Sakurai, Modern Quantum Mechanics
First edition

BS

CLASS SCHEDULE

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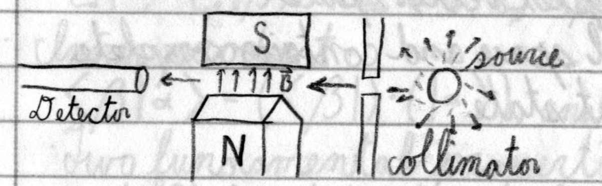
Our description of the physical world is dynamic in nature and undergoes frequent change. At any given time, we summarize our knowledge of natural phenomenon by means of certain laws. These laws adequately describe the phenomenon studied up to that time, to an accuracy then attainable. As time passes, we enlarge the domain of observation and improve the accuracy of measurement. As we do so, we constantly check to see if the laws continue to be valid. Those laws that do remain valid gain stature, and those that do not must be abandoned in favour of new ones that do. - R. Shankar

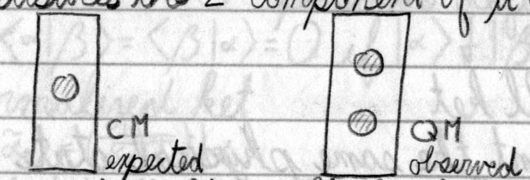
Chapter 1, Fundamental Concepts

Section 1 The Stern-Gerlach Experiment

Subsection Description of the experiment

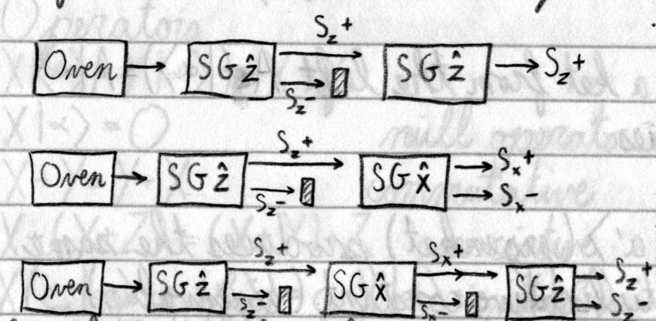


- Silver atom source
- 47 electrons ⇒ angular momentum
- Magnetic moment ($\vec{\mu}$) proportional to electron spin ($\vec{S}$), $\vec{\mu} \propto \vec{S}$
- $F_z = \frac{\partial}{\partial z}(\vec{\mu}\cdot\vec{B}) \simeq \mu_z \frac{\partial B_z}{\partial z}$
- Measures the z-component of $\vec{\mu}$ (or $\vec{S}$)



- $S_z = \pm \hbar/2$
- Quantization of electron spin angular momentum

Subsection Sequential Stern-Gerlach Experiment



- Cannot determine S_z & S_x simultaneously i.e., selection of an S_x state destroys any previous information about S_z

Subsection Analogy with polarization of light

Follow the example in the book, the main takeaway is that QM states are to be represented by vectors in an abstract complex vector space

Section 2. Kets, Bras, and Operators

Subsection Ket Space

- Ket: state vector in a complex vector space
  - Represents a physical space and contains complete information about that state
  - Denoted $|\alpha\rangle$
  - Some properties of kets
    - $|\alpha\rangle + |\beta\rangle = |\gamma\rangle$
    - $c|\alpha\rangle = |\alpha\rangle c$
    - $0|\alpha\rangle = |0\rangle$ null ket
    - $|\alpha\rangle$ + $c|\alpha\rangle$ represent the same physical state
- Operator: represents an observable (e.g. spin)
  - Denoted A
  - Generally acts on a ket from the left $A\cdot(|\alpha\rangle) = A|\alpha\rangle$
- Eigenkets + Eigenvalues
  - $A|a'\rangle = a'|a'\rangle$
  - Applying A to $|a'\rangle$ (eigenket) produces the same $|a'\rangle$ with a multiplicative constant (eigenvalue)
  - So named because "Eigen" means "Same" in German
  - Physical state corresponding to Eigenket is called the Eigenstate
- $|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle$
  - Any ket can be written as a linear combination of eigenkets
  - There are a number of eigenvalues (+ eigenkets) equal to dimensionality of the vector space

Subsection Bra Space + Inner Products

- There exists a vector space dual to ket space with a one-to-one correspondence:
  - $|\alpha\rangle \overset{DC}{\leftrightarrow} \langle\alpha|$
  - $|a'\rangle, |a''\rangle, \ldots \overset{DC}{\leftrightarrow} \langle a'|, \langle a''|, \ldots$
  - $c_\alpha|\alpha\rangle + c_\beta|\beta\rangle \overset{DC}{\leftrightarrow} c_\alpha^*\langle\alpha| + c_\beta^*\langle\beta|$
- Inner product
  - $\langle\beta|\alpha\rangle = (\langle\beta|)\cdot(|\alpha\rangle)$ analogous to scalar product
  - Two fundamental properties
    - $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$ complex conjugates
    - $\langle\alpha|\alpha\rangle \geq 0$ positive definite metric
  - $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle = 0$ if $|\alpha\rangle$ + $|\beta\rangle$ are orthogonal
- Normalized ket
  - $|\tilde{\alpha}\rangle = \left(\frac{1}{\sqrt{\langle\alpha|\alpha\rangle}}\right)|\alpha\rangle$
  - $\langle\tilde{\alpha}|\tilde{\alpha}\rangle = 1$

Subsection Operators

- $X|\alpha\rangle = Y|\alpha\rangle \Rightarrow X = Y$
- $X|\alpha\rangle = 0$ null operator
- $X + Y = Y + X$ commutative
- $X + (Y + Z) = (X + Y) + Z$ associative
- $X(c_\alpha|\alpha\rangle + c_\beta|\beta\rangle) = c_\alpha X|\alpha\rangle + c_\beta X|\beta\rangle$ linear
- $X|\alpha\rangle \overset{DC}{\leftrightarrow} \langle\alpha|X^\dagger$
- $X = X^\dagger$ Hermitian

Subsection Multiplication

- $XY \neq YX$ noncommutative
- $X(YZ) = (XY)Z = XYZ$ associative
- $X(Y|\alpha\rangle) = (XY)|\alpha\rangle = XY|\alpha\rangle$ associative
- $(XY)^\dagger = Y^\dagger X^\dagger$
- Outer Product
  - $|\beta\rangle\langle\alpha| = (|\beta\rangle)\cdot(\langle\alpha|)$
  - Returns an operator

Subsection The Associative Axiom

- Multiplication among kets, bras, and operators is associative (n.b., multiplication must be legal)
- $(|\beta\rangle\langle\alpha|)\cdot|\gamma\rangle = |\beta\rangle\cdot(\langle\alpha|\gamma\rangle)$
- $X = |\beta\rangle\langle\alpha| \Rightarrow X^{\dagger} = |\alpha\rangle\langle\beta|$
- $\langle\beta|X|\alpha\rangle = \langle\alpha|X|\beta\rangle^*$ for Hermitian X

Section 3 Base Kets and Matrix Representations

Subsection Eigenkets of an Observable

- Operators representing some physical observable are often Hermitian
- Theorem: eigenvalues of Hermitian operator A are real. Eigenkets of A are orthogonal
- $\langle a''|a'\rangle = \delta_{a''a'}$
  - Orthonormal set of eigenkets
  - Forms a complete set

Subsection Eigenkets as Base Kets

- From before, $|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle$
  - $\langle a''|\alpha\rangle = \sum_{a'}\langle a''|c_{a'}|a'\rangle = \sum_{a'} c_{a'}\langle a''|a'\rangle$
  - $\langle a'|\alpha\rangle = c_{a'}$
  - $|\alpha\rangle = \sum_{a'}|a'\rangle\langle a'|\alpha\rangle$
- $\sum_{a'}|a'\rangle\langle a'| = I$ Completeness relation
- $\langle\alpha|\alpha\rangle = \langle\alpha|I|\alpha\rangle = \langle\alpha|\left(\sum_{a'}|a'\rangle\langle a'|\right)|\alpha\rangle$
  $= \sum_{a'}\langle\alpha|a'\rangle\langle a'|\alpha\rangle$
  $= \sum_{a'}|\langle a'|\alpha\rangle|^2$

  $\Rightarrow$ if $|\alpha\rangle$ is normalized, $\sum_{a'}|c_{a'}|^2 = 1$
- $\Lambda_{a'} = |a'\rangle\langle a'|$ Projection operator
  - Selects portion of $|\alpha\rangle$ parallel to $|a'\rangle$
  - $\sum_{a'}\Lambda_{a'} = I$ Rewriting the Completeness relation

Subsection Matrix Representations

- $X = I\cdot X\cdot I = \sum_{a''}\sum_{a'}|a''\rangle\langle a''|X|a'\rangle\langle a'|$

$$X = \begin{pmatrix} \langle a^{(1)}|X|a^{(1)}\rangle & \langle a^{(1)}|X|a^{(2)}\rangle & \cdots \\ \langle a^{(2)}|X|a^{(1)}\rangle & \langle a^{(2)}|X|a^{(2)}\rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

i.e. each slot in the matrix can be represented by $\langle a''|X|a'\rangle = \langle a'|X^{\dagger}|a''\rangle^*$

Subsection Spin-½ systems

- In the S_z basis, define:
  - $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$
  - $|-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$
- $I = (|+\rangle\langle+|) + (|-\rangle\langle-|)$
- $S_z = \hbar/2\,[(|+\rangle\langle+|) - (|-\rangle\langle-|)] = \hbar/2\begin{pmatrix}1&0\\0&-1\end{pmatrix}$
- $S_+ = \hbar|+\rangle\langle-| = \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $\quad S_+$ will raise spin component by $\hbar$,
  $S_- = \hbar|-\rangle\langle+| = \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}$ $\quad$ and will return 0 if it can't be raised any more. S_- lowers the spin component

Section 4. Measurements, Observables, and the Uncertainty Relations

Subsection Measurements

- Before a measurement is made, $|\alpha\rangle = \sum_{a'}|a'\rangle\langle a'|\alpha\rangle$
- After A is measured, $|\alpha\rangle \rightarrow |a'\rangle$
- $P(a') = |\langle a'|\alpha\rangle|^2$ Probability for going to the state $|a'\rangle$ from $|\alpha\rangle$. $|\alpha\rangle$ must be normalized
- $\langle A\rangle = \langle\alpha|A|\alpha\rangle$ Expectation value. Average measured value
- Selective measurement (filtration)
  - Selects only one eigenket of A
  - Represented by $\Lambda_{a'}|a'\rangle = |a'\rangle\langle a'|\alpha\rangle$

Subsection Spin-½ systems, once again

- Before going into this, remember the S_z basis and the previous discussion of spin-½ systems
  - $|S_x;\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle$
  - $|S_y;\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$
  - $|S_z;\pm\rangle = |\pm\rangle$
  - $S_x = \hbar/2\,[(|+\rangle\langle-|) + (|-\rangle\langle+|)]$
  - $S_y = \hbar/2\,[-i(|+\rangle\langle-|) + i(|-\rangle\langle+|)]$
  - $S_z = \hbar/2\,[(|+\rangle\langle+|) - (|-\rangle\langle-|)]$
- Some important relations
  - $S_\pm = S_x \pm i S_y$
  - $[S_i, S_j] = i\varepsilon_{ijk}\hbar S_k$
  - $\{S_i, S_j\} = \frac{1}{2}\hbar^2\delta_{ij}$
  - $\vec{S}^2 = S_x^2 + S_y^2 + S_z^2 = 3\hbar^2/4$
  - $[\vec{S}^2, S_i] = 0$

Subsection Compatible observables

- $[A, B] = 0$ Compatible
- $[A, B] \neq 0$ Incompatible
- Degeneracy
  - If two or more linear independent eigenkets of some observable A have the same eigenvalue, said eigenvalue is considered degenerate
  - Can use compatible observables to label degenerate eigenkets
- Theorem: If A + B are compatible observables and eigenvalues of A are non-degenerate, then the matrix elements $\langle a''|B|a'\rangle$ are diagonal
  - This theorem states that A + B can be represented by the same set of base kets
  - $|a'\rangle$ is a simultaneous eigenket (also written as $|a', b'\rangle$)
  - $A|a', b'\rangle = a'|a', b'\rangle$
  - $B|a', b'\rangle = b'|a', b'\rangle$
- Measurements of A + measurements of B do not interfere regardless of degeneracy assuming $[A, B] = 0$

Subsection Incompatible observables

- Incompatible observables do not have a complete set of simultaneous eigenkets
- This subsection contains a proof of the above

Subsection The Uncertainty Relation

- $\Delta A = A - \langle A\rangle$ Operator of A
- $\langle(\Delta A)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$ Dispersion of A
- $\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A, B]\rangle|^2$ Uncertainty relation
- Lemma (Schwarz inequality): $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$
- Lemma: the expectation value of a Hermitian operator is purely real
- Lemma: the expectation value of an anti-Hermitian operator is purely imaginary

Section 5 Change of basis

Subsection Transformation operator

- Suppose A + B do not commute. The two sets of base kets span the same ket space. How do we relate the two sets?
- Theorem: given two sets of base kets, there exists a unitary operator $\mathcal{U}$ such that
  - $|b^{(1)}\rangle = \mathcal{U}|a^{(1)}\rangle, \ldots |b^{(N)}\rangle = \mathcal{U}|a^{(N)}\rangle$
  - (n.b. unitary operators satisfy $\mathcal{U}\mathcal{U}^\dagger = \mathcal{U}^\dagger\mathcal{U} = 1$

Subsection Transformation matrix

- $\langle a^{(k)}|U|a^{(l)}\rangle = \langle a^{(k)}|b^{(l)}\rangle$, transformation matrix from $\{|a'\rangle\}$ basis to $\{|b'\rangle\}$ basis
- Given $|\alpha\rangle$ in the $|a'\rangle$ basis, we can get to $|b'\rangle$ basis by

$$\langle b^{(k)}|\alpha\rangle = \sum_l \langle b^{(k)}|a^{(l)}\rangle\langle a^{(l)}|\alpha\rangle$$
$$= \sum_l \langle a^{(k)}|U^\dagger|a^{(l)}\rangle\langle a^{(l)}|\alpha\rangle$$

i.e. $(New) = (U^\dagger)(Old)$

- $X' = U^\dagger X U$ Similarity transformation
- $\sum_{a'} \langle a'|X|a'\rangle$ Trace of an operator X
  - Independent of representation i.e. $\sum_{a'}\langle a'|X|a'\rangle = \sum_{b'}\langle b'|X|b'\rangle$
  - $tr(XY) = tr(YX)$
  - $tr(U^\dagger X U) = tr(X)$
  - $tr(|a'\rangle\langle a''|) = \delta_{a'a''}$
  - $tr(|b'\rangle\langle a'|) = \langle a'|b'\rangle$

Subsection Diagonalization

- Solving the eigenvalue problem $B|b'\rangle = b'|b'\rangle$

$$\sum_{a'} \langle a''|B|a'\rangle\langle a'|b'\rangle = b'\langle a''|b'\rangle$$

$$\begin{pmatrix} B_{11} & B_{12} & \cdots \\ B_{21} & B_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} C_1^{(l)} \\ C_2^{(l)} \\ \vdots \end{pmatrix} = b^{(l)}\begin{pmatrix} C_1^{(l)} \\ C_2^{(l)} \\ \vdots \end{pmatrix} \qquad B_{ij} = \langle a^{(i)}|B|a^{(j)}\rangle \quad C_k^{(l)} = \langle a^{(k)}|b^{(l)}\rangle$$

Solve $\det(B - \lambda I) = 0$

λ returns the eigenvalues $b^{(l)}$

Subsection Unitary equivalent observables

- Theorem: given two orthonormal bases $\{|a'\rangle\}$ & $\{|b'\rangle\}$ connected by U operator. May construct UAU^{-1}, unitary transform of A. A & UAU^{-1} are unitary equivalent observables.

$$A|a^{(l)}\rangle = a^{(l)}|a^{(l)}\rangle \Rightarrow UAU^{-1}U|a^{(l)}\rangle = a^{(l)}U|a^{(l)}\rangle$$

or, $(UAU^{-1})|b^{(l)}\rangle = a^{(l)}|b^{(l)}\rangle$

Section 6. Position, momentum, and translation

Subsection Continuous Spectra

- This subsection contains examples of moving from a discrete to a continuous world. Most of it can be boiled down to: treat the two cases the same way with some minor changes in notation (e.g. $\Sigma \to \int$)

Subsection Position eigenkets and position measurements

- Consider the position operator x
  - $x|x'\rangle = x'|x'\rangle$ where $|x'\rangle$ form a complete set
  - $|\alpha\rangle = \int_{-\infty}^{\infty} dx'|x'\rangle\langle x'|\alpha\rangle$
- $P = \int_{-\infty}^{\infty} dx' |\langle x'|\alpha\rangle|^2$ Probability of recording particle somewhere between $-\infty$ & ∞
- Moving to three dimensions
  - $|\alpha\rangle = \int d^3x' |\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$

  where $|\vec{x}'\rangle = |x', y', z'\rangle$

  $x|\vec{x}'\rangle = x'|\vec{x}'\rangle$, $y|\vec{x}'\rangle = y'|\vec{x}'\rangle$, $z|\vec{x}'\rangle = z'|\vec{x}'\rangle$

  $[x_i, x_j] = 0$

Subsection Translation

$\mathcal{J}(d\vec{x}')|\vec{x}'\rangle = |\vec{x}' + d\vec{x}'\rangle$ Infinitesimal translation by $d\vec{x}'$

$\mathcal{J}(d\vec{x}')|\alpha\rangle = \int d^3x' |\vec{x}' + d\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$
$= \int d^3x' |\vec{x}'\rangle\langle\vec{x}' - d\vec{x}'|\alpha\rangle$

- Properties of $\mathcal{J}(d\vec{x}')$
  - $\mathcal{J}^\dagger(d\vec{x}')\mathcal{J}(d\vec{x}') = 1$ Unitary
  - $\mathcal{J}(d\vec{x}'')\mathcal{J}(d\vec{x}') = \mathcal{J}(d\vec{x}' + d\vec{x}'')$ Adding infinitesimal transformation
  - $\mathcal{J}(-d\vec{x}') = \mathcal{J}^{-1}(d\vec{x}')$ Inverse
  - $\lim_{d\vec{x}'\to 0}\mathcal{J}(d\vec{x}') = 1$ To first order, translation operator is identity

$[\vec{x}, \mathcal{J}(d\vec{x}')] = d\vec{x}'$ Operator identity

Subsection Momentum as a generator of translation

· Infinitesimal translation in classical mechanics

· $\vec{x}_{new} = \vec{x} + d\vec{x}$

· $\vec{p}_{new} = \vec{p}$

· $\mathcal{J}(d\vec{x}') = 1 - i\vec{p}\cdot d\vec{x}'/\hbar$

$[x_i, p_j] = i\hbar\delta_{ij}$

Subsection The canonical commutation relations

· Fundamental commutation relations

· $[x_i, x_j] = 0$

· $[p_i, p_j] = 0$

· $[x_i, p_j] = i\hbar\delta_{ij}$

· $[\ ,\ ]_{cl} = \frac{[\ ,\ ]}{i\hbar}$

· Replace classical Poisson brackets with commutators

· $[A(q,p), B(q,p)]_{cl} = \sum_s \left(\frac{\partial A}{\partial q_s}\frac{\partial B}{\partial p_s} - \frac{\partial A}{\partial p_s}\frac{\partial B}{\partial q_s}\right)$

· $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ Jacobi identity

Section 7. Wave functions in position and momentum space

Subsection Position-space wave function

· $\psi_\alpha(x') = \langle x'|\alpha\rangle$ wave function for state $|\alpha\rangle$

· $\langle\beta|\alpha\rangle = \int dx'\, \psi_\beta^*(x')\psi_\alpha(x')$

Represents the probability amplitude for state $|\alpha\rangle$ to be found in state $|\beta\rangle$

Going back to $|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$

· $\psi_\alpha(x') = \sum_{a'} c_{a'} u_{a'}(x')$

$u_{a'}(x') = \langle x'|a'\rangle$ eigenfunction with eigenvalue a'

· $\langle\beta|f(x)|\alpha\rangle = \int dx'\, \psi_\beta^*(x') f(x') \psi_\alpha(x')$

Subsection Momentum operator in the position basis

· $\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle$

· $\langle x'|p|x''\rangle = -i\hbar\frac{\partial}{\partial x'}\delta(x'-x'')$

· $\langle\beta|p|\alpha\rangle = \int dx'\, \psi_\beta^*(x')\left(-i\hbar\frac{\partial}{\partial x'}\right)\psi_\alpha(x')$

Subsection Momentum-space wave function

· Work in the p-basis

· $p|p'\rangle = p'|p'\rangle$

· $\langle p'|p''\rangle = \delta(p'-p'')$

· $|\alpha\rangle = \int dp'\, |p'\rangle\langle p'|\alpha\rangle$

· $\phi_\alpha(p') = \langle p'|\alpha\rangle$ momentum-space wave function

· $\int dp'\, |\phi_\alpha(p')|^2 = 1$ if $|\alpha\rangle$ is normalized

· $\langle x'|p'\rangle$ transformation from x-representation to p-representation

· $\langle x'|p|p'\rangle = p'\langle x'|p'\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle$

· $\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip'x'}{\hbar}\right)$

Subsection Gaussian wave packets

· $\langle x'|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}}\exp\left(ikx' - \frac{x'^2}{2d^2}\right)$ Gaussian wave packet

· $\langle x\rangle = 0$

· $\langle x^2\rangle = d^2/2$

· $\langle p\rangle = \hbar k$

· $\langle p^2\rangle = \frac{\hbar^2}{2d^2} + \hbar^2k^2$

· $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = d^2/2$

· $\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2 = \hbar^2/2d^2$

· $\langle p'|\alpha\rangle = \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\exp\left(\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2}\right)$

Subsection Generalization to three dimensions

· This subsection details moving from the one-dimensional case to the three-dimensional case. Things stay largely the same, now you just need to repeat three times.

Problems

1. $[AB,CD] = ABCD - CDAB$

$-AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB$

$= -AC(DB+BD) + A(CB+BC)D - C(DA+AD)B + (CA+AC)DB$

$= \cancel{-ACDB} - \cancel{ACBD} + \cancel{ACBD} + ABCD - CDAB - \cancel{CADB} + \cancel{CADB} + \cancel{ACDB}$

$= ABCD - CDAB = [AB, CD]$

2.

a. $X = a_0 + \vec{\sigma}\cdot\vec{a}$

$tr(X) = tr(a_0 I) + tr(\vec{\sigma}\cdot\vec{a}) = 2a_0 + 0 = 2a_0$

$\sigma_k X = a_0\sigma_k + a_1\sigma_k\sigma_1 + a_2\sigma_k\sigma_2 + a_3\sigma_k\sigma_3$

$tr(\sigma_k X) = 2a_k$ since only the σ_k term survives.

$a_0 = tr(X)/2$

$a_k = tr(\sigma_k X)/2$

b. $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_1 X = \begin{pmatrix} X_{21} & X_{22} \\ X_{11} & X_{12} \end{pmatrix}$

$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_2 X = \begin{pmatrix} -iX_{21} & -iX_{22} \\ iX_{11} & iX_{12} \end{pmatrix}$

$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \sigma_3 X = \begin{pmatrix} X_{11} & X_{12} \\ -X_{21} & -X_{22} \end{pmatrix}$

$a_0 = \frac{1}{2}(X_{11} + X_{22})$

$a_1 = \frac{1}{2}(X_{21} + X_{12})$

$a_2 = \frac{1}{2}(-iX_{21} + iX_{12})$

$a_3 = \frac{1}{2}(X_{11} - X_{22})$

3. $\vec{\sigma}\cdot\vec{a} \to \vec{\sigma}\cdot\vec{a}' = \exp\left(i\vec{\sigma}\cdot\vec{n}\phi/2\right)\vec{\sigma}\cdot\vec{a}\exp\left(-i\vec{\sigma}\cdot\vec{n}\phi/2\right)$

$\det(\vec{\sigma}\cdot\vec{a}') = \det\left(\exp\left(i\vec{\sigma}\cdot\vec{n}\phi/2\right)\right)\det(\vec{\sigma}\cdot\vec{a})\det\left(\exp\left(-i\vec{\sigma}\cdot\vec{n}\phi/2\right)\right)$

$= \det\left(\exp\left(i\vec{\sigma}\cdot\vec{n}\phi/2\right)\right)\det\left(\exp\left(-i\vec{\sigma}\cdot\vec{n}\phi/2\right)\right)\det(\vec{\sigma}\cdot\vec{a})$

$= \det\left(\exp\left(i\vec{\sigma}\cdot\vec{n}\phi/2\right)\exp\left(-i\vec{\sigma}\cdot\vec{n}\phi/2\right)\right)\det(\vec{\sigma}\cdot\vec{a})$

$= \det(\vec{\sigma}\cdot\vec{a})$

3. $\hat{n} = \hat{z} = (0,0,1)$

$\vec{\sigma}\cdot\vec{a}' = \exp\left(i\vec{\sigma}_3\phi/2\right)\vec{\sigma}\cdot\vec{a}\exp\left(-i\vec{\sigma}_3\phi/2\right)$

$= \begin{pmatrix} \exp(i\phi/2) & \\ & \exp(-i\phi/2) \end{pmatrix}\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}\begin{pmatrix} \exp(-i\phi/2) & \\ & \exp(i\phi/2) \end{pmatrix}$

$= \begin{pmatrix} \exp(i\phi/2) & \\ & \exp(-i\phi/2) \end{pmatrix}\begin{pmatrix} a_3\exp(-i\phi/2) & (a_1 - ia_2)\exp(i\phi/2) \\ (a_1 + ia_2)\exp(-i\phi/2) & -a_3\exp(i\phi/2) \end{pmatrix}$

$= \begin{pmatrix} a_3 & (a_1 - ia_2)\exp(i\phi) \\ (a_1 + ia_2)\exp(-i\phi) & -a_3 \end{pmatrix}$

$a_0' = {a_3 - a_3}/2 = 0$

$a_1' = \frac{1}{2}\left[(a_1 + ia_2)\exp(-i\phi) + (a_1 - ia_2)\exp(i\phi)\right]$

$= \frac{1}{2}\left[a_1\cos\phi + \cancel{ia_2\cos\phi} - \cancel{ia_1\sin\phi} - i^2 a_2\sin\phi + a_1\cos\phi + \cancel{ia_1\sin\phi} - \cancel{ia_2\cos\phi} - i^2 a_2\sin\phi\right]$

$= \frac{1}{2}\left[2a_1\cos\phi + 2a_2\sin\phi\right] = a_1\cos\phi + a_2\sin\phi$

$a_2' = \frac{1}{2}\left[-i(a_1 + ia_2)\exp(-i\phi) + i(a_1 - ia_2)\exp(i\phi)\right]$

$= \frac{1}{2}\left[-ia_1\cos\phi - i^2 a_2\cos\phi + i^2 a_1\sin\phi + i^3 a_2\sin\phi + ia_1\cos\phi + i^2 a_1\sin\phi - i^2 a_2\cos\phi - i^3 a_2\sin\phi\right]$

$= \frac{1}{2}\left[\cancel{-ia_1\cos\phi} + a_2\cos\phi - a_1\sin\phi - \cancel{ia_2\sin\phi} + \cancel{ia_1\cos\phi} - a_1\sin\phi + a_2\cos\phi + \cancel{ia_2\sin\phi}\right]$

$= \frac{1}{2}\left[2a_2\cos\phi - 2a_1\sin\phi\right] = -a_1\sin\phi + a_2\cos\phi$

$a_3' = \frac{1}{2}(a_3 + a_3) = a_3$

Since $a_3' = a_3$, this represents a rotation about the z-axis

4.

a. $Tr(XY) = Tr(X)\cdot Tr(Y) = Tr(Y)\,Tr(X) = Tr(YX)$

b. $(XY)|\alpha\rangle = \langle\alpha|(XY)^\dagger$

$XY|\alpha\rangle = X(Y|\alpha\rangle) \leftrightarrow \langle\alpha|Y^\dagger X^\dagger$

$(XY)^\dagger = Y^\dagger X^\dagger$

c. $\exp(i f(A))|\alpha\rangle = [\cos(f(A)) + i\sin(f(A))]|\alpha\rangle$

$= [\cos(f(\alpha)) + i\sin(f(\alpha))]|\alpha\rangle$

$\exp(if(A)) = \cos(f(A)) + i\sin(f(A))$

d. $\sum_{a'}\langle a'|\vec{x}'\rangle\langle\vec{x}''|a'\rangle = \delta_{\vec{x}'\vec{x}''}$

5.

a. $|\alpha\rangle\langle\beta| = \begin{pmatrix} \langle a^{(1)}|\alpha\rangle\langle a^{(1)}|\beta\rangle^* & \langle a^{(1)}|\alpha\rangle\langle a^{(2)}|\beta\rangle^* & \cdots \\ \langle a^{(2)}|\alpha\rangle\langle a^{(1)}|\beta\rangle^* & \langle a^{(2)}|\alpha\rangle\langle a^{(2)}|\beta\rangle^* & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

b. $|S_z = \hbar/2\rangle\langle S_x = \hbar/2| = |+\rangle \frac{1}{\sqrt{2}}(\langle +| + \langle -|)$

$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 1 \end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

6. $A|i\rangle = a|i\rangle$

$A|j\rangle = a'|j\rangle$

$A(|i\rangle + |j\rangle) = A|i\rangle + A|j\rangle = a|i\rangle + a'|j\rangle$

$= a''(|i\rangle + |j\rangle)$

$|i\rangle + |j\rangle$ is an eigenket of A if $a = a'$ i.e. degeneracy

7.

a. $A|\psi\rangle = a'|\psi\rangle$

$A|\psi\rangle - a'|\psi\rangle = 0$

$A - a' = 0$ for at least one case. Since product over all a' if $A - a' = 0$ for one case, $\prod_{a'}(A - a') = 0$

b. $\prod_{a''\neq a'} \frac{(A-a'')}{(a'-a'')}|a'\rangle = \prod_{a''\neq a'} \frac{(a'-a'')}{(a'-a'')}|a'\rangle = |a'\rangle$

$\prod_{a''\neq a'} \frac{(A-a'')}{(a'-a'')}|\psi\rangle = \prod_{a''\neq a'} \frac{(A-a'')}{(a'-a'')}|a'\rangle\langle a'|\psi\rangle = |a'\rangle\langle a'|\psi\rangle$

Projection operator of $|a'\rangle$

c. $S_z = \begin{pmatrix} \hbar/2 & \\ & -\hbar/2 \end{pmatrix}$ $\quad a' = \hbar/2$, $\quad a'' = -\hbar/2$

$(S_z - \hbar/2)(S_z + \hbar/2) = \begin{pmatrix} 0 & \\ & -\hbar \end{pmatrix}\begin{pmatrix} \hbar & \\ & 0 \end{pmatrix} = 0$

$\frac{S_z + \hbar/2}{\hbar} = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$

8. For this, I am only going to do $i = x$ & $j = y$ case. The other cases can be worked out in a similar manner

$[S_x, S_y] = S_xS_y - S_yS_x = \frac{i\hbar^2}{4}[(|+\rangle\langle -|) + (|-\rangle\langle +|)][-(|+\rangle\langle -|) + (|-\rangle\langle +|)]$
$\quad - \frac{i\hbar^2}{4}[-(|+\rangle\langle -|) + (|-\rangle\langle +|)][(|+\rangle\langle -|) + (|-\rangle\langle +|)]$

$= \frac{i\hbar^2}{4}[\cancel{-(|+\rangle\langle -|)(|+\rangle\langle -|)} + (|+\rangle\langle -|-\rangle\langle +|) - (|-\rangle\langle +|+\rangle\langle -|) + \cancel{(|-\rangle\langle +|-\rangle\langle +|)}$
$\quad + \cancel{(|+\rangle\langle -|+\rangle\langle -|)} + (|+\rangle\langle -|-\rangle\langle +|) - (|-\rangle\langle +|+\rangle\langle -|) - \cancel{(|-\rangle\langle +|-\rangle\langle +|)}]$

$= \frac{i\hbar^2}{4}[2(|+\rangle\langle +|) - 2(|-\rangle\langle -|)] = \frac{i\hbar^2}{2}[(|+\rangle\langle +|) - (|-\rangle\langle -|)]$

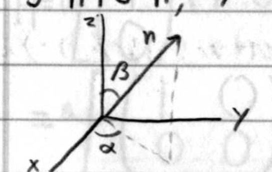
$= i\hbar S_z$

$\{S_x, S_y\} = S_xS_y + S_yS_x = \frac{i\hbar^2}{4}[(|+\rangle\langle -|) + (|-\rangle\langle +|)][-(|+\rangle\langle -|) + (|-\rangle\langle +|)]$
$\quad + \frac{i\hbar^2}{4}[-(|+\rangle\langle -|) + (|-\rangle\langle +|)][(|+\rangle\langle -|) + (|-\rangle\langle +|)]$

$= \frac{i\hbar^2}{4}[\cancel{-(|+\rangle\langle -|+\rangle\langle -|)} + (|+\rangle\langle -|-\rangle\langle +|) - (|-\rangle\langle +|+\rangle\langle -|) + \cancel{(|-\rangle\langle +|-\rangle\langle +|)}$
$\quad \cancel{-(|+\rangle\langle -|+\rangle\langle -|)} - (|+\rangle\langle -|-\rangle\langle +|) + (|-\rangle\langle +|+\rangle\langle -|) + \cancel{(|-\rangle\langle +|-\rangle\langle +|)}]$

$= \frac{i\hbar^2}{4}[0] = 0$

9. $\vec{S}\cdot\hat{n}|\vec{S}\cdot\hat{n};+\rangle = (\hbar/2)|\vec{S}\cdot\hat{n};+\rangle$



$\vec{S} = \hbar/2(\sigma_x, \sigma_y, \sigma_z)$

$\hat{n} = (\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta)$

$\vec{S}\cdot\hat{n} = \hbar/2\left[\begin{pmatrix} 0 & \cos\alpha\sin\beta \\ \cos\alpha\sin\beta & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\alpha\sin\beta \\ i\sin\alpha\sin\beta & 0 \end{pmatrix} + \begin{pmatrix} \cos\beta & 0 \\ 0 & \cos\beta \end{pmatrix}\right]$

$= \hbar/2\begin{bmatrix} \cos\beta & \sin\beta(\cos\alpha - i\sin\alpha) \\ \sin\beta(\cos\alpha + i\sin\alpha) & \cos\beta \end{bmatrix}$

$= \hbar/2\begin{pmatrix} \cos\beta & \sin\beta\exp(-i\alpha) \\ \sin\beta\exp(i\alpha) & \cos\beta \end{pmatrix}$

$|\vec{S}\cdot\hat{n};+\rangle = \begin{pmatrix} x \\ y \end{pmatrix}$

$\vec{S}\cdot\hat{n}|\vec{S}\cdot\hat{n};+\rangle = \hbar/2\, |\vec{S}\cdot\hat{n};+\rangle$

$$\begin{pmatrix} \cos\beta & \sin\beta \exp(-i\alpha) \\ \sin\beta \exp(i\alpha) & -\cos\beta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x\cos\beta + y\sin\beta\exp(-i\alpha) \\ x\sin\beta\exp(i\alpha) - y\cos\beta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Because $|\vec{S}\cdot\hat{n};+\rangle$ needs to be normalized, $|x|^2+|y|^2=1$

$x\cos\beta + y\sin\beta\exp(-i\alpha) = x$

$x(\cos\beta - 1) = -y\sin\beta\exp(-i\alpha)$ From here, we work with lengths because that erases the exponential term

$|y| = \frac{(1-\cos\beta)|x|}{\sin\beta}$

$|y|^2 = \frac{(1-\cos\beta)^2|x|^2}{\sin^2\beta}$

$|x|^2+|y|^2=1$

$|x|^2 + \frac{|x|^2 - 2|x|^2\cos\beta + |x|^2\cos^2\beta}{\sin^2\beta} = 1$

$\frac{|x|^2\sin^2\beta + |x|^2\cos^2\beta + |x|^2 - 2|x|^2\cos\beta}{\sin^2\beta} = 1$

$2|x|^2 - 2|x|^2\cos\beta = \sin^2\beta$

$|x|^2 = \frac{\sin^2\beta}{2(1-\cos\beta)} = \frac{(1+\cos\beta)(1-\cos\beta)}{2(1-\cos\beta)} = \frac{1+\cos\beta}{2}$

$x = \cos(\beta/2)$

$x\cos\beta + y\sin\beta\exp(-i\alpha) = x$

$y = \frac{\cos(\beta/2)(1-\cos\beta)}{\sin\beta}\exp(i\alpha)$

$|x|^2+|y|^2=1$

$|y|^2 = 1-\cos^2(\beta/2) = \sin^2(\beta/2)$

$|y| = \sin(\beta/2)$

$y = \sin(\beta/2)\exp(i\alpha)$

$|\vec{S}\cdot\hat{n};+\rangle = \cos(\beta/2)|+\rangle + \sin(\beta/2)\exp(i\alpha)|-\rangle$

10. $H|\psi\rangle = E|\psi\rangle$

$H = a[(|1\rangle\langle 1|) - (|2\rangle\langle 2|) + (|1\rangle\langle 2|) + (|2\rangle\langle 1|)]$

$|\psi\rangle = b|1\rangle + c|2\rangle$

$H|\psi\rangle = a[b(|1\rangle\langle 1|1\rangle) \cancel{+c(|1\rangle\langle 1|2\rangle)} \cancel{-b(|2\rangle\langle 2|1\rangle)} - c(|2\rangle\langle 2|2\rangle)$
$\cancel{+b(|1\rangle\langle 2|1\rangle)} + c(|1\rangle\langle 2|2\rangle) + b(|2\rangle\langle 1|1\rangle) \cancel{+c(|2\rangle\langle 1|2\rangle)}$

$= ab|1\rangle + ac|1\rangle - ac|2\rangle + ab|2\rangle = E[b|1\rangle + c|2\rangle]$

$ab + ac = Eb$

$-ac + ab = Ec$

Say $a=1$ for simplicity

$\frac{b+c}{b} = E$

$\frac{b-c}{c} = E$

Alternatively, could write a matrix representation, which I am much more comfortable with.

$$H = a\left[\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}\right]$$

$$= a\left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\right]$$

$$= \begin{pmatrix}a & a\\a & -a\end{pmatrix}$$

$\det(H-\lambda I) = \det\begin{pmatrix}a-\lambda & a\\a & -a-\lambda\end{pmatrix} = (a-\lambda)(-a-\lambda) - a^2$

$= -a^2 + \lambda^2 - a^2 = \lambda^2 - 2a^2$

$\lambda = \pm\sqrt{2}\,a$

$|\sqrt{2}a\rangle$: $\begin{pmatrix}a-\sqrt{2}a & a\\a & -a-\sqrt{2}a\end{pmatrix}\begin{pmatrix}b\\c\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$(1-\sqrt{2})b + c = 0$

$b - (1+\sqrt{2})c = 0$ $\quad b = (1+\sqrt{2})c$

$|\sqrt{2}a\rangle = N\begin{bmatrix}1+\sqrt{2}\\1\end{bmatrix}$ $\quad N^2 = [(1+\sqrt{2})^2 + 1^2]^{-1}$

$= \frac{1}{1+2+2\sqrt{2}+1} = \frac{1}{4+2\sqrt{2}}$

$|-\sqrt{2}a\rangle$: $\begin{pmatrix} a+\sqrt{2}a & a \\ a & -a+\sqrt{2}a \end{pmatrix}\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$(1+\sqrt{2})b + c = 0$$
$$b + (-1+\sqrt{2})c = 0$$
$$b = (1-\sqrt{2})c$$

$|-\sqrt{2}a\rangle = N'\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$ $\quad N'^2 = \left[(1-\sqrt{2})^2 + 1^2\right]^{-1} = \frac{1}{1-2\sqrt{2}+2+1} = \frac{1}{4-2\sqrt{2}}$

Taking these solutions, we can then go back and verify that the bra-ket method actually works.

$E = \sqrt{2}$, $b = 1+\sqrt{2}$, $c = 1$

$$E = \frac{b+c}{b} = \frac{1+\sqrt{2}+1}{1+\sqrt{2}} = \frac{2+\sqrt{2}}{1+\sqrt{2}} = \sqrt{2}$$
$$2+\sqrt{2} = \sqrt{2}+2 \ \checkmark$$
$$= \frac{b-c}{c} = \frac{1+\sqrt{2}-1}{1} = \sqrt{2}$$
$$\sqrt{2} = \sqrt{2} \ \checkmark$$

$E = -\sqrt{2}$, $b = 1-\sqrt{2}$, $c = 1$

$$E = \frac{b+c}{b} = \frac{1-\sqrt{2}+1}{1-\sqrt{2}} = -\sqrt{2}$$
$$2-\sqrt{2} = -\sqrt{2}(1-\sqrt{2})$$
$$2-\sqrt{2} = -\sqrt{2}+2 \ \checkmark$$
$$= \frac{b-c}{c} = \frac{1-\sqrt{2}-1}{1} = -\sqrt{2} \ \checkmark$$

11. Again, this can be solved in bra-ket notation, but I'm more comfortable with matrices.

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}$$

$$\det(H-\lambda I) = \det\begin{pmatrix} H_{11}-\lambda & H_{12} \\ H_{12} & H_{22}-\lambda \end{pmatrix}$$
$$= (H_{11}-\lambda)(H_{22}-\lambda) - H_{12}^2 = \lambda^2 - H_{11}\lambda - H_{22}\lambda + H_{11}H_{22} - H_{12}^2$$
$$\lambda = \frac{(H_{11}+H_{22}) \pm \sqrt{(H_{11}+H_{22})^2 - 4(H_{11}H_{22}-H_{12}^2)}}{2}$$
$$\lambda_1 = \frac{(H_{11}+H_{22}) + \sqrt{(H_{11}+H_{22})^2 - 4(H_{11}H_{22}-H_{12}^2)}}{2}$$
$$\lambda_2 = \frac{(H_{11}+H_{22}) - \sqrt{(H_{11}+H_{22})^2 - 4(H_{11}H_{22}-H_{12}^2)}}{2}$$

$|\lambda_1\rangle$: $\begin{pmatrix} H_{11}-\lambda_1 & H_{12} \\ H_{12} & H_{22}-\lambda_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$(H_{11}-\lambda_1)x - H_{12}y = 0$$
$$H_{12}x + (H_{22}-\lambda_1)y = 0$$

$$\frac{(H_{11}-\lambda_1)(H_{22}-\lambda_1)}{H_{12}}x - (H_{22}-\lambda_1)y = 0$$
$$H_{12}x + (H_{22}-\lambda_1)y = 0$$

$$x = \frac{1}{H_{12} + \frac{(H_{11}-\lambda)(H_{22}-\lambda)}{H_{12}}} = \frac{H_{12}}{H_{12}^2 + (H_{11}-\lambda)(H_{22}-\lambda)}$$
$$y = \frac{(H_{11}-\lambda_1)}{H_{12}^2 + (H_{11}-\lambda_1)(H_{22}-\lambda_1)}$$

Repeat for λ_2. I really don't want to repeat this process, so I'll just go on to the next part.

$H_{12} = 0$

$H = \begin{pmatrix} H_{11} & \\ & H_{22} \end{pmatrix}$

$\lambda = H_{11}, H_{22}$

$|H_{11}\rangle : \begin{pmatrix} 0 & 0 \\ 0 & H_{22} - H_{11} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

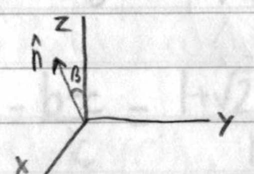
$|H_{11}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|H_{22}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

which is much easier to solve and more reasonable.

12.

a.



$|S_x\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$|\vec{S}\cdot\hat{n};+\rangle = \cos(\beta/2)|+\rangle + \sin(\beta/2)|-\rangle$

$P(\hbar/2) = |\langle S_x | \vec{S}\cdot\hat{n};+\rangle|^2 = \frac{1}{2}|(\langle +| + \langle -|)(\cos(\beta/2)|+\rangle + \sin(\beta/2)|-\rangle)|^2$

$= \frac{1}{2}|\cos(\beta/2)\langle +|+\rangle + \cancel{\sin(\beta/2)\langle +|-\rangle} + \cancel{\cos(\beta/2)\langle -|+\rangle} + \sin(\beta/2)\langle -|-\rangle|^2$

$= \frac{1}{2}|\cos(\beta/2) + \sin(\beta/2)|^2$

$= \frac{1}{2}(\cos^2(\beta/2) + 2\cos(\beta/2)\sin(\beta/2) + \sin^2(\beta/2))$

$= \frac{1}{2}(1 + 2\cos(\beta/2)\sin(\beta/2)) = \frac{1}{2}(1 + \sin\beta)$

since $\beta = \gamma$, let's re-write this

$P(\hbar/2) = \frac{1}{2}(1 + \sin\gamma)$

We should see half the particles in the $+\hbar/2$ state if $\hat{n}$ is aligned in the x-axis and all the particles in the $+\hbar/2$ state if $\hat{n}$ is orthogonal to the x-axis.

$P(\hbar/2, \gamma = 0) = \frac{1}{2}$

$P(\hbar/2, \gamma = \pi/2) = \frac{1}{2}(1+1) = 1$

$P(\hbar/2, \gamma = \pi) = \frac{1}{2}(1+0) = \frac{1}{2}$

b. $\langle (S_x - \langle S_x\rangle)^2\rangle = \langle (S_x)^2\rangle - \langle S_x\rangle^2$

$S_x = \frac{\hbar}{2}[(|+\rangle\langle -|) + (|-\rangle\langle +|)]$

$S_x^2 = \frac{\hbar^2}{4}[(|+\rangle\langle +|) + (|-\rangle\langle -|)]$

$|\psi\rangle = \cos(\gamma/2)|+\rangle + \sin(\gamma/2)|-\rangle$

$\langle S_x\rangle = \langle\psi|S_x|\psi\rangle = \frac{\hbar}{2}[\cos(\gamma/2)\langle +| + \sin(\gamma/2)\langle -|][(|+\rangle\langle -|) + (|-\rangle\langle +|)][\cos(\gamma/2)|+\rangle + \sin(\gamma/2)|-\rangle]$

$= \frac{\hbar}{2}[\cos(\gamma/2)\langle +| + \sin(\gamma/2)\langle -|][\sin(\gamma/2)|+\rangle + \cos(\gamma/2)|-\rangle]$

$= \frac{\hbar}{2}[\cos(\gamma/2)\sin(\gamma/2) + \sin(\gamma/2)\cos(\gamma/2)] = \frac{\hbar}{2}\sin\gamma$

$\langle (S_x)^2\rangle = \langle\psi|(S_x)^2|\psi\rangle = \frac{\hbar^2}{4}[\cos(\gamma/2)\langle +| + \sin(\gamma/2)\langle -|][(|+\rangle\langle +|) + (|-\rangle\langle -|)][\cos(\gamma/2)|+\rangle + \sin(\gamma/2)|-\rangle]$

$= \frac{\hbar^2}{4}[\cos(\gamma/2)\langle +| + \sin(\gamma/2)\langle -|][|+\rangle\cos(\gamma/2) + \sin(\gamma/2)|-\rangle]$

$= \frac{\hbar^2}{4}(\cos^2(\gamma/2) + \sin^2(\gamma/2)) = \frac{\hbar^2}{4}$

$\langle (S_x)^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4}(1 - \sin^2\gamma) = \frac{\hbar^2}{4}\cos^2\gamma$

$\gamma = 0, \Delta S_x = \frac{\hbar^2}{4}$

$\gamma = \pi/2, \Delta S_x = 0$

$\gamma = \pi, \Delta S_x = \frac{\hbar^2}{4}$

13. *First measurement*: $M = |+\rangle\langle +|$ as only the $|+\rangle$ state particles survive

Second measurement: $M' = |\hat{n};+\rangle\langle\hat{n};+|$ where

$|\hat{n};+\rangle = \cos(\beta/2)|+\rangle + \sin(\beta/2)|-\rangle$

Third measurement: $M'' = |-\rangle\langle -|$

Total will be $M''M'M$

$|\hat{n};+\rangle\langle\hat{n};+| = \begin{pmatrix} \cos(\beta/2) \\ \sin(\beta/2) \end{pmatrix} \begin{pmatrix} \cos(\beta/2) & \sin(\beta/2) \end{pmatrix}$

$= \begin{pmatrix} \cos^2(\beta/2) & \cos(\beta/2)\sin(\beta/2) \\ \cos(\beta/2)\sin(\beta/2) & \sin^2(\beta/2) \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos^2(\beta/2) & \cos(\beta/2)\sin(\beta/2) \\ \cos(\beta/2)\sin(\beta/2) & \sin^2(\beta/2) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos^2(\beta/2) & 0 \\ \cos(\beta/2)\sin(\beta/2) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \cos(\beta/2)\sin(\beta/2) & 0 \end{pmatrix}$$

$M^T = M''M'M = \cos(\beta/2)\sin(\beta/2)|-\rangle\langle+|$

$M^T(|+\rangle + |-\rangle) = \cos(\beta/2)\sin(\beta/2)|-\rangle$

Intensity is related to the beam squared, so

$I = \cos^2(\beta/2)\sin^2(\beta/2) = \sin^2\beta/4$

which is maximized when $\beta = \pi/2$, which gives an intensity of $1/4$ the initial surviving beam

14.

a. $\Omega = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}$

$\det(\Omega - \lambda I) = \det\begin{pmatrix} -\lambda & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -\lambda & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -\lambda \end{pmatrix} = -\lambda(\lambda^2 - 1/2) - 1/\sqrt{2}(-\lambda/\sqrt{2})$

$= -\lambda^3 + \lambda/2 + \lambda/2 = -\lambda(\lambda^2 - 1)$

$\lambda = 0, \pm 1$

$|0\rangle: \begin{pmatrix} & 1/\sqrt{2} & \\ 1/\sqrt{2} & & 1/\sqrt{2} \\ & 1/\sqrt{2} & \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$a + c = 0 \qquad a = -c \qquad |0\rangle = 1/\sqrt{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$b = 0$

$|1\rangle: \begin{pmatrix} -1 & 1/\sqrt{2} & \\ 1/\sqrt{2} & -1 & 1/\sqrt{2} \\ & 1/\sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-a + b/\sqrt{2} = 0 \qquad a\sqrt{2} = b \qquad |1\rangle = 1/2\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

$b/\sqrt{2} - c = 0 \qquad c\sqrt{2} = b$

$|-1\rangle: \begin{pmatrix} +1 & 1/\sqrt{2} & \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ & 1/\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$a + b/\sqrt{2} = 0 \qquad -a\sqrt{2} = b \qquad |-1\rangle = 1/2\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$

$b/\sqrt{2} + c = 0 \qquad -c\sqrt{2} = b$

No degeneracy

b. Looking this up, these are the eigenvalues + eigenvectors of $\hbar A$ for the spin-1 particle. I believe this is further explained in Chapter 3

15. $[A,B] = \sum_{a'',b''}\sum_{a',b'} |a'',b''\rangle\langle a'',b''|[AB - BA]|a',b'\rangle\langle a',b'|$

$= \sum_{a'',b''}\sum_{a',b'} |a'',b''\rangle\langle a'',b''|(a'b' - b'a')|a',b'\rangle\langle a',b'|$

$a'b' - b'a' = 0$ since these are not operators.

$\therefore [A,B] = 0$ if the simultaneous eigenkets of $A + B$ form a complete orthonormal set of base kets

16. $\{A,B\} = \langle a''|AB|a'\rangle + \langle a''|BA|a'\rangle = a''\langle a''|B|a'\rangle + a'\langle a''|B|a'\rangle$

$= (a'' + a')\langle a''|B|a'\rangle$

since $(a'' + a') \neq 0 \Rightarrow \langle a''|B|a'\rangle = 0$ for both $a'' = a'$ + $a'' \neq a'$, which implies they don't have simultaneous eigenkets

17. $H|n\rangle = E|n\rangle$ is unique since there is no degeneracy.

$[A_1, H]|n\rangle = 0$

$A_1H|n\rangle - HA_1|n\rangle = 0$

$E(A_1|n\rangle) = H(A_1|n\rangle)$ also, $E(A_2|n\rangle) = H(A_2|n\rangle)$

$A_1|n\rangle = a_1|n\rangle \qquad A_2|n\rangle = a_2|n\rangle$

$(A_1A_2 - A_2A_1)|n\rangle = (a_1a_2 - a_2a_1)|n\rangle = 0$ which is not true since $[A_1, A_2] \neq 0$. Thus, energy eigenstates must be degenerate.

18.

a. $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$ *Schwarz inequality*

$(\langle\alpha| + \lambda^*\langle\beta|)\cdot(|\alpha\rangle + \lambda|\beta\rangle) \geq 0$

$\langle\alpha|\alpha\rangle + \lambda\langle\alpha|\beta\rangle + \lambda^*\langle\beta|\alpha\rangle + \lambda^*\lambda\langle\beta|\beta\rangle \geq 0$

$\lambda = \frac{-\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$

$\langle\alpha|\alpha\rangle - \frac{\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle} - \frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} + \frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle\langle\beta|\beta\rangle}{\langle\beta|\beta\rangle} \geq 0$

$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$

b. $\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq |\langle\Delta A\Delta B\rangle|^2$

$|\langle\Delta A\Delta B\rangle|^2 = \frac{1}{4}|\langle[A,B]\rangle|^2 + \frac{1}{4}|\langle\{\Delta A,\Delta B\}\rangle|^2$

$\Delta A = A - \langle A\rangle \qquad A = \Delta A + \langle A\rangle$

$\Delta B = B - \langle B\rangle \qquad B = \Delta B + \langle B\rangle$

Taking the generalized uncertainty relation, let's take each term in $|\langle\Delta A\Delta B\rangle|^2$ and try to figure out what they are.

$[A,B] = AB - BA = (\Delta A + \langle A\rangle)(\Delta B + \langle B\rangle) - (\Delta B + \langle B\rangle)(\Delta A + \langle A\rangle)$

$= \Delta A\Delta B + \cancel{\Delta A\langle B\rangle} + \langle A\rangle\Delta B + \cancel{\langle A\rangle\langle B\rangle}$

$- \Delta B\Delta A - \Delta B\langle A\rangle - \cancel{\Delta A\langle B\rangle} - \cancel{\langle B\rangle\langle A\rangle} = \Delta A\Delta B - \Delta B\Delta A$

$= [\Delta A, \Delta B]$

$\therefore \langle[A,B]\rangle = \langle\alpha|(\Delta A\Delta B - \Delta B\Delta A)|\alpha\rangle = \lambda^*\langle\alpha|(\Delta B)^2|\alpha\rangle - \lambda\langle\alpha|(\Delta B)^2|\alpha\rangle$

since λ is purely imaginary $\quad = -2\lambda\langle\alpha|(\Delta B)^2|\alpha\rangle$

$= -2\lambda\langle(\Delta B)^2\rangle$

$\{A,B\} = \{\Delta A,\Delta B\}$

$\langle\{\Delta A,\Delta B\}\rangle = \lambda^*\langle\alpha|(\Delta B)^2|\alpha\rangle - \lambda\langle\alpha|(\Delta B)^2|\alpha\rangle = 0$

$\therefore |\langle\Delta A\Delta B\rangle|^2 = \frac{1}{4}\cdot 4\lambda^2\langle(\Delta B)^2\rangle^2$

$= \lambda^2\langle(\Delta B)^2\rangle^2$

$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle = \lambda^2\langle(\Delta B)^2\rangle^2$

c. $\Delta x = x - \langle x\rangle$

$\langle x'|\Delta x|\alpha\rangle = \int dx''\langle x'|x''\rangle\langle x''|x|\alpha\rangle - \int dx''\langle x'|x''\rangle\langle x''|\langle x\rangle|\alpha\rangle$

$= \int dx''\,\delta(x'-x'')x''\langle x''|\alpha\rangle - \int dx''\,\delta(x'-x'')\langle x\rangle\langle x''|\alpha\rangle$

$\Delta p = p - \langle p\rangle \qquad p = -i\hbar\frac{d}{dx}$

$\langle x'|\Delta p|\alpha\rangle = \int dx''\langle x'|x''\rangle\langle x''|-i\hbar\frac{d}{dx}|\alpha\rangle - \int dx''\langle x'|x''\rangle\langle x''|\langle p\rangle|\alpha\rangle$

$= \int dx''\,\delta(x'-x'')\cdot -i\hbar\frac{d}{dx}\langle x''|\alpha\rangle - \int dx''\,\delta(x'-x'')\langle p\rangle\langle x''|\alpha\rangle$

$\langle x'|\Delta x|\alpha\rangle = \int dx''\,\delta(x'-x'')x''\cdot(2\pi d^2)^{-1/4}\exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right)$

$- \int dx''\,\delta(x'-x'')\langle x\rangle\cdot(2\pi d^2)^{-1/4}\exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right)$

This is not an integral I want to evaluate, so let's leave everything in integral formula.

$\langle x'|\Delta p|\alpha\rangle = \int dx''\,\delta(x'-x'')\cdot -i\hbar\frac{d}{dx''}\left[(2\pi d^2)^{-1/4}\exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right)\right]$

$- \int dx''\,\delta(x'-x'')\langle p\rangle(2\pi d^2)^{-1/4}\exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right)$

$= \int dx''\,\delta(x'-x'')\cdot -i\hbar\cdot\left[(2\pi d^2)^{-1/4}\exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right)\right]\cdot\left(\frac{i\langle p\rangle}{\hbar} - \frac{2(x''-\langle x\rangle)}{4d^2}\right)$

$- \int dx''\,\delta(x'-x'')\langle p\rangle\cdot(2\pi d^2)^{-1/4}\exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right)$

$= \int dx''\,\delta(x'-x'')(2\pi d^2)^{-1/4}\langle p\rangle\exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right)$

$- \int dx''\,\delta(x'-x'')\langle p\rangle(2\pi d^2)^{-1/4}\exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right)$

$+ \int dx''\,\delta(x'-x'')\cdot\frac{i\hbar x''}{2d^2}(2\pi d^2)^{-1/4}\exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right)$

$- \int dx''\,\delta(x'-x'')\cdot\frac{i\hbar\langle x\rangle}{2d^2}(2\pi d^2)^{-1/4}\exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x''-\langle x\rangle)^2}{4d^2}\right)$

$\langle x'|\Delta x|\alpha\rangle = -\frac{2id^2}{\hbar}\langle x'|\Delta p|\alpha\rangle$

19.

a. $S_x = \hbar/2[(|+\rangle\langle-|)+(|-\rangle\langle+|)]$

$S_x^2 = \hbar^2/4[(|+\rangle\langle+|)+(|-\rangle\langle-|)]$

$S_y = -i\hbar/2[(|+\rangle\langle-|)-(|-\rangle\langle+|)]$

$S_y^2 = \hbar^2/4[(|+\rangle\langle+|)+(|-\rangle\langle-|)]$

$|S_z;+\rangle = |+\rangle$

$[S_x,S_y] = i\hbar S_z = i\hbar/2[(|+\rangle\langle+|)-(|-\rangle\langle-|)]$

$\langle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$

$\langle S_x^2\rangle = \langle S_z;+|S_x^2|S_z;+\rangle = \hbar^2/4\langle+|[(|+\rangle\langle+|)+(|-\rangle\langle-|)]|+\rangle$

$= \hbar^2/4\langle+|+\rangle = \hbar^2/4$

$\langle S_x\rangle = \hbar/2\langle+|[(|+\rangle\langle-|)+(|-\rangle\langle+|)]|+\rangle$

$= \hbar/2\langle+|-\rangle = 0$

$\langle(\Delta S_x)^2\rangle = \hbar^2/4$

$\langle(\Delta S_y)^2\rangle = \langle S_y^2\rangle - \langle S_y\rangle^2$

$\langle S_y^2\rangle = \langle S_x^2\rangle = \hbar^2/4$

$\langle S_y\rangle = -i\hbar/2\langle+|[(|+\rangle\langle-|)-(|-\rangle\langle+|)]|+\rangle$

$= +i\hbar/2\langle+|-\rangle = 0$

$\langle(\Delta S_y)^2\rangle = \hbar^2/4$

$\langle[S_x,S_y]\rangle = \langle+|i\hbar^2/2[(|+\rangle\langle+|)-(|-\rangle\langle-|)]|+\rangle$

$= i\hbar^2/2\langle+|+\rangle = i\hbar^2/2$

$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle \geq 1/4|\langle[S_x,S_y]\rangle|^2$

$\hbar^2/4\cdot\hbar^2/4 \geq 1/4\cdot\hbar^4/4$

$\hbar^4/16 \geq \hbar^4/16$

b. $|S_x;+\rangle = 1/\sqrt{2}|+\rangle + 1/\sqrt{2}|-\rangle$

$\langle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$

$\langle S_x^2\rangle = \hbar^2/8[\langle+|+\langle-|][(|+\rangle\langle+|)+(|-\rangle\langle-|)][|+\rangle+|-\rangle]$

$= \hbar^2/8[\langle+|+\langle-|][|+\rangle+|-\rangle] = \hbar^2/4$

$\langle S_x\rangle = \hbar/4[\langle+|+\langle-|][(|+\rangle\langle-|)+(|-\rangle\langle+|)][|+\rangle+|-\rangle]$

$= \hbar/4[\langle+|+\langle-|][|+\rangle+|-\rangle] = \hbar/2$

$\langle(\Delta S_x)^2\rangle = 0$

don't need to do $\langle(\Delta S_y)^2\rangle$

$\langle[S_x,S_y]\rangle = i\hbar^2/4[\langle+|+\langle-|][(|+\rangle\langle+|)-(|-\rangle\langle-|)][|+\rangle+|-\rangle]$

$= i\hbar^2/4[\langle+|+\langle-|][|+\rangle-|-\rangle] = 0$

20. As a reminder,

$\langle(\Delta S_x)^2\rangle = \langle\psi|\hbar^2/4[(|+\rangle\langle+|)+(|-\rangle\langle-|)]|\psi\rangle$
$- (\langle\psi|\hbar/2[(|+\rangle\langle-|)+(|-\rangle\langle+|)]|\psi\rangle)^2$

$\langle(\Delta S_y)^2\rangle = \langle\psi|\hbar^2/4[(|+\rangle\langle+|)+(|-\rangle\langle-|)]|\psi\rangle$
$- (\langle\psi|-i\hbar/2[(|+\rangle\langle-|)-(|-\rangle\langle+|)]|\psi\rangle)^2$

$|\psi\rangle = (a|+\rangle + (1-a^2)^{1/2}\exp(i\beta)|-\rangle$

This is functionally equivalent to $|\vec{S}\cdot\hat{n};+\rangle$

$\langle(\Delta S_x)^2\rangle = \hbar^2/4[a\langle+| + (1-a^2)^{1/2}\exp(-i\beta)\langle-|][a|+\rangle + (1-a^2)^{1/2}\exp(-i\beta)|-\rangle]$

$- \hbar^2/4([a\langle+| + (1-a^2)^{1/2}\exp(-i\beta)\langle-|][(1-a^2)^{1/2}\exp(i\beta)|+\rangle + a|-\rangle])^2$

$= \hbar^2/4(a^2 + (1-a^2) - (a(1-a^2)^{1/2}\exp(i\beta) + a(1-a^2)^{1/2}\exp(-i\beta))^2)$

$= \hbar^2/4[1 - a^2(1-a^2)(\exp(i\beta)+\exp(-i\beta))^2]$

$= \hbar^2/4[1 - 4a^2(1-a^2)\cos^2\beta]$

$\langle(\Delta S_y)^2\rangle = \hbar^2/4 + \hbar^2/4([a\langle+| + (1-a^2)^{1/2}\exp(-i\beta)\langle-|][(1-a^2)^{1/2}\exp(i\beta)|+\rangle - a|-\rangle])^2$

$= \hbar^2/4[1 + a^2(1-a^2)(\exp(i\beta)-\exp(-i\beta))^2]$

$= \hbar^2/4[1 - 4a^2(1-a^2)\sin^2\beta]$

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \hbar^4/16\,[1-4a^2(1-a^2)\cos^2\beta][1-4a^2(1-a^2)\sin^2\beta]$$

$$= \hbar^4/16\,[1-4a^2(1-a^2)\sin^2\beta - 4a^2(1-a^2) + 16a^4(1-a^2)^2\cos^2\beta\sin^2\beta]$$

$$= \hbar^4/16\,[1-4a^2(1-a^2) + 4a^4(1-a^2)^2\sin^2(2\beta)]$$

$\beta = \pi/4$

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \hbar^4/16\,[1-4a^2(1-a^2)+4a^4(1-a^2)^2]$$

maximized when $a^2 = 0$ or 1

$\pm|+\rangle$ or $\exp(i\pi/4)|-\rangle$

$\pm|+\rangle$ has already been done in the previous question

$$\langle \cancel{\exp(-i\pi/4)}\langle -|[\,i\hbar^2/2\,(|+\rangle\langle+|) - i\hbar^2/2\,(|-\rangle\langle-|)]\,|-\rangle \cancel{\exp(-i\pi/4)}$$

$$= -i\hbar^2/2$$

$$\hbar^4/16 = \hbar^4/16$$

21. $\psi = \sqrt{2/a}\,\sin(n\pi x/a)$ this is the solution for the particle in a box (Appendix A. 2)

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty}\psi^* x^2 \psi\,dx = \int_0^a 2/a\, x^2 \sin^2(n\pi x/a)\,dx$$

$$= 1/a\int_0^a x^2 - x^2\cos(2n\pi x/a)\,dx = a^2/3 - 1/a\int_0^a x^2\cos(2n\pi x/a)\,dx$$

$u = x^2$ $\quad v' = \cos(2n\pi x/a)$

$u' = 2x$ $\quad v = \frac{a}{2n\pi}\sin(2n\pi x/a)$

$$= a^2/3 - 1/a\,(uv - \int v\,du\,dx)$$

$$= a^2/3 - 1/a\left(\cancel{\frac{ax^2}{2n\pi}\sin(2n\pi x/a)\Big|_0^a} - \int_0^a \frac{a\,x}{n\pi}\sin(2n\pi x/a)\,dx\right)$$

$u = x$ $\quad v' = \sin(2n\pi x/a)$

$u' = 1$ $\quad v = -\frac{a}{2n\pi}\cos(2n\pi x/a)$

$$= a^2/3 + \frac{1}{n\pi}\left(\frac{-ax}{2n\pi}\cos\left(\frac{2n\pi x}{a}\right)\Big|_0^a + \cancel{\frac{a}{2n\pi}\int_0^a \cos(2n\pi x/a)\,dx}\right)$$

$$= a^2/3 + \frac{a^2}{2n^2\pi^2} = a^2\left(1/3 - \frac{1}{2n^2\pi^2}\right)$$

$$\langle x\rangle = 2/a\int_0^a x\sin^2(n\pi x/a)\,dx = 1/a\int_0^a x(1-\cos(2n\pi x/a))\,dx$$

$$= a/2 - \int_0^a x\cos(2n\pi x/a)\,dx$$

$u = x$ $\quad v' = \cos(2n\pi x/a)$

$u' = 1$ $\quad v = \frac{a}{2n\pi}\sin(2n\pi x/a)$

$$= a/2 - \left(\cancel{\frac{ax}{2n\pi}\sin(2n\pi x/a)\Big|_0^a} - \cancel{\int_0^a \frac{a}{2n\pi}\sin(2n\pi x/a)\,dx}\right) = a/2$$

$$\langle(\Delta x)^2\rangle = a^2/3 - \frac{a^2}{2n^2\pi^2} - a^2/4 = a^2\left(1/12 - \frac{1}{2n^2\pi^2}\right)$$

$$\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2$$

$p = -i\hbar\frac{\partial}{\partial x}$ $\quad p = -\hbar^2\frac{\partial^2}{\partial x^2}$

$$p|\psi\rangle = -i\hbar\sqrt{\tfrac{2}{a}}\cdot\frac{n\pi}{a}\cdot\cos(n\pi x/a)$$

$$p^2|\psi\rangle = +\hbar^2\sqrt{\tfrac{2}{a}}\cdot\frac{n^2\pi^2}{a^2}\sin(n\pi x/a)$$

$$\langle p^2\rangle = \hbar^2 n^2\pi^2/a^2\cdot 2/a\int_0^a \sin^2(n\pi x/a)\,dx = 2\hbar^2 n^2\pi^2/a^3\int_0^a 1-\cos(2n\pi x/a)\,dx$$

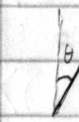
$$= \hbar^2 n^2\pi^2/a^2$$

$$\langle p\rangle = \frac{-2i\hbar n\pi}{a^2}\int_0^a \sin(n\pi x/a)\cos(n\pi x/a)\,dx = \frac{-i\hbar n\pi}{a^2}\int_0^a\sin(2n\pi x/a)\,dx = 0$$

$$\langle(\Delta p)^2\rangle = \hbar^2 n^2\pi^2/a^2$$

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{\hbar^2 n^2\pi^2}{\cancel{a^2}}\cdot\cancel{a^2}\left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right)$$

$$= \frac{\hbar^2 n^2\pi^2}{12} - \frac{\hbar^2}{2} = \frac{\hbar^2}{2}\left(\frac{n^2\pi^2}{6} - 1\right)$$

22.  ℓ with mass m $\quad \tau = m\ell^2\frac{d^2\theta}{dt^2} = mg\theta\ell$

$$\ddot\theta = g/\ell\,\theta$$

$$\theta(t) = a\exp(\sqrt{g/\ell}\,t) + b\exp(-\sqrt{g/\ell}\,t)$$

$$\Delta x = \ell\theta = (a+b)\ell$$

$$\Delta p = m\ell\frac{d\theta}{dt} = \sqrt{g/\ell}\,(a-b)m\ell = m\sqrt{g\ell}\,(a-b)$$

$$\Delta x\Delta p \geq \hbar/2$$

$$(a^2-b^2)\,m\sqrt{g\ell^3} \geq \hbar/2$$

23.

a. Calculate eigenvalues of B to determine degeneracy

$$\det(B-\lambda I) = \begin{pmatrix} b-\lambda & & \\ & -\lambda & -ib \\ & ib & -\lambda \end{pmatrix} = (b-\lambda)(\lambda^2 - b^2)$$

$$\lambda = \pm b$$

Since there are repeated eigenvalues, there is degeneracy

b. $[A,B] = AB - BA = 0$

$$AB = BA$$

$$AB = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}$$

c. $|a\rangle$: $A - aI = \begin{pmatrix} 0 & & \\ & -2a & \\ & & -2a \end{pmatrix}$

$$\begin{pmatrix} 0 & & \\ & -2a & \\ & & -2a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|a\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$|-a\rangle$: $A + aI = \begin{pmatrix} 2a & & \\ & 0 & \\ & & 0 \end{pmatrix}$

$$|-a\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

part of the reason this is imaginary is because I know the solution for $|b\rangle$, and I want the eigenvectors to be orthonormal

$|b\rangle$: $B - bI = \begin{pmatrix} 0 & & \\ & -b & -ib \\ & ib & -b \end{pmatrix}$

$x_2 = -i x_3$ $\quad |b\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$

$A|a\rangle = a|a\rangle \qquad B|a\rangle = b|a\rangle$

$A|-a\rangle = -a|-a\rangle \qquad B|-a\rangle = -b|-a\rangle$

$A|b\rangle = -a|b\rangle \qquad B|b\rangle = b|b\rangle$

24.

a. Rotation matrix is given by 3.2.44

$$\cos(\phi/2) - i\vec{\sigma}\cdot\hat{n}\sin(\phi/2)$$

Clockwise rotation about x-axis through $-\pi/2$, $\phi = -\pi/2$

Rotation matrix is $\tfrac{1}{\sqrt{2}}(1 + i\sigma_x)$

b. $S_z = \tfrac{\hbar}{2}\cdot\tfrac{1}{\sqrt{2}}(1 - i\sigma_x)\,\sigma_z\,\cdot\tfrac{1}{\sqrt{2}}(1 + i\sigma_x)$

$$= \tfrac{\hbar}{4}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$= \tfrac{\hbar}{4}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix} = \tfrac{\hbar}{4}\begin{pmatrix} 1-1 & i+i \\ -i-i & 1-1 \end{pmatrix}$$

$$= \tfrac{\hbar}{2}\begin{pmatrix} & i \\ -i & \end{pmatrix}$$

25. Given a basis $\{|c'\rangle\}$

$$|c'\rangle = \sum_{b'} |b'\rangle\langle b'|c'\rangle$$

$$\langle c'|A|c''\rangle = \sum_{b'}\sum_{b''}\langle c'|b'\rangle\langle b'|A|b''\rangle\langle b''|c''\rangle$$

$$= \sum_{b'}\sum_{b''}\langle c'|b'\rangle\langle b''|c''\rangle\langle b'|A|b''\rangle$$

$\langle c'|b'\rangle\langle b''|c''\rangle$ needs to be real, but the individual components don't need to be real

26. $|S_z;+\rangle = |+\rangle \qquad |S_x;+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$

$|S_z;-\rangle = |-\rangle \qquad |S_x;-\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$

In the S_x basis, this turns into

$$|S_x;+\rangle' = |+\rangle$$
$$|S_x;-\rangle' = |-\rangle$$

$|S_x;+\rangle' = U|S_x;+\rangle$

$|S_x;-\rangle' = U|S_x;-\rangle$

The transformation matrix takes it from the S_z basis to the S_x basis

$$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}$$

$1 = U_{11}/\sqrt{2} + U_{12}/\sqrt{2} \qquad 0 = U_{11}/\sqrt{2} - U_{12}/\sqrt{2}$

$0 = U_{21}/\sqrt{2} + U_{22}/\sqrt{2} \qquad 1 = U_{21}/\sqrt{2} - U_{22}/\sqrt{2}$

$U_{21} = -U_{22} \qquad U_{11} = U_{12}$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$U = \sum_r |b^{(r)}\rangle\langle a^{(r)}| = |+\rangle\left(\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|\right) + |-\rangle\left(\frac{1}{\sqrt{2}}\langle +| - \frac{1}{\sqrt{2}}\langle -|\right)$

$= \frac{1}{\sqrt{2}}|+\rangle\langle +| + \frac{1}{\sqrt{2}}|+\rangle\langle -| + \frac{1}{\sqrt{2}}|-\rangle\langle +| - \frac{1}{\sqrt{2}}|-\rangle\langle -|$

$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

27

a. $\langle b''|f(A)|b'\rangle$

$= \sum_{a',a''}\langle b''|a''\rangle\langle a''|f(A)|a'\rangle\langle a'|b'\rangle$

$= \sum_{a'} f(a')\langle b''|a'\rangle\langle a'|b'\rangle$

b. $\langle \vec{p}''|F(r)|\vec{p}'\rangle$

$= \int d\vec{r}'\, F(r')\langle \vec{p}''|\vec{r}'\rangle\langle \vec{r}'|\vec{p}'\rangle \qquad \langle \vec{r}'|\vec{p}'\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\exp\left(i\vec{p}'\cdot\vec{r}'/\hbar\right)$

$= \frac{1}{(2\pi\hbar)^3}\int d\vec{r}'\, F(\vec{r}')\exp\left(i(\vec{p}'-\vec{p}'')\vec{r}'/\hbar\right)$

$= \frac{2\pi}{(2\pi\hbar)^3}\int_{-1}^{1} d(\cos\theta)\int_0^\infty dr'\, r'^2 F(r')\exp\left(i(\vec{p}'-\vec{p}'')r'\cos\theta/\hbar\right)$

$= \frac{1}{2\pi^2\hbar^2 q}\int_0^\infty dr'\, \sin(qr'/\hbar)F(r')$

$q = |\vec{p}' - \vec{p}''|$

28.

a. $[x, F(p_x)]_{cl} = \frac{\partial x}{\partial x}\cdot\frac{\partial F(p_x)}{\partial p_x} - \frac{\partial x}{\partial p_x}\cdot\frac{\partial F(p_x)}{\partial x} = \frac{\partial F(p_x)}{\partial p_x}$

b. $[\ ,\ ]_{cl} = [\ ,\ ]/i\hbar$

$[x, \exp(ip_x a/\hbar)]_{cl} = \frac{\partial \exp(ip_x a/\hbar)}{\partial p_x} = \frac{ia}{\hbar}\exp(ip_x a/\hbar)$

$[x, \exp(ip_x a/\hbar)] = i\hbar\cdot\frac{ia}{\hbar}\exp(ip_x a/\hbar) = -a\exp(ip_x a/\hbar)$

c. $|\psi\rangle = \exp(ip_x a/\hbar)|x'\rangle \qquad [x, F(p_x)] = xF(p_x) - F(p_x)x = -aF(p_x)$

$x|\psi\rangle = x\exp(ip_x a/\hbar)|x'\rangle$

$= -a\exp(ip_x a/\hbar)|x'\rangle + \exp(ip_x a/\hbar)x'|x'\rangle$

$= (x'-a)\exp(ip_x a/\hbar)|x'\rangle$

$= (x'-a)|\psi\rangle$

29.

a. $[x_i, G(\vec{p})]_{cl} = \frac{\partial G}{\partial p_i} = \frac{[x_i, G(\vec{p})]}{i\hbar}$

$[x_i, G(\vec{p})] = i\hbar\frac{\partial G}{\partial p_i}$

and you can make the same argument for $[p_i, F(\vec{x})]$

b. $[x^2, p^2] = x^2p^2 - p^2x^2$ $p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$

$[x^2, p^2]|\psi\rangle$

$= -x^2\hbar^2 \frac{\partial^2\psi}{\partial x^2} + \hbar^2 \frac{\partial^2(x^2\psi)}{\partial x^2}$

$\frac{\partial(x^2\psi)}{\partial x} = 2x\psi + x^2\psi'$

$\frac{\partial^2(x^2\psi)}{\partial x^2} = 2\psi + 2x\psi' + 2x\psi' + x^2\psi''$

$[x^2, p^2]|\psi\rangle = \hbar^2[2\psi + 4x\psi']$

$= 2i\hbar\{x, p\}$

$[x^2, p^2]_{cl} = \frac{\partial x^2}{\partial x}\frac{\partial p^2}{\partial p} - \frac{\partial x^2}{\partial p}\frac{\partial p^2}{\partial x} = 2x \cdot 2p = 4xp$

In the classical limit, $\{x, p\} = 2xp$

$4xp = \frac{4i\hbar xp}{i\hbar}$

$4xp = 4xp$

30.

a. $[x_i, \mathcal{J}(\vec{l})]_{cl} = \frac{[x_i, \mathcal{J}(\vec{l})]}{i\hbar}$

$= \sum_i \frac{\partial}{\partial p_i}\left(\exp\left(\frac{-i\vec{p}\cdot\vec{l}}{\hbar}\right)\right) = \frac{-i\vec{l}}{\hbar}\exp\left(\frac{-i\vec{p}\cdot\vec{l}}{\hbar}\right)$

$[x_i, \mathcal{J}(\vec{l})] = \sum_i l_i \mathcal{J}(\vec{l})$

b. $|\psi'\rangle = \mathcal{J}(\vec{l})|\psi\rangle$

$\langle\psi'|x|\psi'\rangle = \langle\psi|[\mathcal{J}(\vec{l})^\dagger x \mathcal{J}(\vec{l})]|\psi\rangle$

$= \langle\psi|\cancel{\mathcal{J}^\dagger(\vec{l})\mathcal{J}(\vec{l})}x|\psi\rangle + \langle\psi|\mathcal{J}^\dagger(\vec{l})\, l_i \mathcal{J}(\vec{l})|\psi\rangle$

$= \langle\psi|x|\psi\rangle + \sum_i l_i$

$= \langle x\rangle + \vec{l}$

31. $|\alpha\rangle' = \mathcal{J}(d\vec{x}')|\alpha\rangle$

$[\vec{x}, \mathcal{J}(d\vec{x}')] = d\vec{x}'$ 1.6.25

$[\vec{p}, \mathcal{J}(d\vec{x}')] = 0$ 1.6.45

$\langle\alpha'|x|\alpha'\rangle = \langle\alpha|\mathcal{J}^\dagger(d\vec{x}')\, x\, \mathcal{J}(d\vec{x}')|\alpha\rangle$

$= \langle\alpha|\mathcal{J}^\dagger(d\vec{x}')(\mathcal{J}(d\vec{x}')x + d\vec{x}')|\alpha\rangle$

$= \langle\alpha|\mathcal{J}^\dagger(d\vec{x}')\mathcal{J}(d\vec{x}')x + \mathcal{J}^\dagger(d\vec{x}')d\vec{x}'|\alpha\rangle$

$= \langle x\rangle + d\vec{x}'$

$\langle\alpha'|p|\alpha'\rangle = \langle\alpha|\mathcal{J}^\dagger(d\vec{x}')\vec{p}\,\mathcal{J}(d\vec{x}')|\alpha\rangle$

$= \langle\alpha|\mathcal{J}^\dagger(d\vec{x}')\mathcal{J}(d\vec{x}')\vec{p}|\alpha\rangle$

$= \langle\alpha|\vec{p}|\alpha\rangle = \langle p\rangle$

32.

a. $\langle x'|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}}\exp(ikx' - x'^2/2d^2)$

$\langle p\rangle = \langle\alpha|x'\rangle(-i\hbar\frac{d}{dx})\langle x'|\alpha\rangle$

$= \int_{-\infty}^{\infty}\frac{1}{\pi^{1/2}d}\exp(-ikx' - x'^2/2d^2) \cdot -i\hbar(\exp(ikx' - x'^2/2d^2)(ik - x'/d^2))\,dx'$

$= \int_{-\infty}^{\infty}\frac{-i\hbar(ik - x'/d^2)}{\pi^{1/2}d}\exp(-x'^2/d^2)\,dx'$

I'm going to take a little detour here and talk about the Gaussian integral. This is a really cool math trick that I was first introduced to while reading Shankar. Essentially, it involves switching between polar and cartesian coordinates to evaluate this integral. The general form of the solution is

$\int_{-\infty}^{\infty}\exp(-\alpha x^2) = \sqrt{\pi/\alpha}$

$\int_{-\infty}^{\infty}x^2\exp(-\alpha x^2) = \frac{1}{2\alpha}\sqrt{\pi/\alpha}$

$$\langle p\rangle = \int_{-\infty}^{\infty} \frac{\hbar k \exp(-x'^2/d^2)}{\pi^{1/2} d}\, dx' + \cancel{\int_{-\infty}^{\infty} \frac{i\hbar x' \exp(-x'^2/d^2)}{\pi^{1/2} d^3} dx'}$$

$$= \frac{\hbar k}{\pi^{1/2} d} \cdot \sqrt{\pi d^2} = \hbar k$$

$$\langle p^2\rangle = -\hbar^2 \langle \alpha|x'\rangle \frac{d^2}{dx^2}\langle x'|\alpha\rangle$$

$$\frac{d}{dx}\langle x'|\alpha\rangle = \frac{1}{\pi^{1/4} d^{1/2}} \exp(ikx' - x'^2/2d^2)(ik - x'/d^2)$$

$$= \frac{ik}{\pi^{1/4} d^{1/2}} \exp(ikx' - x'^2/2d^2) - \frac{x'}{\pi^{1/4} d^{5/2}} \exp(ikx' - x'^2/2d^2)$$

$$\frac{d^2}{dx^2}\langle x'|\alpha\rangle = \frac{ik}{\pi^{1/4} d^{1/2}} \exp(ikx' - x'^2/2d^2)(ik - x'/d^2)$$

$$- \frac{\exp(ikx' - x'^2/2d^2)}{\pi^{1/4} d^{5/2}} - \frac{x' \exp(ikx' - x'^2/2d^2)(ik - x'/d^2)}{\pi^{1/4} d^{5/2}}$$

Since we know $\int_{-\infty}^{\infty} x \exp^{-x^2} dx = 0$, we can only write the terms that don't die.

$$\int_{-\infty}^{\infty} \frac{\hbar^2 k^2}{\pi^{1/2} d} \exp(-x'^2/d^2) + \frac{\hbar^2}{\pi^{1/2} d^3} \exp(-x'^2/d^2) - \frac{\hbar^2 x'^2}{\pi^{1/2} d^5} \exp(-x'^2/d^2)\, dx$$

$$= \frac{\hbar^2 k^2}{\pi^{1/2} d} \cdot \sqrt{\pi d^2} + \frac{\hbar^2}{\pi^{1/2} d^3} \sqrt{\pi d^2} - \frac{\hbar^2}{\pi^{1/2} d^5} \cdot \frac{d^2}{2} \sqrt{\pi d^2}$$

$$= \hbar^2/2d^2 + \hbar^2 k^2$$

b. $\langle p'|\alpha\rangle = \frac{d^{1/2}}{\hbar^{1/2}\pi^{1/4}} \exp(-(p' - \hbar k)^2 d^2 / 2\hbar^2)$

$$\langle p\rangle = \langle \alpha|p\rangle p \langle p|\alpha\rangle$$

$$= \int_{-\infty}^{\infty} \frac{d}{\hbar\pi^{1/2}}\, p \exp(-(p' - \hbar k)^2 d^2/\hbar^2)\, dp \qquad u = p - \hbar k \quad du = dp$$

$$= \int_{-\infty}^{\infty} \frac{d \hbar k}{\hbar \pi^{1/2}} \exp(-u^2 d^2/\hbar^2)\, du$$

$$= \frac{dk}{\pi^{1/2}} \cdot \sqrt{\frac{\pi \hbar^2}{d^2}} = \hbar k$$

$$\langle p^2\rangle = \langle \alpha|p\rangle p^2 \langle p|\alpha\rangle$$

$$= \int_{-\infty}^{\infty} \frac{d}{\hbar\pi^{1/2}} p^2 \exp(-(p - \hbar k)^2 d^2/\hbar^2)\, dp \qquad u = p - \hbar k \quad du = dp$$

$$p = u + k\hbar \qquad p^2 = u^2 + 2uk\hbar + k^2\hbar^2$$

$$= \int_{-\infty}^{\infty} \frac{d}{\hbar\pi^{1/2}} u^2 \exp(-u^2 d^2/\hbar^2)\, du + \int_{-\infty}^{\infty} \frac{d \cdot k^2 \hbar^2}{\hbar \pi^{1/2}} \exp(-u^2 d^2/\hbar^2)\, du$$

$$= \frac{d}{\hbar\pi^{1/2}} \cdot \frac{\hbar^2}{2d^2} \sqrt{\frac{\pi\hbar^2}{d^2}} + \frac{d\hbar^2 k^2}{\hbar\pi^{1/2}} \cdot \sqrt{\frac{\pi\hbar^2}{d^2}}$$

$$= \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$

33.

a. $\langle p'|x|\alpha\rangle = \langle p'|x|p''\rangle\langle p''|\alpha\rangle = \langle p'|x|p'\rangle\langle p'|\alpha\rangle$

$$= i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle$$

$$\langle \beta|x|\alpha\rangle = \langle \beta|p'\rangle\langle p'|x|p'\rangle\langle p'|\alpha\rangle$$

$$= \int dp'\, \phi_\beta^*(p')\, i\hbar \frac{\partial}{\partial p'} \phi_\alpha(p')$$

b. $\exp(\frac{ix\Xi}{\hbar})$ is the momentum translation operator