The darkness crumbles away.
It is the same old druid Time as ever,
Only a live thing leaps my hand,
A queer sardonic rat,

As I pulled the parapet's poppy
To stick behind my ear.
Droll rat, they would shoot you if they knew
Your cosmopolitan sympathies.
Now you have touched this English hand

Chapter 6: Oscillations

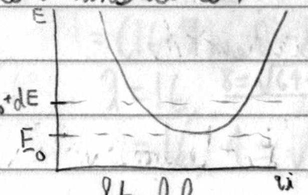
Section 1. Formulation of the Problem

Imagine we have a conservative system whose potential energy is solely a function of position. The system is said to be at equilibrium if

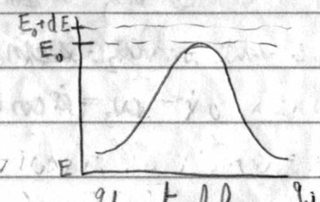
$$Q_i = -\left(\frac{\partial V}{\partial q_i}\right)_0 = 0 \qquad (6.1)$$

We can imagine that if the system starts at equilibrium, it will stay at equilibrium (if you buy a spring and set it on a table, will it spontaneously start to move?). However, what will happen if perturb the system slightly?

If we plot the potential, we can determine if the motion is stable or unstable.



Stable



Unstable

We can imagine a particle at equilibrium (E_0) is pushed a little to the right. If the motion is stable, it will return to equilibrium. If the motion is unstable, the particle will fly off. For small deviation, we can expand the potential as a Taylor series about the equilibrium

$$V(q_1,\dots,q_n) = V(q_{01},\dots,q_{0n}) + \left(\frac{\partial V}{\partial q_i}\right)_0 \eta_i + \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j + \dots \qquad (6.3)$$

$$\approx \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j = \frac{1}{2} V_{ij}\, \eta_i\, \eta_j \qquad (6.4)$$

since we set the equilibrium position to 0 and from the condition set in (6.1).

We can similarly treat the kinetic energy

$$T = \frac{1}{2} T_{ij}\, \dot\eta_i\, \dot\eta_j \qquad (6.6)$$

$$L = \frac{1}{2} T_{ij}\, \dot\eta_i\, \dot\eta_j - \frac{1}{2} V_{ij}\, \eta_i\, \eta_j \qquad (6.7)$$

$$= \frac{1}{2}\left(T_{ij}\, \dot\eta_i^2 - V_{ij}\, \eta_i\, \eta_j\right) \qquad (6.9)$$

You will do the same to a German
Soon, no doubt, if it be your pleasure
To cross the sleeping green between.
It seems as if you inwardly grin as you pass
Strong eyes, fine limbs, haughty athletes,

Less chanced than you for life,
Bonds to the whims of murder,
Sprawled in the bowels of the earth,
The torn fields of France.
What do you see in our eyes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\eta_i}\right) - \frac{\partial L}{\partial \eta_i} = 0$$

$$\frac{d}{dt}\left(T_{ij}\, \dot\eta_j\right) - \left(-V_{ij}\, \eta_j\right) = 0$$

$$T_{ij}\, \ddot\eta_j + V_{ij}\, \eta_j = 0 \qquad (6.10)$$

Section 2. The Eigenvalue Equation and the Principal Axis Transformation

The solution to the harmonic oscillator is a known quantity, and one we've used in the past. Say we have a spring whose Lagrangian can be written as

$$L = \frac{1}{2} m\dot x^2 - \frac{1}{2} k x^2$$

$$m\ddot x + kx = 0$$

$$x(t) = A\exp(i\omega t) + B\exp(-i\omega t)$$

$$\omega = \sqrt{k/m}$$

Now let's say we had some damping force $\vec F = -\gamma \dot x$

$$m\ddot x + \gamma\dot x + kx = 0$$

$$x(t) = \exp(-\beta t)\left[A\exp\left(\sqrt{\beta^2-\omega_0^2}\, t\right) + B\exp\left(-\sqrt{\beta^2-\omega_0^2}\, t\right)\right]$$

$$\omega = \sqrt{k/m}\,;\ \beta = \gamma/2m$$

There are three interesting cases for damped harmonic motion

- Underdamped: $\beta^2 < \omega_0^2$

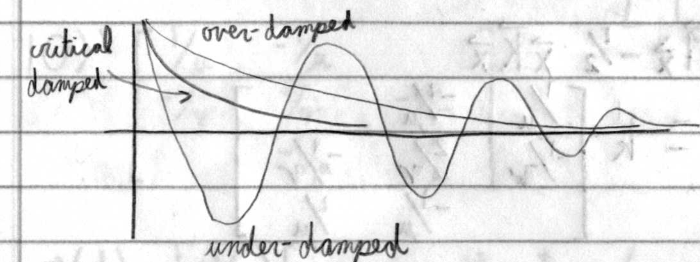
$$x(t) = C\exp(-\beta t)\cos\left(\sqrt{\omega_0^2-\beta^2}\, t + \phi\right)$$

$$x(\infty) = 0$$

And we can convince ourselves that energy is not conserved because of the dissipative forces.

- Overdamped: $\beta^2 > \omega_0^2$
- Critical damping: $\beta^2 = \omega_0^2$

$$x(t) = C\exp(-\beta t)$$



At the shrieking iron and flame
Hurled through still heavens?
What quaver—what heart aghast?
Poppies, whose roots are in man's veins
Drop, and are ever dropping;
But mine in my ear is safe—
Just a little white with the dust.
– Isaac Rosenberg (Break of Day in the Trenches, 1916)

If we have some external driving force

$$m\ddot{x}+\gamma\dot{x}+kx=F(t)$$

Section 3 Frequencies of Free Vibration, and Normal Coordinates

The general oscillator solution is satisfied for a set of n frequencies ω_k. Thus, in order to form a complete solution, we need to take a superposition of all the allowed frequencies.

$$\eta_i = C_k a_{ik} \exp(-i\omega_k t) \qquad (6.35)$$

If we want to trigger only a single frequency, we must find the normal modes. In vector notation,

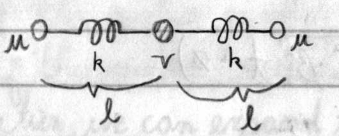
$$\mathcal{L} = \tfrac{1}{2}(\dot{\vec{x}}, \vec{M}\dot{\vec{x}}) - \tfrac{1}{2}(\vec{x}, \vec{K}\vec{x}) \qquad \text{(Jose 4.44)}$$

$$\ddot{\vec{x}} + \Lambda\vec{x} = 0 \qquad \Lambda = M^{-1}K \qquad \text{(Jose 4.45)}$$

The normal frequencies are the square roots of the eigenvalues, and the corresponding eigenvectors are the initial condition.

Section 4. Free Vibrations of a Linear Triatomic Molecule (drawn from Jose)

We can treat a water molecule as three particles in a line. connected by equal springs as shown below



$$\mathcal{L} = \tfrac{1}{2}\mu(\dot{x}_1^2+\dot{x}_3^2) + \tfrac{1}{2}\nu\dot{x}_2^2 - \tfrac{1}{2}k(x_2-x_1)^2 - \tfrac{1}{2}k(x_3-x_2)^2$$

$$= \tfrac{1}{2}(\dot{\vec{x}}, \vec{M}\dot{\vec{x}}) - \tfrac{1}{2}(\vec{x}, \vec{K}\vec{x})$$

$$\vec{M} = \begin{bmatrix} \mu & & \\ & \nu & \\ & & \mu \end{bmatrix} \qquad \vec{K} = k\begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{bmatrix}$$

i.e.

$$\mathcal{L} = \tfrac{1}{2}\,\dot{\vec{x}}\vec{M}\dot{\vec{x}} - \tfrac{1}{2}\,\vec{x}\vec{K}\vec{x}$$

$$\vec{\Lambda} = \vec{K}/\vec{M} = k\begin{bmatrix} 1/\mu & -1/\mu & \\ -1/\nu & 2/\nu & -1/\nu \\ & -1/\mu & 1/\mu \end{bmatrix}$$

$$\det(\Lambda - \lambda I) = \det\begin{bmatrix} 1/\mu-\lambda & -1/\mu & \\ -1/\nu & 2/\nu-\lambda & -1/\nu \\ & -1/\mu & 1/\mu-\lambda \end{bmatrix}$$

$$= (1/\mu-\lambda)\left[(2/\nu-\lambda)(1/\mu-\lambda) - 1/\mu\nu\right] + 1/\mu\,(-1/\nu)(1/\mu-\lambda)$$

$$= (1/\mu-\lambda)\left[(2/\nu-\lambda)(1/\mu-\lambda) - 2/\mu\nu\right]$$

$$= (1/\mu-\lambda)(\lambda^2 - 2\lambda/\nu - \lambda/\mu) = 0$$

$$= (1/\mu-\lambda)\,\lambda\,(\lambda - (2/\nu + 1/\mu)) = 0$$

$$\lambda = 1/\mu,\ 0,\ 2/\nu + 1/\mu$$

$$\omega_1 = \sqrt{k/\mu}$$

$$\omega_2 = \sqrt{k(2/\nu + 1/\mu)}$$

$$\omega_3 = 0$$

$$|\sqrt{k/\mu}\rangle: \begin{bmatrix} 0 & 1/\mu & 0 \\ -1/\nu & 2/\nu - 1/\mu & -1/\nu \\ 0 & -1/\mu & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$b = 0$$

$$a = -c$$

$$|\sqrt{k/\mu}\rangle = 1/\sqrt{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$|\sqrt{k(2/\nu + 1/\mu)}\rangle: \begin{bmatrix} -2/\nu & -1/\mu & \\ -1/\nu & 1/\mu & -1/\nu \\ & -1/\mu & -2/\nu \end{bmatrix}$$

$$-2a/\nu - b/\mu = 0$$

$$-b/\mu - 2c/\nu = 0$$

$$|\sqrt{k(2/\nu + 1/\mu)}\rangle = n\begin{bmatrix} 1 \\ -2\mu/\nu \\ 1 \end{bmatrix}$$

$$|0\rangle: \begin{bmatrix} 1/\mu & -1/\mu & \\ -1/\nu & 2/\nu & -1/\nu \\ & -1/\mu & 1/\mu \end{bmatrix} \qquad |0\rangle = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{x}(t) = |\sqrt{k/\mu}\rangle\left[\alpha_1 \exp(i\omega_1 t) + \alpha_1^* \exp(-i\omega_1 t)\right] + |\sqrt{k(2/\nu + 1/\mu)}\rangle\left[\alpha_2 \exp(i\omega_2 t) + \alpha_2^* \exp(-i\omega_2 t)\right] + |0\rangle\left[\alpha_3 \exp(i\omega_3 t) + \alpha_3^* \exp(-i\omega_3 t)\right]$$

Section 5 Forced Vibrations and the Effect of Dissipative Forces

Derivations

1. $\dot{y}_1 = -\dot{x}_1$

$\dot{y}_2 = \dot{x}_3$

$$\mathcal{L} = \frac{\mu}{2}\left(\dot{y}_1^2 + \dot{y}_2^2\right) - \frac{k}{2}\left(y_1^2 + y_2^2\right)$$

$$M = \begin{bmatrix} \mu & \\ & \mu \end{bmatrix} \quad K = \begin{bmatrix} k & \\ & k \end{bmatrix}$$

$$\mathcal{L} = \tfrac{1}{2}\vec{\dot{y}}\,\overleftrightarrow{M}\,\vec{\dot{y}} - \tfrac{1}{2}\vec{x}\,\overleftrightarrow{K}\,\vec{x}$$

$$\Lambda = M^{-1}K = \begin{bmatrix} 1/\mu & \\ & 1/\mu \end{bmatrix}\begin{bmatrix} k & \\ & k \end{bmatrix} = \begin{bmatrix} k/\mu & \\ & k/\mu \end{bmatrix}$$

$$\det\Lambda = (k/\mu - \lambda)^2 = 0$$

$$\lambda = k/\mu$$

$$\omega_1 = \sqrt{k/\mu}$$

$$|\sqrt{k/\mu}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

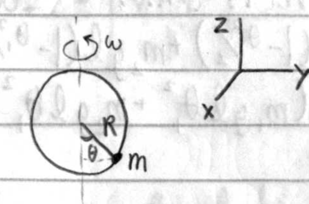
$$|\sqrt{k/\mu}'\rangle = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Which corresponds to moving one of the outside particle

$$\vec{x}(t) = |\sqrt{k/\mu}\rangle\left(\alpha_1 \exp(i\omega_1 t) + \alpha_1^* \exp(-i\omega_1 t)\right)$$
$$+ |\sqrt{k/\mu}'\rangle\left(\alpha_2 \exp(i\omega_1 t) + \alpha_2^* \exp(-i\omega_1 t)\right)$$

Problems

3.



a. $x = -R\sin(\omega t)\sin\theta$

$y = R\cos(\omega t)\sin\theta$

$z = -R\cos\theta$

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$
$$= \frac{m}{2}\left(R^2\omega^2\sin^2\theta + R^2\dot{\theta}^2\right) + mgR\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

$$\frac{d}{dt}\left(mR^2\dot{\theta}\right) - \left(mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta\right) = 0$$

$$\ddot{\theta} - \omega^2\sin\theta\cos\theta + \frac{g}{R}\sin\theta = 0$$

b. At equilibrium, $\dot{\theta} = \ddot{\theta} = 0$

$$\omega^2\cos\theta = g/R$$

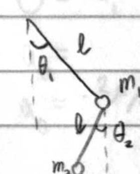
$$\omega_c = \sqrt{g/R}$$ since the bottom of the hoop is the equilibrium position ($\theta = 0$)

c. $\omega > \sqrt{g/R}$

$$\omega^2 = \frac{\omega_c^2}{\cos\theta}$$

$$\theta_{eq} = \cos^{-1}\left(\omega_c/\omega\right)$$

4.



$x_1 = l\sin\theta_1$ $\qquad y_1 = l\cos\theta_1$

$x_2 = l\sin\theta_1 + l\sin\theta_2$ $\qquad y_2 = l\cos\theta_1 + l\cos\theta_2$

$\dot{x}_1 = l\dot{\theta}_1\cos\theta_1$ $\qquad \dot{y}_1 = -l\dot{\theta}_1\sin\theta_1$

$\dot{x}_2 = l\dot{\theta}_1\cos\theta_1 + l\dot{\theta}_2\cos\theta_2$ $\qquad \dot{y}_2 = -l\dot{\theta}_1\sin\theta_1 - l\dot{\theta}_2\sin\theta_2$

$$\mathcal{L} = \frac{m_1}{2}\left(l^2\dot{\theta}_1^2\right) + \frac{m_2}{2}\left(l^2\dot{\theta}_1^2 + 2l^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + l^2\dot{\theta}_2^2\right)$$
$$- m_1 g l\cos\theta_1 - m_2 g\left(l\cos\theta_1 + l\cos\theta_2\right)$$

Working in small angle approximation $\quad -m_2 l^2 \dot\theta_1 \dot\theta_2$

$$\mathcal{L} \simeq \frac{m_1}{2}(l^2\dot\theta_1^2) + \frac{m_2}{2}(l^2\dot\theta_1^2 + l^2\dot\theta_2^2) + m_1 g l(1-\theta_1^2/2) + m_2 g l(1-\theta_1^2/2 + 1-\theta_2^2/2)$$

$$\approx \tfrac{1}{2}(m_1\dot\theta_1^2 + m_2\dot\theta_1^2 + m_2\dot\theta_2^2 - 2m_2\dot\theta_1\dot\theta_2) - \tfrac{1}{2}(m_1 g l\theta_1^2 + m_2 g l\theta_1^2 + m_2 g l \theta_2^2)$$

$$M = \begin{bmatrix} l^2(m_1+m_2) & -m_2 l^2 \\ -m_2 l^2 & l^2 m_2 \end{bmatrix}$$

$$K = \begin{bmatrix} gl(m_1+m_2) & \\ & g l m_2 \end{bmatrix}$$

$$\mathcal{L} = \tfrac{1}{2}\dot{\vec{x}}\, M\, \dot{\vec{x}} - \tfrac{1}{2}\vec{x} K \vec{x}$$

$$\det(\Lambda - \lambda I) = \det(K - \lambda M) = 0$$

$$\det\begin{bmatrix} gl(m_1+m_2) - l^2\lambda(m_1+m_2) & m_2 l^2\lambda \\ m_2 l^2 \lambda & g l m_2 - l^2\lambda m_2 \end{bmatrix}$$

$$= (m_1+m_2)(gl - l^2\lambda)\, m_2 (gl - l^2\lambda) - m_2^2 l^4 \lambda^2 = 0$$

$$\lambda = \frac{g(m_1+m_2)}{l m_1} \pm \frac{g}{l}\sqrt{\frac{m_1 m_2 + m_2^2}{m_1^2}}$$

$$= \omega_0^2\left(M/m_1 \pm \sqrt{M m_2/m_1^2}\right)$$

$$\omega_0^2 = g/l$$

$$M = m_1 + m_2$$

$$|\omega_0(M/m_1 + \sqrt{M m_2/m_1})^{1/2}\rangle : \begin{bmatrix} g l M - \omega^2 l^2 M & \omega^2 m_2 l^2 \\ \omega^2 m_2 l^2 & g l m_2 - \omega^2 l^2 m_2 \end{bmatrix}\begin{bmatrix} a \\ b\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$a(g l M - \omega^2 l^2 M) + b\omega^2 m_2 l^2 = 0$$

$$a\omega^2 m_2 l^2 + b(g l m_2 - \omega^2 l^2 m_2) = 0$$

$$a = \frac{-b m_2 (gl - \omega^2 l^2)}{m_2 \omega^2 l^2}$$

$$\frac{M(gl - \omega^2 l^2)^2}{\omega^2 l^2} + b\omega^2 m_2 l^2 = 0$$

$$b = -\frac{(gl - \omega^2 l^2)^2}{\omega^4 l^4}$$

$$|\omega_+\rangle = \begin{bmatrix} \sqrt{m_2/M} \\ 1 \end{bmatrix} \qquad |\omega_-\rangle = \begin{bmatrix} \sqrt{m_2/M} \\ -1 \end{bmatrix}$$

5.

a. *The general solution is*

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}(\alpha_1 \exp(i\omega_1 t) + \alpha_1^* \exp(-i\omega_1 t)) + \begin{pmatrix} 1 \\ -2\mu/r \\ 1 \end{pmatrix}(\alpha_2 \exp(i\omega_2 t) + \alpha_2^* \exp(-i\omega_2 t)) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(a + vt)$$

$$\begin{pmatrix} 0 \\ a_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}(\alpha_1 + \alpha_1^*) + \begin{pmatrix} 1 \\ -2\mu/r \\ 1 \end{pmatrix}(\alpha_2 + \alpha_2^*) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} a$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}(i\omega_1\alpha_1 - i\omega_1\alpha_1^*) + \begin{pmatrix} 1 \\ -2\mu/r \\ 1 \end{pmatrix}(i\omega_2\alpha_2 - i\omega_2\alpha_2^*) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} v$$

$$0 = \alpha_1 + \alpha_1^* + \alpha_2 + \alpha_2^* + a \qquad 0 = i\omega_1\alpha_1 - i\omega_1\alpha_1^* + i\omega_2\alpha_2 - i\omega_2\alpha_2^* + v$$

$$0 = -\alpha_1 - \alpha_1^* + \alpha_2 + \alpha_2^* + a \qquad 0 = -i\omega_1\alpha_1 + i\omega_1\alpha_1^* + i\omega_2\alpha_2 - i\omega_2\alpha_2^* + v$$

$$a_0 = -2\mu/r\,(\alpha_2 + \alpha_2^*) + a \qquad 0 = -2\mu/r\,(i\omega_2\alpha_2 - i\omega_2\alpha_2^*) + v$$

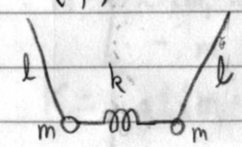
$$a_0 = -2\mu/r\,(\alpha_2 + \alpha_2^*) - (\alpha_2 + \alpha_2^*) \qquad 0 = -2\mu/r\,(i\omega_2\alpha_2 - i\omega_2\alpha_2^*) - (i\omega_2\alpha_2 + i\omega_2\alpha_2^*)$$

$$0 = (-2\mu/r - 1)(i\omega_2)(a_2 - a_2^*)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}(\exp(i\omega_1 t) - \exp(-i\omega_1 t)) + \begin{pmatrix} 1 \\ -2\mu/r \\ 1 \end{pmatrix}(\exp(i\omega_2 t) - \exp(-i\omega_2 t)) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(a_0)$$

b. $\vec{x} = \begin{pmatrix}1\\0\\1\end{pmatrix}(\exp(i\omega_1 t)-\exp(-i\omega_1 t)) + \begin{pmatrix}1\\-2m/M\\1\end{pmatrix}(\exp(i\omega_2 t)-\exp(-i\omega_2 t))$

$+\begin{pmatrix}1\\1\\1\end{pmatrix}V_0 t$

6. 

$$\frac{m\,d^2x}{dt^2} + kx + mg\sin\theta = 0$$

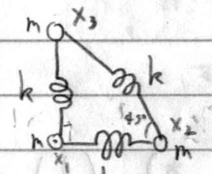
$$\frac{m\,d^2x}{dt^2} + kx + mg\frac{x}{\ell} = 0$$

$$m\ddot{x} = -\left(k/m + g/\ell\right)x$$

$$\omega = \sqrt{k/m + g/\ell}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{3/.23 + 10/.8}} = 1.24s$$

7.

8. 

$(\eta_1, \varepsilon_1) = (x_1, y_1)$

$(\eta_2, \varepsilon_2) = (x_2 - a, y_2)$

$(\eta_3, \varepsilon_3) = (x_3, y_3 - a)$

$$T = m/2\,(\dot\eta_1^2 + \dot\varepsilon_1^2 + \dot\eta_2^2 + \dot\varepsilon_2^2 + \dot\eta_3^2 + \dot\varepsilon_3^2)$$

$$V_{12} = k/2\left[\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} - a\right]^2$$
$$= k/2\left[\sqrt{(\eta_2 + a - \eta_1)^2 + (\varepsilon_2 - \varepsilon_1)^2} - a\right]^2$$
$$\approx k/2\left[\left(a^2\left(1 + \tfrac{\eta_2-\eta_1}{a}\right)^2\right)^{1/2} - a\right]^2$$
$$\approx k/2\left[a\left(1 + \eta_2 - \eta_1/a\right) - a\right]^2 = k/2\,(\eta_2 - \eta_1)^2$$

$$V_{13} = k/2\,(\varepsilon_3 - \varepsilon_1)^2$$

$$V_{23} = k/2\left[\sqrt{(x_3-x_2)^2 + (y_3-y_2)^2} - \sqrt{2}\,a\right]^2$$
$$= k/2\left[\left((\eta_3 - \eta_2 - a)^2 + (\varepsilon_3 + a - \varepsilon_2)^2\right)^{1/2} - \sqrt{2}\,a\right]^2$$
$$\approx k/2\left[\left(a^2\left(1 - \tfrac{\eta_3-\eta_2}{a}\right)^2 + a^2\left(1 - \varepsilon_3 - \varepsilon_2/a\right)^2\right)^{1/2} - \sqrt{2}\,a\right]^2$$
$$= k/4\left[(\eta_3 - \eta_2) - (\varepsilon_3 - \varepsilon_2)\right]^2$$

$$V = k/2\left[(\eta_2 - \eta_1)^2 + (\varepsilon_3 - \varepsilon_1)^2 + 1/2\,(\eta_3 - \eta_2 - \varepsilon_3 + \varepsilon_2)^2\right]$$

$$V - \lambda T = \begin{bmatrix} k - m\lambda & 0 & -k & 0 & 0 & 0 \\ 0 & k - m\lambda & 0 & 0 & 0 & -k \\ -k & 0 & 3k/2 - m\lambda & -k/2 & -k/2 & k/2 \\ 0 & 0 & -k/2 & k/2 - m\lambda & k/2 & -k/2 \\ 0 & 0 & -k/2 & k/2 & k/2 - m\lambda & -k/2 \\ 0 & -k & k/2 & -k/2 & -k/2 & 3k/2 - m\lambda \end{bmatrix}$$

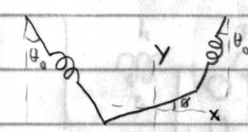
9. $\Delta E = 5J$

$\Delta E = 1/2\,kx^2$

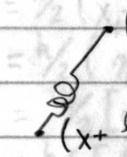
$k = 10\ N/m$

10.

a.

11. 

$$T = m/2\,(\dot x^2 + \dot y^2) + 1/2\,I\dot\phi^2$$
$$I = \int_{-\ell/2}^{\ell/2} x^2 \rho\, dx = m/\ell \cdot 1/3\, x^3\Big|_{-\ell/2}^{\ell/2} = m/3\ell \cdot 2\ell^3/8 = m\ell^2/12$$
$$T = m/2\,(\dot x^2 + \dot y^2) + m\ell^2\dot\phi^2/24$$



$$V = k/2\left[\sqrt{(\ell/2 + b\sin\theta_0 - x - \ell\cos\phi/2)^2 + (b\cos\theta_0 - y - \ell\sin\phi/2)^2} - b\right]^2$$
$$\approx k/2\left[\left((b\sin\theta_0 - x)^2 + (b\cos\theta_0 - y - \ell\phi/2)^2\right)^{1/2} - b\right]^2$$

Keeping linear terms in x, y, ϕ

$$\approx k/2\left[x\sin\theta_0 + (y + \ell\phi/2)\cos\theta_0\right]^2$$
$$+ k/2\left[x\sin\theta_0 - (y - \ell\phi/2)\cos\theta_0\right]^2$$

$$T=\begin{bmatrix} m & & \\ & m & \\ & & m/3 \end{bmatrix}$$

$$V=\begin{bmatrix} \sin^2\theta_0 & 0 & -\sin\theta_0\cos\theta_0 \\ 0 & \cos^2\theta_0 & 0 \\ -\sin\theta_0\cos\theta_0 & 0 & \cos^2\theta_0 \end{bmatrix}\cdot 2k$$

$$\det(V-\lambda T)=\begin{vmatrix} 2k\sin^2\theta_0-\lambda m & 0 & -2k\sin\theta_0\cos\theta_0 \\ 0 & 2k\cos^2\theta_0-\lambda m & 0 \\ -2k\sin\theta_0\cos\theta_0 & 0 & 2k\cos^2\theta_0-{}^{m\lambda}/_3 \end{vmatrix}$$

$$=(2k\sin^2\theta_0-\lambda m)(2k\cos^2\theta_0-\lambda m)(2k\cos^2\theta_0-{}^{m\lambda}/_3)$$
$$-(2k\sin\theta_0\cos\theta_0)^2(2k\cos^2\theta_0-\lambda m)$$
$$=(2k\cos^2\theta_0-\lambda m)[4k^2\sin^2\theta_0\cos^2\theta-{}^{2km\lambda}/_3\sin^2\theta-2km\lambda\cos^2\theta+{}^{m^2\lambda^2}/_3-4k^2\sin^2\theta_0\cos^2\theta]$$
$$=(2k\cos^2\theta_0-\lambda m)\cdot{}^{m}/_3\,(m\lambda-2k\sin^2\theta_0-6k\cos^2\theta_0)$$
$$=(2k\cos^2\theta_0-\lambda m)\cdot{}^{m}/_3\,(m\lambda-2k-4k\cos^2\theta_0)$$

$$\lambda=0,\ {}^{2k}/_m\cos^2\theta_0,\ {}^{2k}/_m(1+2\cos^2\theta_0)$$

$$|0\rangle:\ \begin{bmatrix} 2k\sin^2\theta_0 & 0 & -2k\sin\theta_0\cos\theta_0 \\ 0 & 2k\cos^2\theta_0 & 0 \\ -2k\sin\theta\cos\theta_0 & 0 & 2k\cos^2\theta_0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$b=0$$
$$2ka\sin^2\theta_0-2kc\sin\theta_0\cos\theta_0=0$$
$$a\sin\theta_0=c\cos\theta_0$$

$$|0\rangle=\begin{bmatrix} \cos\theta_0 \\ 0 \\ -\sin\theta_0 \end{bmatrix}$$

$$|{}^{2k}/_m\cos^2\theta_0\rangle:\ \begin{bmatrix} 2k\sin^2\theta_0-2k\cos^2\theta_0 & 0 & -2k\sin\theta_0\cos\theta_0 \\ 0 & 0 & 0 \\ -2k\sin\theta_0\cos\theta_0 & 0 & {}^{4k}/_3\cdot\cos^2\theta_0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

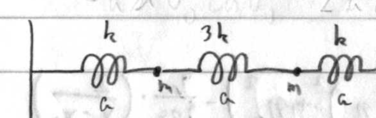
$$|{}^{2k}/_m\cos^2\theta_0\rangle=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|{}^{2k}/_m(1+2\cos^2\theta_0)\rangle:\ \begin{bmatrix} 2k\sin^2\theta-2k-4k\cos^2\theta_0 & 0 & -2k\sin\theta_0\cos\theta_0 \\ 0 & -2k-2k\cos^2\theta_0 & 0 \\ -2k\sin\theta_0\cos\theta_0 & 0 & {}^{4k}/_3\cdot\cos^2\theta_0-{}^{2k}/_3 \end{bmatrix}$$

$$a(\sin^2\theta_0-1-2\cos^2\theta_0)-c\sin\theta_0\cos\theta_0=0$$
$$-3a\cos^2\theta_0=c\sin\theta_0\cos\theta_0$$
$$c\sin\theta_0=-3a\cos\theta_0$$

$$|{}^{2k}/_m(1+2\cos^2\theta_0)\rangle=\begin{bmatrix} \sin\theta_0 \\ 0 \\ -3\cos\theta_0 \end{bmatrix}$$

12.



$$\eta_1=x_1-a$$
$$\eta_2=x_2-2a$$

$$T={}^{m}/_2(\dot x_1^2+\dot x_2^2)={}^{m}/_2(\dot\eta_1^2+\dot\eta_2^2)$$
$$V={}^{k}/_2(x_1-a)^2+{}^{3k}/_2(x_2-x_1-a)^2+{}^{k}/_2(x_2-2a)^2$$
$$={}^{k}/_2(\eta_1^2)+{}^{3k}/_2(\eta_2-\eta_1)^2+{}^{k}/_2(\eta_2^2)$$
$$={}^{k}/_2[\eta_1^2+3(\eta_2^2-2\eta_1\eta_2+\eta_1^2)+\eta_2^2]$$
$$={}^{k}/_2[4\eta_1^2-3\eta_1\eta_2-3\eta_2\eta_1+4\eta_1^2]$$

$$T=\begin{bmatrix} m & \\ & m \end{bmatrix}\qquad V=\begin{bmatrix} 4k & -3k \\ -3k & 4k \end{bmatrix}$$

$$\mathcal{L}={}^{1}/_2\cdot\dot{\vec\eta}\,T\,\dot{\vec\eta}+{}^{1}/_2\cdot\vec\eta\cdot\vec V\cdot\vec\eta$$

$$\det(V-\lambda T) = \det\begin{bmatrix} 4k-\lambda m & -3k \\ -3k & 4k-\lambda m \end{bmatrix}$$

$$= (4k-\lambda m)^2 - 9k^2 = \lambda^2 m^2 - 8k\lambda m + 7k^2$$

$$= (\lambda m - 7k)(\lambda m - k)$$

$$\lambda = 7k/m,\ k/m$$

$$|7k/m\rangle : \begin{bmatrix} -3k & -3k \\ -3k & -3k \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$|7k/m\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|k/m\rangle : \begin{bmatrix} 3k & -3k \\ -3k & 3k \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$|k/m\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

13. $\eta_1 = x_1 - a$

$\eta_2 = x_2 - a$

$$\mathcal{L} = m/2\,(\dot\eta_1^2 + \dot\eta_2^2) - k/2\,(\eta_1^2 + (\eta_2-\eta_1)^2 + \eta_2^2) - \frac{q}{4\pi\varepsilon_0}\cdot\left(\frac{2}{x_2-x_1}\right)$$

16. $y = ax^4$

$\dot y = 4ax^3\dot x$

$$\mathcal{L} = m/2\,(\dot x^2 + \dot y^2) - mgy$$

$$= m/2\,(\dot x^2 + 16a^2x^6\dot x^2) - mgax^4$$

$x = x_0 + \eta \qquad y = a(x_0+\eta)^4$

$\dot x = \dot\eta \qquad \dot y = 4a(x_0+\eta)^3\cdot\dot\eta$

$$\mathcal{L} = m/2\,(\dot\eta^2 + 16a^2(x_0+\eta)^6\dot\eta^2) - mga(x_0+\eta)^4$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\eta}\right) - \frac{\partial\mathcal{L}}{\partial\eta} = 0$$

$$\frac{d}{dt}\left(m\dot\eta + 16ma^2(x_0+\eta)^6\dot\eta\right) - \left(3m\cdot16a^2(x_0+\eta)^5\dot\eta^2 - 4mga(x_0+\eta)^3\right) = 0$$

$$m\ddot\eta + 96ma^2(x_0+\eta)^5\dot\eta^2 + 16ma^2(x_0+\eta)^6\ddot\eta - 48ma^2(x_0+\eta)^5\dot\eta^2 + 4mga(x_0+\eta)^3 = 0$$

$$\ddot\eta(1+16a^2(x_0+\eta)^6) + 48a^2(x_0+\eta)^5\dot\eta^2 + 4ga(x_0+\eta)^3 = 0$$

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