle formulas,

$$x = \cos(\beta/2)$$

Plugging this into the second line,

$$x\sin(\beta)\exp(i\alpha) - y\cos(\beta) = y$$

$$\cos(\beta/2)\sin(\beta)\exp(i\alpha) - y\cos(\beta) = y$$

$$y = \cos(\beta/2)\frac{\sin(\beta)}{1+\cos(\beta)}\exp(i\alpha)$$

$$= \sqrt{\frac{1+\cos(\beta)}{2}}\frac{\sin(\beta)}{1+\cos(\beta)}\exp(i\alpha)$$

$$= \sqrt{\frac{\sin^2(\beta)}{2(1+\cos(\beta))}} \exp(i\alpha)$$

$$= \sqrt{\frac{1-\cos^2(\beta)}{2(1+\cos(\beta))}} \exp(i\alpha)$$

$$y = \sin(\beta/2) \exp(i\alpha)$$

Combining,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\beta/2) \\ \sin(\beta/2) \exp(i\alpha) \end{pmatrix}$$

Which, when we write using $|+\rangle$ and $|-\rangle$ gives the solution provided by Sakurai.

1.10 Energy Eigenvalues

The Hamiltonian operator for a two-state system is given by

$$H = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)$$

where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$).

To find the energy eigenvalues,

$$H|\Psi\rangle = E|\Psi\rangle$$

It is probably easiest to do this in matrix representation. Setting

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In this basis,

$$H = \begin{bmatrix} a & a \\ a & -a \end{bmatrix}$$

Solving the characteristic equation (1.20),

$$\det(H - \lambda I) = \det \begin{bmatrix} a-\lambda & a \\ a & -a-\lambda \end{bmatrix} = 0$$

$$-(a-\lambda)(a+\lambda) - a^2 = 0$$

$$\lambda^2 - 2a^2 = 0$$

Our eigenvalues are $\lambda = \pm a\sqrt{2}$. Solving for the eigenvectors,

$$|a\sqrt{2}\rangle = \frac{1}{4+2\sqrt{2}} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}$$

$$|-a\sqrt{2}\rangle = \frac{1}{4-2\sqrt{2}} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}$$

In the $|1\rangle$, $|2\rangle$ basis,

$$\begin{cases} |a\sqrt{2}\rangle = \frac{1}{4+2\sqrt{2}}[(1+\sqrt{2}|1\rangle + |2\rangle] \\ |-a\sqrt{2}\rangle = \frac{1}{4-2\sqrt{2}}[(1-\sqrt{2}|1\rangle + |2\rangle] \end{cases}$$

1.11 Energy Eigenvalues

A two-state system is characterized by the Hamiltonian

$$H = H_{11} |1\rangle \langle 1| + H_{22} |2\rangle \langle 2| + H_{12}[|1\rangle \langle 2| + |2\rangle \langle 1|]$$

where H_{11}, H_{22}, and H_{12} are real numbers with the dimension of energy, and $|1\rangle$ and $|2\rangle$ are eigenkets of some observable ($\neq H$). Find the energy eigenkets and corresponding energy eigenvalues. Make sure that your answer makes good sense for $H_{12} = 0$. (You need not solve this problem from scratch. The following fact may be used without proof:

$$(\vec{S} \cdot \hat{n}) |\hat{n}; +\rangle = \frac{\hbar}{2} |\hat{n}; +\rangle$$

with $|\hat{n}; +\rangle$ given by

$$|\hat{n}; +\rangle = \cos\left(\frac{\beta}{2}\right) |+\rangle + \exp(i\alpha) \sin\left(\frac{\beta}{2}\right) |-\rangle$$

where β and α are the polar and azimuthal angles, respectively, that characterize $\hat{n}$.

This problem could be probably be solved in braket notation, but I'm more comfortable with matrices,

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix}$$

Solving the characteristic equation (1.20) gives two eigenvalues,

$$\begin{cases} \lambda_1 = \dfrac{(H_{11} + H_{22}) + \sqrt{(H_{11} + H_{22})^2 - 4(H_{11}H_{22} - H_{12})^2}}{2} \\ \\ \lambda_2 = \dfrac{(H_{11} + H_{22}) - \sqrt{(H_{11} + H_{22})^2 - 4(H_{11}H_{22} - H_{12}^2)}}{2} \end{cases}$$

To find the eigenvalues, we have

$$(H_{11} - \lambda_1)x - H_{12}y = 0$$

Let's set $x = 1$, which gives us an eigenket of

$$|\lambda_1\rangle = \begin{bmatrix} 1 \\ \dfrac{H_{11} - \lambda_1}{H_{12}} \end{bmatrix}$$

Similarly,

$$|\lambda_2\rangle = \begin{bmatrix} \dfrac{H_{12}}{H_{11} - \lambda_{11}} \\ 1 \end{bmatrix}$$

If $H_{12} = 0$,

$$H = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix}$$

Our eigenvalues are $\lambda = H_{11}, H_{22}$ with

$$|H_{11}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|H_{22}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

It can be verified that these follow the formulas laid out above.

1.12 1.12

1.13 1.13

1.14 Eigenvalues

A certain observable in quantum mechanics has a 3×3 matrix representation as follows:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

1.14.a Find the normalized eigenvectors of this observable and the corresponding eigenvalues. Is there any degeneracy?

Solving the characteristic equation (1.20) gives the eigenvalues $\lambda = 0, \pm 1$. There is no degeneracy since we have three eigenvalues for a 3×3 matrix. Solving for the eigenvectors,

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}; \quad |1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}; \quad |-1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

1.14.b Give a physical example where all this is relevant

Looking this up, these are the eigenvalues and eigenvectors for the spin-1 particle. I believe this is further explained in chapter 3 of Sakurai.

1.15 Simultaneous Eigenkets

Let A and B be observables. Suppose the simultaneous eigenkets of A and B $\{|a', b'\rangle\}$ form a complete orthonormal set of base kets. Can we always conclude that

$$[A, B] = 0$$

If your answer is yes, prove the assertion. If your answer is no, give a counterexample.

We start by writing $[A, B]$ out and inserting identity on both sides,

$$[A, B] = \sum_{a',b'} \sum_{a',b'} |a'', b''\rangle \langle a'', b''|(AB - BA)|a', b'\rangle \langle a', b'|$$

If we act the operators on our ket, we use the eigenvalue,

$$AB\,|a', b'\rangle = a'b'\,|a', b'\rangle$$

$$[A, B] = \sum_{a',b'} \sum_{a'',b''} |a'', b''\rangle \langle a'', b''|(a'b' - b'a')|a', b'\rangle \langle a', b'|$$

We know that $a'b' - b'a' = 0$ since these are not operators, so the order doesn't matter. $[A, B] = 0$ if the simultaneous eigenkets of A and B form a complete orthonormal set of base kets.

1.16 Simultaneous Eigenkets

Two Hermitian operators anticommute:

$$\{A, B\} = AB + BA = 0$$

Is it possible to have a simultaneous (that is, common) eigenket of A and B? Prove or illustrate your assertion.

Let's act some eigenket of A on our anti-commutator,

$$\langle a''|AB|a'\rangle + \langle a''|BA|a'\rangle$$

$$= a'' \langle a''|B|a'\rangle + a' \langle a''|B|a'\rangle$$

$$= (a'' + a') \langle a''|B|a'\rangle$$

We expect this should be equal to 0 if A and B anti-commute. Since $(a'' + a') \neq 0$, this implies $\langle a''|B|a'\rangle = 0$ for both $a'' = a'$ and $a'' \neq a'$, which implies they do not have simultaneous eigenkets.

1.17 Degenerate Eigenkets

Two observables A_1 and A_2, which do not involve time explicitly, are known not to commute,

$$[A_1, A_2] \neq 0$$

yet we also know that A_1 and A_2 both commute with the Hamiltonian:

$$[A_1, H] = 0; \quad [A_2, H] = 0$$

Prove that the energy eigenstates are, in general, degenerate. Are there exceptions? As an example, you may think of the central -force problem $H = \vec{p}^2/2m + V(r)$, with $A_1 \to L_z$, $A_2 \to L_x$.

We'll start by assuming the Hamiltonian is not degenerate, i.e.,

$$H |n\rangle = E |n\rangle$$

$|n\rangle$ is unique for each E.

Using the fact that our operators commute with the Hamiltonian, we can act the commutator on our eigenket $|n\rangle$,

$$[A_1, H] |n\rangle = 0 |n\rangle$$

$$A_1 H |n\rangle - H A_1 |n\rangle = 0$$

$$E(A_1 |n\rangle) = H(A_1 |n\rangle)$$

Since A_1 commutes with the Hamiltonian, they must share a complete set of eigenstates,

$$A_1 |n\rangle = a_1 |n\rangle$$

We can do the same for A_2.

If we act the commutator on the eigenstate,

$$[A_1, A_2] |n\rangle = A_1 A_2 |n\rangle - A_2 A_1 |n\rangle$$

$$= (a_1 a_2 - a_2 a_1) |n\rangle$$

Since a_1 and a_2 are scalars, the order doesn't matter, which means $[A_1, A_2] = 0$. This is in contradiction with the statement in the problem, so the energy eigenstates must be degenerate.

1.18 1.18

1.19 1.19

1.20 1.20

1.21 1.21

1.22 1.22

1.23 Degenerate Eigenstates

Consider a three-dimensional ket space. If a certain set of orthonormal kets-say, $|1\rangle$, $|2\rangle$, $|3\rangle$-are used as the base kets, the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}; \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

with a and b both real.

1.23.a Obviously A exhibits a degenerate spectrum. Does B also exhibit a degenerate spectrum?

To determine if B is degenerate, we solve the characteristic equation (1.20), which gives us repeated eigenvalues $\lambda = b, b, -b$. There is degeneracy.

1.23.b Show that A and B commute

To show that A and B commute, we use brute force,

$$AB = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}$$

1.23.c Find a new set of orthonormal kets which are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

A has eigenvalues $\lambda = -a, -a, a$, so we'll have three eigenvalues, $\lambda = -a, a, b$. These have eigenkets,

$$|a\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad |-a\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}; \quad |b\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

If we act the eigenkets on each operator,

$$\begin{cases} A\,|a\rangle = a\,|a\rangle \\ A\,|-a\rangle = -a\,|-a\rangle \\ A\,|b\rangle = -a\,|b\rangle \end{cases}$$

$$\begin{cases} B\left|a\right\rangle = b\left|a\right\rangle \\ B\left|-a\right\rangle = -b\left|-a\right\rangle \\ B\left|b\right\rangle = b\left|b\right\rangle \end{cases}$$

We see that no eigenket shares the same eigenvalues, which means that this forms a CSCO.

1.24 Spinors

1.24.a Prove that $(1/\sqrt{2})(1+i\sigma_x)$ acting on a two-component spinor can be regarded as the matrix representation of the rotation operator about the x-axis by angle $-\pi/2$. (The minus sign signifies that the rotation is clockwise.)

The rotation matrix is given by

$$\cos(\phi/2) - i\vec{\sigma}\cdot\hat{n}\sin(\phi/2)$$

Clockwise rotation about the x-axis by $-\pi/2$ implies that $\phi = -\phi/2$ and $\hat{n} = \hat{i}$,

$$\cos(-\pi/4) - i\vec{\sigma}\cdot\hat{x}\sin(-\pi/4)$$

$$= 1/\sqrt{2}(1+i\sigma_x)$$

1.24.b Construct the matrix representation of S_z when the eigenkets of S_y are used as base vectors.

We can write S_z as

$$S_z = \frac{\hbar}{2}\frac{1}{\sqrt{2}}(1-i\sigma_x)\sigma_z\frac{1}{\sqrt{2}}(1+i\sigma_x)$$

$$= \frac{\hbar}{4}\begin{bmatrix}1 & -i\\ -i & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix}$$

$$S_z = \frac{\hbar}{2}\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$$