Solutions to Principles of Quantum Mechanics, 2ed. by R. Shankar

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Chapter 1

Mathematical Introduction

1.1 Linear Vector Spaces: Basics

1.1.1 Vector Proofs

Verify these claims.

$|0\rangle$ **is unique, i.e., if** $|0'\rangle$ **has all the properties of** $|0\rangle$**, then** $|0\rangle = |0'\rangle$**. For the first consider** $|0\rangle + |0'\rangle$ **and use the advertised properties of the two null vectors in turn.**

Let's do as suggested. Starting with $|0\rangle + |0'\rangle$ and using the definition of the null vector for $|0\rangle$, $|V\rangle + |0\rangle = |V\rangle$,

$$|0\rangle + |0'\rangle = |0'\rangle \tag{1}$$

If $|0'\rangle$ is also a null vector,

$$|0\rangle + |0'\rangle = |0\rangle \tag{2}$$

Because the left side is the same, we can see $|0\rangle = |0'\rangle$.

$0\,|V\rangle = |0\rangle$**. For the second start with** $|0\rangle = (0+1)\,|V\rangle + |-V\rangle$

Using distributive in the scalars,

$$(0+1)\,|V\rangle + |-V\rangle = 0\,|V\rangle + |V\rangle + |-V\rangle \tag{3}$$

Using vector inverse and the definition of the null vector,

$$= 0\,|V\rangle + |0\rangle = 0\,|V\rangle \tag{4}$$

$$(0+1)\,|V\rangle + |-V\rangle = 0\,|V\rangle \tag{5}$$

Alternatively, we can perform the scalar addition from the get-go,

$$(0+1)\,|V\rangle + |-V\rangle = |V\rangle + |-V\rangle = |0\rangle \tag{6}$$

Again, we started with the same input, and got two answers. $0\,|V\rangle = |0\rangle$.

$|-V\rangle = -|V\rangle$**. For the third, begin with** $|V\rangle + (-|V\rangle) = 0\,|V\rangle = |0\rangle$**.**

We compare to the inverse under addition requirement,

$$|V\rangle + |-V\rangle = |0\rangle \tag{7}$$

Comparing this to the suggestion, we see $-|V\rangle = |-V\rangle$.

$|-V\rangle$ **is the unique additive inverse of** $|V\rangle$**. For the last, let** $|W\rangle$ **also satisfy** $|V\rangle + |W\rangle = |0\rangle$**. Since** $|0\rangle$ **is unique, this means** $|V\rangle + |W\rangle = |V\rangle + |-V\rangle$**. Take it from here.**

Shankar has sort of solved it all already for us. Since we know that both $|V\rangle + |W\rangle$ and $|V\rangle + |-V\rangle$ return 0, we can compare the two and see that $|W\rangle = |-V\rangle$.

I tend to have trouble with proofs because I never really know what I'm allowed to assume or what sort of common sense, ”it's right there” steps I'm allowed to take. I'm also far from mathematically rigorous.

1.1.2 Vector Space Example

Consider the set of all entities of the form (a, b, c) where the entries are real numbers. Addition and scalar multiplication are defined as follows:

$$(a, b, c) + (d, e, f) = (a + d, b + e, c + f)$$

$$\alpha(a, b, c) = (\alpha a, \alpha b, \alpha c)$$

Write down the null vector and inverse of (a, b, c). Show that vectors of the form $(a, b, 1)$ do not form a vector space.

By observation, the null vector,

$$|0\rangle = (0, 0, 0) \tag{1}$$

and the inverse,

$$|-V\rangle = (-a, -b, -c) \tag{2}$$

To show $(a, b, 1)$ do not from a vector space, let's go through the list of requirements. First closure, let's add two vectors together,

$$(a, b, 1) + (c, d, 1) = (a + c, b + d, 2) \tag{3}$$

We see that the resultant vector is not of the form $(a, b, 1)$ so we break closure.

Next, let's try to find a null vector. We quickly realize that the only vector that satisfies this is $(0, 0, 0)$, which is not of the form $(a, b, 1)$. The null vector must also exist as part of the vector space so we break that rule as well.

Assuming we were allowed to have a null vector $(0, 0, 0)$, we would also break the inverse under addition rule since an inverse of $(a, b, 1)$ is $(-a, -b, -1)$, which is not the correct form.

1.1.3 Functions as Vectors

Do functions that vanish at the end points $x = 0$ and $x = L$ form a vector space? How about periodic functions obeying $f(0) = f(L)$? How about functions that obey $f(0) = 4$? If the functions do not qualify, list the things that go wrong.

We can convince ourselves that the first two form a vector space since we have the null vector of $f(x) = 0$ and an inverse function $f(x) + (-f(x)) = 0$. The periodic condition would normally disqualify us from closure, think of $\sin(x)$ and $\sin(x/\sqrt{2})$. However, since we return to the same point after a length L, the two functions will always have the same period, which allows them to have closure.

Functions that obey $f(0) = 4$ does not qualify as a vector space. It breaks closure since adding two functions will give $f(0) = 8$. We can't find a null vector or an inverse.

1.1.4 Linear Independence

Consider three elements from the vector space of real 2×2 matrices:

$$|1\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad |2\rangle = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad |3\rangle = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}$$

Are they linearly independent? Support your answer with details. (Notice we are calling these matrices vectors and using kets to represent them to emphasize their role as elements of a vector space.)

We can quickly see that none of the vectors are scalar multiples of the other, but let's see if $|3\rangle$ is a linear combination of $|1\rangle$ and $|2\rangle$

$$a\,|1\rangle + b\,|2\rangle = |3\rangle \tag{1}$$

Looking at each element, we have the system of equations,

$$\begin{cases} b = -2 \\ a + b = -1 \\ 0 = 0 \\ b = -2 \end{cases} \tag{2}$$

From this, we get $a = 1$ and $b = -2$ or $|1\rangle - 2\,|2\rangle = |3\rangle$. These three elements are not linearly independent.

1.1.5 Linear Independence

Show that the following row vectors are linearly dependent: $(1,1,0)$**,** $(1,0,1)$**,** $(3,2,1)$**.**

Let's label these vectors $|1\rangle$, $|2\rangle$, $|3\rangle$.

$$a\,|1\rangle + b\,|2\rangle = |3\rangle \tag{1}$$

We have the system of equations,

$$\begin{cases} a+b=3 \\ a=2 \\ b=1 \end{cases} \tag{2}$$

We see that $2(1,1,0)+(1,0,1)-(3,2,1)=(0,0,0)$, making this combination linearly dependent.

Show the opposite for $(1,1,0)$**,** $(1,0,1)$**,** $(0,1,1)$

Following the same steps, we have the system of equations,

$$\begin{cases} a+b=0 \\ a=1 \\ b=1 \end{cases} \tag{3}$$

We can convince ourselves there is no set of a and b that satisfy these conditions, so this set is linearly independent.

1.2 Inner Product Spaces

1.3 Dual Spaces and the Dirac Notation

1.3.1 Gram-Schmidt Application

Form an orthonormal basis in two dimensions starting with $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = 2\hat{i} - 6\hat{j}$. Can you generate another orthonormal basis starting with these two vectors? If so, produce another.

Applying the Gram-Schmidt theorem, we first re-normalize $\vec{A}$,

$$|1\rangle = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j} \tag{1}$$

The second step,

$$|2'\rangle = |B\rangle - |1\rangle \langle 1|B\rangle \tag{2}$$

$$= 2\hat{i} - 6\hat{j} - \frac{3\hat{i} + 4\hat{j}}{25}(3\hat{i} + 4\hat{j}) \cdot (2\hat{i} - 6\hat{j}) \tag{3}$$

$$= (2\hat{i} - 6\hat{j}) + \frac{18}{25}(3\hat{i} + 4\hat{j}) = \frac{1}{25}(104\hat{i} - 78\hat{j}) \tag{4}$$

Renormalizing this,

$$|2\rangle = \frac{4}{5}\hat{i} - \frac{3}{5}\hat{j} \tag{5}$$

To generate a different orthonormal basis, we start by renormalizing $\vec{B}$,

$$|1\rangle = \frac{1}{\sqrt{10}}\hat{i} - \frac{3}{\sqrt{10}}\hat{j} \tag{6}$$

Subtracting out the projection,

$$|2'\rangle = 3\hat{i} + 4\hat{j} - \frac{\hat{i} - 3\hat{j}}{10}(\hat{i} - 3\hat{j}) \cdot (3\hat{i} + 4\hat{j}) \tag{7}$$

$$= \frac{1}{10}(39\hat{i} + 13\hat{j}) \tag{8}$$

Renormalizing,

$$|2\rangle = \frac{1}{\sqrt{10}}(3\hat{i} + \hat{j}) \tag{9}$$

1.3.2 Another Gram-Schmidt Application

Show how to go from the basis

$$|I\rangle = \begin{bmatrix}3\\0\\0\end{bmatrix}; \quad |II\rangle = \begin{bmatrix}0\\1\\2\end{bmatrix}; \quad |III\rangle = \begin{bmatrix}0\\2\\5\end{bmatrix}$$

to the orthonormal basis

$$|1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}; \quad |2\rangle = \begin{bmatrix}0\\1/\sqrt{5}\\2\sqrt{5}\end{bmatrix}; \quad |3\rangle = \begin{bmatrix}0\\-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$$

To start, we can normalize $|I\rangle$,

$$|1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix} \tag{1}$$

$$|2'\rangle = |II\rangle - |1\rangle\langle 1|II\rangle = \begin{bmatrix}0\\1\\2\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix} \tag{2}$$

$$|2\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\end{bmatrix} \tag{3}$$

$$|3'\rangle = |III\rangle - |1\rangle\langle 1|III\rangle - |2\rangle\langle 2|III\rangle = \tag{4}$$

$$= \begin{bmatrix}0\\2\\5\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\2\\5\end{bmatrix} - \frac{1}{5}\begin{bmatrix}0\\1\\2\end{bmatrix}\begin{bmatrix}0 & 1 & 2\end{bmatrix}\begin{bmatrix}0\\2\\5\end{bmatrix} = \frac{1}{5}\begin{bmatrix}0\\-2\\1\end{bmatrix} \tag{5}$$

$$|3\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix}0\\-2\\1\end{bmatrix} \tag{6}$$

1.3.3 Schwarz Equality

When will this equality be satisfied? Does this agree with your experience with arrows?

$$\langle V|V\rangle \geq \frac{\langle W|V\rangle \langle V|W\rangle}{|W|^2}$$

$$\langle V|V\rangle = \frac{\langle W|V\rangle \langle V|W\rangle}{|W|^2} \tag{1}$$

Looking at the top side of the right hand side of the equation, we want $\langle W|V\rangle = |W|$, which happens when

$$|V\rangle = a\,|W\rangle \tag{2}$$

Substituting this back in, we get

$$\langle V|V\rangle = |V|^2 \tag{3}$$

This is what we expect from our experience with vectors. The dot product of two vectors is equal to the length squared. Another way of saying this is that the way to maximize a dot product is to multiply a vector by itself.

1.3.4 Triangle Inequality

Prove the triangle inequality starting with $|V+W|^2$. You must use $\Re \langle V|W\rangle \leq |\langle V|W\rangle|$ and the Schwarz inequality. Show that the final inequality becomes an equality only if $|V\rangle = a|W\rangle$ where a is a real positive scalar.

We can expand $|V+W|^2$,

$$|V+W|^2 = \langle V+W|V+W\rangle \tag{1}$$

$$= \langle V|V\rangle + \langle W|V\rangle + \langle V|W\rangle + \langle W|W\rangle = \langle V|V\rangle + 2\langle V|W\rangle + \langle W|W\rangle \tag{2}$$

Starting with the $\Re \langle V|W\rangle \leq |\langle V|W\rangle|$, we want this to look like above. First, we can use the Schwarz inequality, (Eq. 1.3.15),

$$\Re \langle V|W\rangle \leq |V||W| \tag{3}$$

We then add $|V|^2 + |W|^2$ to both sides,

$$\langle V|V\rangle + 2\Re \langle V|W\rangle + \langle W|W\rangle \leq |V|^2 + |W|^2 + 2|V||W| \tag{4}$$

Simplifying,

$$|V+W|^2 \leq (|V| + |W|)^2 \tag{5}$$

Which then gives the triangle inequality.

To get the triangle equality, let's plug in $|V\rangle = a|W\rangle$. Going back to Equation 2,

$$\langle V|V\rangle + \langle W|V\rangle + \langle V|W\rangle + \langle W|W\rangle = \langle V|V\rangle + a\langle V|V\rangle + a\langle V|V\rangle + a^2\langle V|V\rangle \tag{6}$$

$$= (a^2 + 2a + 1)|V|^2 \tag{7}$$

Comparing to $|V+W|^2$,

$$|V+W|^2 = (1+a)^2|V|^2 \tag{8}$$

1.4 Subspaces

1.4.1 Orthogonal Subspace

In a space $\mathbb{V}^n$, prove that the set of all vectors $\{|V_\perp^1\rangle, |V_\perp^2\rangle, ...\}$, orthogonal to any $|V\rangle \neq |0\rangle$, form a subspace $\mathbb{V}^{n-1}$.

We start with the vector space $\mathbb{V}^n$. Looking at the definition of a vector space, we can convince ourselves that a subspace of $\mathbb{V}^n$ will follow many of the same rules. Let's look at closure. If we take a vector $|V_\perp\rangle$ and add it to another orthogonal vector, we should get a resulting vector orthogonal to $|V\rangle$. There are going to be $n-1$ dimensions because n elements form a linear basis in $\mathbb{V}^n$. One vector is removed, resulting in $n-1$ basis vectors.

1.4.2 Vector Space Addition

Suppose $\mathbb{V}_1^{n_1}$ and $\mathbb{V}_2^{n_2}$ are two subspaces such that any element of $\mathbb{V}_1$ is orthogonal to any element of $\mathbb{V}_2$. Show that the dimensionality of $\mathbb{V}_1 \bigoplus \mathbb{V}_2$ is $n_1 + n_2$. (Hint: Theorem 4.)

The dimensions of each vector space is,

$$\begin{cases} \dim(V_1^{n_1}) = n_1 \\ \dim(V_2^{n_2}) = n_2 \end{cases} \tag{1}$$

Since all vectors between the two are orthogonal, there is no overlap, so the dimension of the resulting vector space is the dimension of each combined.

1.5 Linear Operators

1.6 Matrix Elements of Linear Operators

1.6.1 Operator Action

An operator Ω is given by the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

What is its action?

Let's act Ω on the usual three basis vectors,

$$\begin{cases} \Omega \left|1\right\rangle = \left|2\right\rangle \\ \Omega \left|2\right\rangle = \left|3\right\rangle \\ \Omega \left|3\right\rangle = \left|1\right\rangle \end{cases} \tag{1}$$

Which is the permutation operator.

1.6.2 Hermitian Operators

Given Ω and Λ are Hermitian what can you say about

$\Omega\Lambda$

$$(\Omega\Lambda)^\dagger = \Lambda^\dagger\Omega^\dagger = \Lambda\Omega \tag{1}$$

Hermitian if Ω and Λ commute.

$\Omega\Lambda + \Lambda\Omega$

$$(\Omega\Lambda + \Lambda\Omega)^\dagger = \Lambda^\dagger\Omega^\dagger + \Omega^\dagger\Lambda^\dagger \tag{2}$$

$$= \Lambda\Omega + \Omega\Lambda \tag{3}$$

Hermitian.

$[\Omega, \Lambda]$

$$(\Omega\Lambda - \Lambda\Omega)^\dagger \tag{4}$$

Following the same logic as above, anti-Hermitian.

$i[\Omega, \Lambda]$

$$(i\Omega\Lambda - i\Lambda\Omega)^\dagger = -i\Lambda\Omega + i\Omega\Lambda = i(\Omega\Lambda - \Lambda\Omega) \tag{5}$$

Hermitian.

1.6.3 Unitary Operators

Show that a product of unitary operators is unitary.

A restatement of this is to show,

$$UV(UV)^\dagger = I \tag{1}$$

Using (1.6.16),

$$UV(UV)^\dagger = UVV^\dagger U^\dagger \tag{2}$$

Because U and V are unitary, we can kill these terms,

$$= UIU^\dagger = UU^\dagger = I \tag{3}$$

1.6.4 Determinant

It is assumed that you know (1) what a determinant is, (2) that $\det \Omega^T = \det \Omega$ **(**T **denotes transpose), (3) that the determinant of a product of matrices is the product of the determinants. [If you do not, verify these properties for a two-dimensional case**

$$\Omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\det \Omega = (\alpha\delta - \beta\gamma)$**.] Prove that the determinant of a unitary matrix is a complex number of unit modulus.**

We want to show,

$$\det U = a + ib \tag{1}$$

where $|a + ib| = 1$. We take the determinant of a unitary matrix. We are then left with the identity matrix, which has determinant of one, thus proving unit modulus,

$$\det UU^\dagger = \det I = 1 \tag{2}$$

Because the determinant of a product of matrices is the product of the determinants,

$$\det UU^\dagger = \det U \det U^\dagger \tag{3}$$

$$= \det U * (\det U^T)^* \tag{4}$$

Since the determinant of a matrix transpose is equal to the determinant of the matrix,

$$= \det U (\det U)^* = 1 \tag{5}$$

The only way to satisfy this without complex numbers is if U is the identity matrix. Otherwise, $\det U = a + bi$.

1.6.5 Rotational Matrix Unitarity

Verify that $R(^1\!/_2\pi\hat{i})$ is unitary(orthogonal) by examining its matrix.

From (1.6.4),

$$R(^1\!/_2\pi\hat{i}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{1}$$

Nothing doing,

$$R(^1\!/_2\pi\hat{i})R^\dagger(^1\!/_2\pi\hat{i}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

We could also think about this as $R(^1\!/_2\pi\hat{i})$ being the counter-clockwise rotation around the x-axis and $R(^1\!/_2\pi\hat{i})$ as the clockwise rotation around the x-axis. If we take a vector and rotate it by $\pi/2$ and then rotate it by $\pi/2$ the other way, we get back to where we started.

1.6.6 Unitary Matrices

Verify that the following matrices are unitary. Verify that the determinant is of the form $\exp(i\theta)$ in each case. Are any of the above matrices Hermitian?

$$\frac{1}{2^{1/2}}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix}\begin{bmatrix}1 & -i\\ -i & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} \tag{1}$$

$$\det\Omega = \frac{1}{2}(1-i^2) = 1 = \exp(0) \tag{2}$$

This matrix is not Hermitian.

$$\frac{1}{2}\begin{bmatrix}1+i & 1-i\\ 1-i & 1+i\end{bmatrix}$$

$$\frac{1}{4}\begin{bmatrix}1+i & 1-i\\ 1-i & 1+i\end{bmatrix}\begin{bmatrix}1-i & 1+i\\ 1+i & 1-i\end{bmatrix} = \frac{1}{4}\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix} \tag{3}$$

$$\det\Lambda = \frac{1}{4}((1+i)^2-(1-i)^2) = i = \exp(i\pi/2) \tag{4}$$

This matrix is not Hermitian.

1.7 Active and Passive Transformations

1.7.1 Matrix Trace

The trace of a matrix is defined to be the sum of its diagonal matrix elements

$$\operatorname{Tr}\Omega = \sum_i \Omega_{ii}$$

Show that

$\operatorname{Tr}(\Omega\Lambda) = \operatorname{Tr}(\Lambda\Omega)$

$$\operatorname{Tr}(\Omega\Lambda) = \sum_i (\Omega\Lambda)_{ii} \tag{1}$$

$$= \sum_{ij} \Omega_{ij}\Lambda_{ji} \tag{2}$$

Since Ω_{ij} and Λ_{ji} are matrix elements and scalar, we can move those freely,

$$= \sum_{ij} \Lambda_{ji}\Omega_{ij} = \sum_j (\Lambda\Omega)_{jj} = \operatorname{Tr}(\Lambda\Omega) \tag{3}$$

$\operatorname{Tr}(\Omega\Lambda\theta) = \operatorname{Tr}(\Lambda\theta\Omega) = \operatorname{Tr}(\theta\Omega\Lambda)$. **(The permutations are cyclic)**

$$\operatorname{Tr}(\Omega\Lambda\theta) = \sum_{ijk} \Omega_{ij}\Lambda_{jk}\theta_{ki} \tag{4}$$

Moving these scalar quantities around and collapsing,

$$= \sum_{ijk} \Lambda_{jk}\theta_{ki}\Omega_{ij} = \sum_{jj} (\Lambda\theta\Omega)_{jj} = \operatorname{Tr}(\Lambda\theta\Omega) \tag{5}$$

We also have,

$$= \sum_{ijk} \theta_{ki}\Omega_{ij}\Lambda_{jk} = \sum_{kk} (\theta\Omega\Lambda)_{kk} = \operatorname{Tr}(\theta\Omega\Lambda) \tag{6}$$

The trace of an operator is unaffected by a unitary change of basis $|i\rangle \to U|i\rangle$. [Equivalently, show $\operatorname{Tr}\Omega = \operatorname{Tr}(U^\dagger \Omega U)$]

.

Starting with the unitary change, we can use the previous part and move things around cyclically,

$$\operatorname{Tr}(U^\dagger\Omega U) = \operatorname{Tr}(\Omega U U^\dagger) = \operatorname{Tr}\Omega \tag{7}$$

1.7.2 Determinant of a Unitary Change of Basis

Show that the determinant of a matrix is unaffected by a unitary change of basis. [Equivalently show $\det\Omega = \det(U^\dagger \Omega U)$]

Let's prove the equivalent statement. Since the determinant of a product of matrices is the product of the determinants,

$$\det(U^\dagger \Omega U) = \det U^\dagger \det\Omega \det U \tag{1}$$

Now, since each of these is just a scalar, we can move them around freely,

$$= \det\Omega \det U^\dagger \det U = \det\Omega \det(U^\dagger U) = \det\Omega \tag{2}$$

1.8 The Eigenvalue Problem

1.8.1 An Eigenproblem Example

Find the eigenvalues and normalized eigenvectors of the matrix,

$$\Omega = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

We'll work through one eigenvalue problem and the method of characteristic polynomial in detail. We start by taking the determinant,

$$\det(\Omega - \omega I) = \det \begin{bmatrix} 1-\omega & 3 & 1 \\ 0 & 2-\omega & 0 \\ 0 & 1 & 4-\omega \end{bmatrix} \tag{1}$$

$$= (1-\omega)(2-\omega)(4-\omega) = 0 \tag{2}$$

We can see that our eigenvalues are $\omega = 1, 2, 4$. We can then label each of our eigenkets so. For $|1\rangle$, we need to solve,

$$\begin{bmatrix} 1-1 & 3 & 1 \\ 0 & 2-1 & 0 \\ 0 & 1 & 4-1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

We get the system of equations,

$$\begin{cases} 3b + c \\ b = 0 \\ b + 3c = 0 \end{cases} \tag{4}$$

The normalized solution,

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

In the same manner, we can get the other normalized eigenkets,

$$|2\rangle = \frac{1}{\sqrt{30}} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \tag{6}$$

$$|4\rangle = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \tag{7}$$

Is the matrix Hermitian? Are the eigenvectors orthogonal?

The matrix is not Hermitian, and by Theorem 10, we do not expect(nor do we get) orthogonal eigenvectors.

1.8.2 Eigenvalue Problem

Consider the matrix

$$\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Is it Hermitian?

By observation, yes.

Find its eigenvalues and eigenvectors

Solving the characteristic equation, we get eigenvalues, $\omega = 0, \pm 1$. The corresponding eigenkets,

$$|0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad |1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \quad |-1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{1}$$

Verify that $U^\dagger \Omega U$ is diagonal, U being the matrix of eigenvectors of Ω

$$U = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \tag{2}$$

$$U^\dagger \Omega U = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3}$$

We get a diagonal matrix out with the ii indices corresponding to the eigenvalue of the eigenket that we used as the column in U.

1.8.3 Eigenvalue Problem

Consider the Hermitian matrix,

$$\Omega = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Show that $\omega_1 = \omega_2 = 1$**;** $\omega_3 = 2$

We are able to reduce to the characteristic equation,

$$\det\Omega = (1-\omega)(2-3\omega+\omega^2) = 0 \tag{1}$$

Which gives us the required eigenvalues.

Show that $|\omega = 2\rangle$ **is any vector of the form**

$$\frac{1}{(2a^2)^{1/2}}\begin{bmatrix} 0 \\ a \\ -a \end{bmatrix}$$

The easiest way to show this is by solving for $(\Omega - 2I)\,|\omega = 2\rangle = |0\rangle$, which gives us,

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{2}$$

We can then scale this by a.

Show that the $\omega = 1$ **eigenspace contains all vectors of the form below either by feeding** $\omega = 1$ **into the equations or by requiring that the** $\omega = 1$ **eigenspace by orthogonal to** $|\omega = 2\rangle$

$$\frac{1}{(b^2+2c^2)^{1/2}}\begin{bmatrix} b \\ c \\ c \end{bmatrix}$$

We solve,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

The easiest solution is,

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{4}$$

To make it orthogonal to $|\omega = 2\rangle$, we can vary the first element independently from the other two.

1.8.4 Eigenvalue Problem

An arbitrary $n \times n$ matrix need not have n eigenvectors. Consider as an example

$$\Omega = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

Show that $\omega_1 = \omega_2 = 3$

The characteristic equation,

$$(4 - \omega)(2 - \omega) + 1 = (\omega - 3)^2 = 0 \tag{1}$$

Which gives us our two eigenvalues.

By feeding in this value show that we get only one eigenvector of the form. We cannot finda another one that is LI

$$\frac{1}{(2a^2)^{1/2}} \begin{bmatrix} +a \\ -a \end{bmatrix}$$

We solve,

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

We get,

$$|3\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{3}$$

which can then be scaled to match the solution.

1.8.5 Eigenvalue Problem

Consider the matrix

$$\Omega = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Show that it is unitary

Nothing doing,

$$\Omega\Omega^\dagger = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = I \tag{1}$$

Show that its eigenvalues are $\exp(i\theta)$ and $\exp(-i\theta)$

We have the characteristic polynomial,

$$(\cos(\theta) - \omega)^2 + \sin^2(\theta) = \omega^2 - 2\omega\cos(\theta) + 1 = 0 \tag{2}$$

Using the quadratic formula,

$$\omega = \cos(\theta) \pm i\sin(\theta) = \exp(\pm i\theta) \tag{3}$$

Find the corresponding eigenvectors; show that they are orthogonal

Using the usual suspects,

$$|\exp(i\theta)\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{4}$$

$$|\exp(-i\theta)\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{5}$$

To show that they are orthogonal,

$$\langle\exp(-i\theta)|\exp(i\theta)\rangle = \frac{1}{2} \begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{2}(1 + i^2) = 0 \tag{6}$$

Verify that $U^\dagger\Omega U =$(diagonal matrix), where U is the matrix of eigenvectors of Ω

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{bmatrix} \tag{7}$$

$$U^\dagger\Omega U = \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \exp(i\theta) & 0 \\ 0 & \exp(-i\theta) \end{bmatrix} \tag{8}$$

1.8.6 Determinant and Eigenvalues

We have seen that the determinant of a matrix is unchanged under a unitary change of basis. Argue now that

$$\det \Omega = product\ of\ eigenvalues\ of\ \Omega = \prod_{i=1}^{n} \omega_i$$

for a Hermitian or unitary Ω

We know that performing a passive transformation on a Hermitian matrix $\Omega \to U\Omega U^\dagger$ returns a diagonal matrix whose elements are the eigenvalues of Ω.

$$\det(U^\dagger \Omega U) = \prod \omega_i \tag{1}$$

Finally, we know since the determinant of a matrix is unchanged under a unitary change of basis, the left side is equal to $\det \Omega$.

Using the invariance of the trace under the same transformation, show that

$$\operatorname{Tr} \Omega = \sum_{i=1}^{n} \omega_i$$

Using the same logic as above, we perform that unitary change of basis,

$$\operatorname{Tr} \Omega = \operatorname{Tr}(U^\dagger \Omega U) = \sum \omega_i \tag{2}$$

Since $U^\dagger \Omega U$ gives the diagonal matrix with elements equal to the eigenvalues.

1.8.7 Eigenvalue Problem, Hermitian Matrix

By using the results on the trace and determinant from the last problem, show that the eigenvalues of the matrix

$$\Omega = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

are 3 and -1. Verify this by explicit computation. Note that the Hermitian nature of the matrix is an essential ingredient.

We see

$$\begin{cases} \det \Omega = -3 \\ \operatorname{Tr} \Omega = 2 \end{cases} \tag{1}$$

We can convince ourselves that the only two values which satisfy this are $\omega = 3, -1$. Furthermore, we know the eigenvalues of a Hermitian operator must be real from Theorem 9.

The characteristic polynomial,

$$(1-\omega)^2 - 4 = \omega^2 - 2\omega - 3 = 0 \tag{2}$$

1.8.8 Eigenvalue of Hermitian Matrices

Consider Hermitian matrices M^1, M^2, M^3, M^4 that obey

$$M^i M^j + M^j M^i = 2\delta^{ij} I, \quad i, j = 1, ..., 4$$

Show that the eigenvalues of M^i are ± 1. (Hint: go to the eigenbasis of M^i, and use the equation for $i = j$

For $i = j$,

$$M^i M^i = I \tag{1}$$

The determinant is 1, and the trace is n. The only real numbers which would satisfy these two is if the eigenvalues are some combination of $+1$ and -1.

By considering the relation

$$M^i M^j = -M^j M^i; \quad for\ i \neq j$$

show that M^i are traceless. [Hint: $\text{Tr}(ACB) = \text{Tr}(CBA)$]

We multiply both sides by M^i and use $M^i M^i = I$,

$$M^i M^j M^i = -M^j M^i M^i \tag{2}$$

$$M^i M^j M^i = -M^j \tag{3}$$

Taking the trace of both sides,

$$\text{Tr}(M^i M^j M^i = -\text{Tr}\, M^j \tag{4}$$

On the left hand side, we use the cyclic nature of the trace to get the M^i next to each other and kill them,

$$\text{Tr}\, M^j = -\text{Tr}\, M^j \tag{5}$$

The only way for this to be true is if M^i is traceless.

Show that they cannot be odd-dimensional matrices

We showed that

$$\begin{cases} \det M^i = 1 \\ \text{Tr}\, M^i = 0 \end{cases} \tag{6}$$

We must have an equal number of $+1$ and -1 eigenvalues, which means that M^i is an even-dimensional matrix.

1.8.9 Moment of Inertia

A collection of masses m_α, located at $\vec{r}_\alpha$ and rotating with angular velocity $\vec{\omega}$ around a common axis has an angular momentum

$$\vec{l} = \sum_\alpha m_\alpha (\vec{r}_\alpha \times \vec{v}_\alpha)$$

where $\vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha$ is the velocity of m_α. by using the identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

show that each Cartesian component l_i of $\vec{l}$ is given by

$$l_i = \sum_j M_{ij} \omega_j$$

where

$$M_{ij} = \sum_\alpha m_\alpha [r_\alpha^2 \delta_{ij} - (\vec{r}_\alpha)_i (\vec{r}_\alpha)_j]$$

or in Dirac notation

$$|l\rangle = M\,|\omega\rangle$$

Writing out $\vec{l}$,

$$\vec{l} = \sum_\alpha m_\alpha (\vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)) \tag{1}$$

$$= \sum_\alpha m_\alpha (\vec{\omega}(\vec{r}_\alpha \cdot \vec{r}_\alpha) - \vec{r}_\alpha (\vec{r}_\alpha \cdot \vec{\omega})) \tag{2}$$

Looking at the individual elements,

$$l_i = \sum_\alpha \sum_j m_\alpha [r_\alpha^2 \delta_{ij} - (\vec{r}_\alpha)_i (\vec{r}_\alpha)_j] \omega_j \tag{3}$$

Will the angular momentum and angular velocity always be parallel?

Looking at the Dirac notation, we can convince ourselves that angular momentum and angular velocity will only be parallel for the eigenvalues of M (it looks like the statement of the eigenvalue problem).

Show that the moment of inertia matrix M_{ij} is Hermitian.

What we want to show is

$$M_{ij} = M_{ji} \tag{4}$$

$$m_\alpha (r_\alpha^2 \delta_{ij} - \vec{r}_{\alpha i} \vec{r}_{\alpha j}) = m_\alpha (r_\alpha^2 \delta_{ji} - \vec{r}_{\alpha j} \vec{r}_{\alpha i}) \tag{5}$$

We know this is true because

$$\begin{cases} \delta_{ij} = \delta_{ji} \\ \vec{r}_{\alpha i} \vec{r}_{\alpha j} = \vec{r}_{\alpha j} \vec{r}_{\alpha i} \end{cases} \tag{6}$$

since $\vec{r}_{\alpha i}$ is scalar.

Argue now that there exist three directions for $\vec{\omega}$ such that $\vec{l}$ and $\vec{\omega}$ will be parallel. How are these directions to be found?

These directions can be found by finding the eigenkets of M. There are three directions because M is a three-dimensional matrix and thus has three eigenkets.

Consider the moment of inertia matrix of a sphere. Due to the complete symmetry of the sphere, it is clear that every direction is its eigendirection for rotation. What does this say about the three eigenvalues of the matrix M

The three eigenvalues are degenerate since only one is needed to form a complete basis.

1.8.10 Simultaneous Diagonalization

By considering the commutator, show that the following Hermitian matrices may be simultaneously diagonalized. Find the eigenvectors common to both and verify that under a unitary transformation to this basis, both matrices are diagonalized.

$$\Omega = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}; \quad \Lambda = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Since Ω is degenerate and Λ is not, you must be prudent in deciding which matrix dictates the choice of basis.

We start by making sure Ω and Λ commute.

$$\Omega\Lambda = \Lambda\Omega = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} \tag{1}$$

Let's find the eigenvalues of Λ since it is not degenerate. The characteristic polynomial,

$$(2-\omega)(\omega^2 - 2\omega - 1) - (3-\omega) - (1-\omega) = (2-\omega)(\omega^2 - 2\omega - 3) = 0 \tag{2}$$

gives $\omega = 2, 3, -1$. We can then find the eigenkets of Λ,

$$|2\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}; \quad |3\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \quad |-1\rangle = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \tag{3}$$

To check, let's make sure these are eigenkets of Ω,

$$\begin{cases} \Omega\,|2\rangle = 0\,|2\rangle \\ \Omega\,|3\rangle = 2\,|3\rangle \\ \Omega\,|-1\rangle = 0\,|-1\rangle \end{cases} \tag{4}$$

and so we expect the eigenvalues of Ω to be 0,0,2.
Let's perform the unitary transformation,

$$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \tag{5}$$

$$U^\dagger \Omega U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

$$U^\dagger \Lambda U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

1.8.11 Coupled Mass

Consider the coupled mass problem discussed above

Given that the initial state is $|1\rangle$, in which the first mass is displaced by unity and the second is left alone, calculate $|1(t)\rangle$ by following the algorithm.

Our initial state is given by,

$$|x(0)\rangle = |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1}$$

The first step is to solve the eigenvalue problem of H. We don't expect the equations of motion to change if the initial conditions are changed,

$$\begin{cases} \omega_I = \left(\dfrac{k}{m}\right)^{1/2}; \quad |I\rangle = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}; \\ \\ \omega_{II} = \left(\dfrac{3k}{m}\right)^{1/2}; \quad |II\rangle = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{cases} \tag{2}$$

We then look at the propagator $U(t)$, which is given by (1.8.43). However, we want this in the standard $|1\rangle$, $|2\rangle$ basis, so we go back to (1.8.41),

$$\begin{gathered} U(t) = |I\rangle \langle I| \cos(\omega_I t) + |II\rangle \langle II| \cos(\omega_{II} t) = \\ \frac{1}{2}\begin{bmatrix} \cos(\omega_I t) + \cos(\omega_{II} t) & \cos(\omega_I t) - \cos(\omega_{II} t) \\ \cos(\omega_I t) - \cos(\omega_{II} t) & \cos(\omega_I t) + \cos(\omega_{II} t) \end{bmatrix} \end{gathered} \tag{3}$$

Applying this to our initial conditions,

$$|x(t)\rangle = U(t)\,|x(0)\rangle \tag{4}$$

$$|x(t)\rangle = \begin{bmatrix} \dfrac{1}{2}\left(\cos\left(\sqrt{\dfrac{k}{m}}t\right) + \cos\left(\sqrt{\dfrac{3k}{m}}t\right)\right) \\ \\ \dfrac{1}{2}\left(\cos\left(\sqrt{\dfrac{k}{m}}t\right) - \cos\left(\sqrt{\dfrac{3k}{m}}t\right)\right) \end{bmatrix} \tag{5}$$

Compare your result with that following from Eq.(1.8.39)

We can convince ourselves that we get the same result if we plug in the initial conditions (I'm not going to do it here because it is alot of typing).

1.8.12 Coupled Mass

Consider once again the problem discussed in the previous example.

Assuming that

$$|\ddot{x}\rangle = \Omega\,|x\rangle$$

has a solution

$$|x(t)\rangle = U(t)\,|x(0)\rangle$$

find the differential equation satisfied by $U(t)$. Use the fact that $|x(0)\rangle$ is arbitrary.
Inserting the second given equation into the first,

$$|\ddot{x}\rangle = \Omega U(t)\,|x(0)\rangle \tag{1}$$

$$\frac{d^2}{dt^2}\,|x\rangle = \Omega U(t)\,|x(0)\rangle \tag{2}$$

$$\frac{d^2U}{dt^2}\,|x(0)\rangle = \Omega U(t)\,|x(0)\rangle \tag{3}$$

Matching the sides and removing $|x(0)\rangle$, we get the differential equation,

$$\frac{d^2U}{dt^2} = \Omega U \tag{4}$$

$$\frac{d^2U(t)}{dt^2} - \Omega U(t) = 0 \tag{5}$$

Assuming(as is the case) that Ω and U can be simultaneously diagonalized, solve for the elements of the matrix U in this common basis and regain Eq. (1.8.43). Assume $|\dot{x}(0)\rangle = 0$

Let's work with both Ω and U diagonalized, with the elements being the eigenvalues. Looking at *Equation* 5,

$$\begin{bmatrix}\ddot{U}_{11} & 0\\ 0 & \ddot{U}_{22}\end{bmatrix} - \begin{bmatrix}-\omega_1^2 & 0\\ 0 & -\omega_2^2\end{bmatrix}\begin{bmatrix}U_{11} & 0\\ 0 & U_{22}\end{bmatrix} = 0 \tag{6}$$

We have the system of equations,

$$\begin{cases}\ddot{U}_{11} + \omega_1^2 U_{11} = 0\\ \ddot{U}_{22} + \omega_2^2 U_{22} = 0\end{cases} \tag{7}$$

We can convince ourselves(from remembering harmonic motion) that the solution will take the form,

$$\begin{cases}U_{11} = A_1\cos(\omega_1 t) + B_1\sin(\omega_1 t)\\ U_{22} = A_2\cos(\omega_2 t) + B_2\sin(\omega_2 t)\end{cases} \tag{8}$$

Using the condition, $|\dot{x}(0)\rangle = 0$ and substituting into,

$$\frac{dU(t)}{dt}\,|\dot{x}(0)\rangle = 0 \tag{9}$$

we then use $\dot{U}_{11}(0) = 0$ and $\dot{U}_{22}(0) = 0$ to show that $A_1 = A_2 = 1$ and $B_1 = B_2 = 0$,

$$U(t) = \begin{bmatrix} \cos(\omega_1 t) & 0 \\ 0 & \cos(\omega_2 t) \end{bmatrix} \tag{10}$$

1.9 Functions of Operators and Related Concepts

1.9.1 Hermitian Operator Power Series

We know that the series

$$f(x) = \sum_{n=0}^{\infty} x^n$$

may be equated to the function $f(x) = (1-x)^{-1}$ **if** $|x| < 1$**. By going to the eigenbasis, examine when the** q **number power series**

$$f(\Omega) = \sum_{n=0}^{\infty} \Omega^n$$

of a Hermitian operator Ω **may be identified with** $(1-\Omega)^{-1}$**.**

Going to the eigenbasis,

$$f(\Omega) = \begin{bmatrix} \sum_{n=0}^{\infty} \omega_1^n & & \\ & \ddots & \\ & & \sum_{n=0}^{\infty} \omega_m^n \end{bmatrix} \tag{1}$$

If we want $f(\Omega)$ to go to $(1-\Omega)^{-1}$, we need each element to go to $(1-x)^{-1}$, which converges if $|\omega_n| < 1$ for all eigenvalues ω_m.

1.9.2 Operator Analogy of Complex Numbers

If H is a Hermitian operator, show that $U = \exp(iH)$ is unitary. (Notice the analogy with c numbers: if θ is real, $u = \exp(i\theta)$ is a number of unit modulus.)

We can write,

$$U = \exp(iH) = \begin{bmatrix} \exp(i\omega_1) & & \\ & \ddots & \\ & & \exp(i\omega_n) \end{bmatrix} \tag{1}$$

$$U^\dagger U = \begin{bmatrix} \exp(-i\omega_1) & & \\ & \ddots & \\ & & \exp(-i\omega_n) \end{bmatrix} \begin{bmatrix} \exp(i\omega_1) & & \\ & \ddots & \\ & & \exp(i\omega_n) \end{bmatrix} = I \tag{2}$$

1.9.3 Determinant of a Function of Operators

For the case above, show that $\det U = \exp(i \operatorname{Tr} H)$

Writing out,

$$\det \begin{bmatrix} \exp(i\omega_1 & & \\ & \ddots & \\ & & \exp(i\omega_n) \end{bmatrix} = \exp(i\omega_1) \exp(i\omega_2) ... \exp(i\omega_n) \tag{1}$$

$$= \exp(i(\omega_1 + \omega_2 + ...\omega_n)) = \exp(i \operatorname{Tr} H) \tag{2}$$

1.10 Generalization to Infinite Dimensions

1.10.1 Delta Function

Show that $\delta(ax) = \delta(x)/|a|$**. [Consider** $\int \delta(ax)\ d(ax)$**. Remember that** $\delta(x) = \delta(-x)$**.]**

We start by saying,

$$\int \frac{\delta(x)}{|a|} g(x)\ dx = \frac{g(0)}{|a|} \tag{1}$$

We now want to show this is equivalent to what we want to prove. We consider what the problem wants us to consider,

$$\int \delta(ax) g(x)\ d(ax) \tag{2}$$

We then make the substitution, $y = ax$,

$$\int \delta(y) g(y/a) \frac{1}{a}\ dy = \frac{g(0)}{|a|} \tag{3}$$

Comparing solutions,

$$\frac{\partial(x)}{|a|} = \delta(ax) \tag{4}$$

1.10.2 Delta Function

Show that

$$\delta(f(x)) = \sum_i \frac{\delta(x_i - x)}{|df/dx_i|}$$

where x_i are the zeros of $f(x)$. Hint: Where does $\delta(f(x))$ blow up? Expand $f(x)$ near such points in a Taylor series, keeping the first nonzero term.

On the right side, let's use the results from problem 1.10.1,

$$\int \sum \frac{\delta(x_i - x)}{|df/dx_i|} g(x)\ dx = \sum \frac{g(x_i)}{|df/dx_i|} \tag{1}$$

On the left, if we expand $f(x)$ as a Taylor series,

$$f(x) = f(x_i) + (x - x_i)f'(x_i) = (x - x_i)f'(x_i) \tag{2}$$

If we use the results from problem 1.10.1,

$$\int \delta(f(x))g(x)\ dx = \int \delta((x - x_i)f'(x_i))g(x)\ dx \tag{3}$$

$$\frac{1}{f'(x)} g(x_i) = \sum \frac{g(x_i)}{|df/dx_i|} \tag{4}$$

1.10.3 Theta Function

Consider the theta function $\theta(x - x')$ which vanishes if $x - x'$ is negative and equals 1 if $x - x'$ is positive. Show that $\delta(x - x') = d/dx\ \theta(x - x')$.

We can do this by observation,

$$\frac{d}{dx}\theta(x - x') = 0; \quad if\ x \neq x' \tag{1}$$

Which is the definition of the delta function.

1.10.4 Normal Modes, Continuous Space

A string is displaced as follows at $t = 0$:

$$\begin{cases} \psi(x,0) = \dfrac{2xh}{L}, & 0 \le x \le \dfrac{L}{2} \\ = \dfrac{2h}{L}(L-x), & \dfrac{L}{2} \le x \le L \end{cases}$$

Show that

$$\psi(x,t) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{L}\right)\cos(\omega_m t)\cdot\left(\frac{8h}{\pi^2 m^2}\right)\sin\left(\frac{\pi m}{2}\right)$$

Let's start by calculating,

$$\langle m|\psi(0)\rangle = \left(\frac{2}{L}\right)^{1/2}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\psi(x,0)\ dx \tag{1}$$

$$= \left(\frac{2}{L}\right)^{1/2}\left[\int_0^{L/2}\frac{2h}{L}x\sin\left(\frac{m\pi x}{L}\right)\ dx + \int_{L/2}^{L}\frac{2h}{L}(L-x)\sin\left(\frac{m\pi x}{L}\right)\ dx\right] \tag{2}$$

Integrating by parts, let's look at the first term with $u = x$, $v' = \sin\left(\frac{m\pi x}{L}\right)$,

$$\frac{2h}{L}\int_0^{L/2} x\sin\left(\frac{m\pi x}{L}\right)\ dx = \frac{2h}{L}\left[-\frac{Lx}{m\pi}\cos\left(\frac{m\pi x}{L}\right)\Big|_0^{L/2} - \int_0^{L/2} -\frac{L}{m\pi}\cos\left(\frac{m\pi x}{L}\right)\ dx\right] \tag{3}$$

$$= \frac{2h}{L}\left[-\frac{L^2\cos(m\pi/2)}{2m\pi} + \frac{L^2}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right)\right] \tag{4}$$

We can do the same thing to the second term,

$$\int_{L/2}^{L}\frac{2h}{L}(L-x)\sin\left(\frac{m\pi x}{L}\right)\ dx = -\frac{2hL}{m\pi}\cos(m\pi) + \frac{2hL}{2m\pi}\cos\left(\frac{m\pi}{2}\right) + \frac{L^2}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right) \tag{5}$$

Substituting this into (1.10.59),

$$\psi(x,t) = \sum_{m=1}^{\infty}\left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{m\pi x}{L}\right)\cos(\omega_m t)\,\langle m|\psi(0)\rangle \tag{6}$$

$$= \sum_{m=1}^{\infty}\sin\left(\frac{m\pi x}{L}\right)\cos(\omega_m t)\left(\frac{8h}{m^2\pi^2}\right)\sin\left(\frac{\pi m}{2}\right) \tag{7}$$

Chapter 2

Review of Classical Mechanics

2.1 The Principle of Least Action and Lagrangian Mechanics

2.1.1 Lagrangian of the Harmonic Oscillator

Consider the following system, called a harmonic oscillator. The block has a mass m and lies on a frictionless surface. The spring has a force constant k. Write the Lagrangian and get the equations of motion.

Because we're only working in one-dimension, the kinetic energy,

$$T = \frac{1}{2} m\dot{x}^2 \tag{1}$$

And the potential energy,

$$V = \frac{1}{2} kx^2 \tag{2}$$

Thus the Lagrangian, $\mathscr{L} = T - V$,

$$\mathscr{L} = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \tag{3}$$

We can use (2.1.11) to get the equation of motion,

$$m\ddot{x} = -kx \tag{4}$$

$$\ddot{x} = -\frac{k}{m} x \tag{5}$$

2.1.2 Lagrangian of the Coupled-Mass Problem

Do the same for the coupled-mass problem discussed at the end of Section 1.8. Compare the equations of motion with Eqs. (1.8.24) and (1.8.25).

Referencing fig. (1.5), the kinetic energy is what we would expect for two masses moving in one dimension. For the potential energy, we need the spring potential from the wall to m_1 and the spring potential from the wall to m_2 as well as the spring between the two.

$$\begin{cases} T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) \\ \\ V = \frac{1}{2}k[x_1^2 + x_2^2 + (x_2 - x_1)^2] \end{cases} \tag{1}$$

$$\mathscr{L} = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k[x_1^2 + x_2^2 + (x_2 - x_1)^2] \tag{2}$$

The equations of motion,

$$\begin{cases} m\ddot{x}_1 = -2kx_1 + kx_2 \\ m\ddot{x}_2 = kx_1 - 2kx_2 \end{cases} \tag{3}$$

Which is the same equation of motion we get from Newton's laws (1.8.24),

$$\begin{cases} \ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2 \\ \\ \ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2 \end{cases} \tag{4}$$

2.1.3 Lagrangian, Polar Coordinates

A particle of mass m moves in three dimensions under a potential $V(r,\theta,\phi) = V(r)$. Write its $\mathscr{L}$ and find the equations of motion.

We start with the Lagrangian we're most familiar with,

$$\mathscr{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(r) \tag{1}$$

However, we have to convert the kinetic energy portion into polar coordinates,

$$\begin{cases} x = r\cos(\theta)\sin(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\phi) \end{cases} \tag{2}$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\phi}^2 + r^2\dot{\theta}^2\sin^2(\phi) \tag{3}$$

Thus, the Lagrangian and associated equations of motion,

$$\mathscr{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + r^2\dot{\theta}^2\sin^2(\phi)) - V(r) \tag{4}$$

$$\begin{cases} m\ddot{r} = mr\dot{\phi}^2 + mr\dot{\theta}^2\sin^2(\phi) - \dfrac{d}{dt}V'(r) \\ m\ddot{\phi} = mr^2\dot{\theta}^2\sin(\phi)\cos(\phi) \\ m\ddot{\theta} = 0 \end{cases} \tag{5}$$

2.2 The Electromagnetic Lagrangian

2.3 The Two-Body Problem

2.3.1 Changing to Center of Mass Frame

Derive Eq.(2.3.6) from (2.3.5) by changing variables.

We want to go from

$$\mathscr{L} = \frac{1}{2}m_1|\dot{\vec{r}}_1|^2 + \frac{1}{2}m_2|\dot{\vec{r}}_2|^2 - V(\vec{r}_1 - \vec{r}_2)$$

to

$$\mathscr{L} = \frac{1}{2}(m_1 + m_2)|\dot{\vec{r}}_{cm}|^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}|\dot{\vec{r}}|^2 - V(\vec{r})$$

Let's insert (2.3.3) and (2.3.4) into (2.3.5),

$$\mathscr{L} = \frac{1}{2}m_1\left(\dot{r}_{cm}^2 + \frac{2m_2\dot{r}r_{cm}}{m_1 + m_2} + \frac{m_2^2\dot{r}^2}{(m_1 + m_2)^2}\right) + \frac{1}{2}m_2\left(\dot{r}_{cm}^2 - \frac{2m_1\dot{r}r_{cm}}{m_1 + m_2} + \frac{m_1^2\dot{r}^2}{(m_1 + m_2)^2}\right) - V(\vec{r}) \quad (1)$$

which when reduced, gives us (2.3.6).

2.4 How Smart Is a Particle

2.5 The Hamiltonian Formalism

2.5.1 Kinetic Energy

Show that if $T = \sum_i \sum_j T_{ij}(q)\dot{q}_i\dot{q}_j$, where $\dot{q}$'s are generalized velocities, $\sum_i p_i\dot{q}_i = 2T$.

We look at (2.5.1). Since we assume the potential is not dependant on $\dot{q}$,

$$p_i = \frac{\partial \mathscr{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} \tag{1}$$

Let's look at the derivative of T according to $\dot{q}_i$,

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_i T_{is}(q)\dot{q}_i + \sum_j T_{sj}(q)\dot{q}_j \tag{2}$$

We then insert this,

$$\sum_i p_i\dot{q}_i = \sum_i \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i = \sum_{is} T_{is}(q)\dot{q}_i\dot{q}_s + \sum_{sj} T_{sj}(q)\dot{q}_s\dot{j} = 2T \tag{3}$$

2.5.2 Trajectory of the Oscillator

Using the conservation of energy, show that the trajectories in phase space for the oscillator are ellipses of the form $(x/a)^2 + (p/b)^2 = 1$, where $a^2 = 2E/k$ and $b^2 = 2mE$

The Hamiltonian is given by the total energy of the system,

$$\mathscr{H} = E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{1}$$

Then using (2.5.16) and (2.5.17),

$$= \frac{1}{2}m\left(\frac{p}{m}\right)^2 + \frac{1}{2}kx^2 = \frac{x^2k}{2} + \frac{p^2}{2m} \tag{2}$$

$$\frac{x^2k}{2E} + \frac{p^2}{2mE} = 1 \tag{3}$$

which gives us an ellipses with $a^2 = 2E/k$ and $b^2 = 2mE$.

2.5.3 Hamiltonian of the Coupled-Mass Problem

Solve Exercise 2.1.2 using the Hamiltonian formalism

We can write the total energy,

$$\mathscr{E} = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}k(x_1 + x_2^2 + (x_2 - x_1)^2) \tag{1}$$

Because we want this in terms of momentum and position (p and q), we can use (2.5.16),

$$\mathscr{H} = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}k(x_1^2 + x_2^2 + (x_2 - x_1)^2) \tag{2}$$

We can easily see,

$$\begin{cases} \dfrac{p_1}{m} = \dot{x}_1 \\ \\ \dfrac{p_2}{m} = \dot{x}_2 \end{cases} \tag{3}$$

Let's then use Hamilton's canonical equations (2.5.12) to get the equations of motion,

$$-\frac{\partial \mathscr{H}}{\partial x_1} = -2k_1 + kx_2 = m\ddot{x}_1 \tag{4}$$

$$-\frac{\partial \mathscr{H}}{\partial x_2} = kx_1 - 2kx_2 = m\ddot{x}_2 \tag{5}$$

2.5.4 Hamiltonian of the Center of Mass Problem

Show that $\mathscr{H}$ corresponding to $\mathscr{L}$ in Eq.(2.3.6) is $\mathscr{H} = |\vec{p}_{cm}|^2/2M + |\vec{p}|^2/2\mu + V(\vec{r})$, where M is the total mass, μ is the reduced mass, $\vec{p}_{cm}$ and $\vec{p}$ are the momenta conjugate to $\vec{r}_{cm}$ and $\vec{r}$, respectively.

Equation (2.3.6),

$$\mathscr{L} = \frac{1}{2}(m_1 + m_2)|\dot{\vec{r}}_{cm}|^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}|\dot{\vec{r}}|^2 - V(\vec{r})$$

Let's use (2.5.8),

$$\mathscr{H} = p\dot{r} + p_{cm}\dot{r}_{cm} - \mathscr{L} \tag{1}$$

We then use (2.5.1),

$$\begin{cases} p = \mu\dot{r} \\ p_{cm} = M\dot{r}_{cm} \end{cases} \tag{2}$$

Inserting this and making the changes to the Lagrangian,

$$\mathscr{H} = \frac{p^2}{\mu} + \frac{p_{cm}^2}{M} - \frac{p_{cm}^2}{2M} - \frac{p^2}{2\mu} + V(\vec{r}) \tag{3}$$

$$= \frac{p_{cm}^2}{2M} + \frac{p^2}{2\mu} + V(\vec{r}) \tag{4}$$

2.6 The Electromagnetic Force in the Hamiltonian Scheme

2.7 Cyclic Coordinates, Poisson Brackets, and Canonical Transformations

2.7.1 Poisson Brackets

Show that... Note the similarity between the above and Eqs.(1.5.10) and (1.5.11) for commutators

$\{\omega, \lambda\} = -\{\lambda, \omega\}$

For all of these, let's work over a single indices,

$$\{\omega, \lambda\} = \frac{\partial \omega}{\partial q}\frac{\partial \lambda}{\partial p} - \frac{\partial \omega}{\partial p}\frac{\partial \lambda}{\partial q} \tag{1}$$

$$= -\left(\frac{\partial \omega}{\partial p}\frac{\partial \lambda}{\partial q} - \frac{\partial \omega}{\partial q}\frac{\partial \lambda}{\partial p}\right) = -\{\lambda, \omega\} \tag{2}$$

$\{\omega, \lambda + \sigma\} = \{\omega, \lambda\} + \{\omega, \sigma\}$

$$\{\omega, \lambda + \sigma\} = \frac{\partial \omega}{\partial q}\frac{\partial (\lambda + \sigma)}{\partial p} - \frac{\partial \omega}{\partial p}\frac{\partial (\lambda + \sigma)}{\partial q} \tag{3}$$

$$= \frac{\partial \omega}{\partial q}\left(\frac{\partial \lambda}{\partial p} + \frac{\partial \sigma}{\partial p}\right) - \frac{\partial \omega}{\partial p}\left(\frac{\partial \lambda}{\partial q} + \frac{\partial \sigma}{\partial q}\right) \tag{4}$$

$$= \frac{\partial \omega}{\partial q}\frac{\partial \lambda}{\partial p} - \frac{\partial \omega}{\partial p}\frac{\partial \lambda}{\partial q} + \frac{\partial \omega}{\partial q}\frac{\partial \sigma}{\partial p} - \frac{\partial \omega}{\partial p}\frac{\partial \sigma}{\partial q} = \{\omega, \lambda\} + \{\omega, \sigma\} \tag{5}$$

$\{\omega, \lambda\sigma\} = \{\omega, \lambda\}\sigma + \lambda\{\omega, \sigma\}$

$$\frac{\partial \omega}{\partial q}\frac{\partial (\lambda\sigma)}{\partial p} - \frac{\partial \omega}{\partial p}\frac{\partial (\lambda\sigma)}{\partial q} = \frac{\partial \omega}{\partial q}\left(\lambda\frac{\partial \sigma}{\partial p} + \frac{\partial \lambda}{\partial p}\sigma\right) - \frac{\partial \omega}{\partial p}\left(\lambda\frac{\partial \sigma}{\partial q} - \frac{\partial \lambda}{\partial q}\sigma\right) \tag{6}$$

$$= \lambda\left(\frac{\partial \omega}{\partial q}\frac{\partial \sigma}{\partial p} - \frac{\partial \omega}{\partial p}\frac{\partial \sigma}{\partial q}\right) + \left(\frac{\partial \omega}{\partial q}\frac{\partial \lambda}{\partial p} - \frac{\partial \omega}{\partial p}\frac{\partial \lambda}{\partial q}\right)\sigma = \{\omega, \lambda\}\sigma + \lambda\{\omega, \sigma\} \tag{7}$$

2.7.2 Canonical Poisson Bracket Relations

Verify Eqs.(2.7.4) and (2.7.5)

Start with $\{q_i, q_j\} = \{p_i, p_j\} = 0$,

$$\{q_i, q_j\} = \frac{\partial q_i}{\partial q_i}\frac{\partial q_j}{\partial p_i} - \frac{\partial q_i}{\partial p_i}\frac{\partial q_j}{\partial q_i} + \frac{\partial q_i}{\partial q_j}\frac{\partial q_j}{\partial p_j} - \frac{\partial q_i}{\partial p_j}\frac{\partial q_j}{\partial q_i} \tag{1}$$

$$= 1 \cdot 0 - 0 \cdot \delta_{ij} + \delta_{ij} \cdot 0 - 0 \cdot 1 = 0 \tag{2}$$

$$\{p_i, p_j\} = \frac{\partial p_i}{\partial q_i}\frac{\partial p_j}{\partial p_i} - \frac{\partial p_i}{\partial p_i}\frac{\partial p_j}{\partial q_i} + \frac{\partial p_i}{\partial q_j}\frac{\partial p_j}{\partial p_j} - \frac{\partial p_i}{\partial p_j}\frac{\partial p_j}{\partial q_j} \tag{3}$$

$$= 0 \cdot \delta_{ij} - 1 \cdot 0 + 0 \cdot 1 - \delta_{ij} \cdot 0 = 0 \tag{4}$$

Let's look at $\{q_i, p_j\} = \delta_{ij}$,

$$\sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial p_k}\right) \tag{5}$$

$$= \delta_{ik}\delta_{kj} = \delta ij \tag{6}$$

Show $\dot{q}_i = \{q_i, \mathscr{H}\}$. We can use (2.5.12),

$$\frac{\partial q_i}{\partial q_i}\frac{\partial \mathscr{H}}{\partial p_i} - \frac{\partial q_i}{\partial p_i}\frac{\partial \mathscr{H}}{\partial q_i} = \dot{q}_i \tag{7}$$

$$\frac{\partial p_i}{\partial q_i}\frac{\partial \mathscr{H}}{\partial p_i} - \frac{\partial p_i}{\partial p_i}\frac{\partial \mathscr{H}}{\partial q_i} = \dot{p}_i \tag{8}$$

Consider a problem in two dimensions given by $\mathscr{H} = p_x^2 + p_y^2 + ax^2 + by^2$. Argue that if $a = b$, $\{l_z, \mathscr{H}\}$ must vanish. Verify by explicit computation.

For reference, l_z, or a rotation around the z-axis,

$$l_z = p_x y - p_y x \tag{9}$$

If $a = b$, the Hamiltonian becomes,

$$\mathscr{H} = p_x^2 + p_y^2 + ax^2 + ay^2 \tag{10}$$

which is symmetric about the z-axis.

$$\{l_z, \mathscr{H}\} = -p_y \cdot 2p_x - y \cdot 2ax + p_x \cdot 2p_y - (-x) \cdot 2ay = 0 \tag{11}$$

2.7.3 Canonical Poisson Bracket Proofs

Fill in the missing steps leading to Eq. (2.7.18) starting from Eq. (2.7.14).

We start with (2.7.14),

$$\dot{\bar{q}}_j = \sum_i \left(\frac{\partial \bar{q}_j}{\partial q_i} \frac{\partial \mathscr{H}}{\partial p_i} - \frac{\partial \bar{q}_j}{\partial p_i} \frac{\partial \mathscr{H}}{\partial q_i} \right) \tag{1}$$

We can then follow the text and jump to (2.7.16),

$$= \sum_k \left(\frac{\partial \mathscr{H}}{\partial \bar{q}_k} \{\bar{q}_j, \bar{q}_k\} + \frac{\partial \mathscr{H}}{\partial \bar{p}_k} \{\bar{q}_j, \bar{p}_k\} \right) \tag{2}$$

We compare this to (2.7.10), which implies,

$$\begin{cases} \{\bar{q}_j, \bar{q}_k\} = 0 \\ \{\bar{q}_j, \bar{p}_k\} = \delta_{jk} \end{cases} \tag{3}$$

Similarly, if we compare (2.7.17) to (2.7.10),

$$-\frac{\partial \mathscr{H}}{\partial \bar{q}_j} = \sum_k \left(\frac{\partial \mathscr{H}}{\partial \bar{q}_k} \{\bar{p}_j, \bar{q}_k\} + \frac{\partial \mathscr{H}}{\partial \bar{p}_k} \{\bar{p}_j, \bar{p}_k\} \right) \tag{4}$$

which implies,

$$\begin{cases} \{\bar{p}_j, \bar{q}_k\} = -\delta_{jk} \\ \{\bar{p}_j, \bar{p}_k\} = 0 \end{cases} \tag{5}$$

2.7.4 Canonical Transformation Example

Verify that the change to a rotated frame

$$\bar{x} = x\cos(\theta) - y\sin(\theta)$$

$$\bar{y} = x\sin(\theta) + y\cos(\theta)$$

$$\bar{p}_x = p_x\cos(\theta) - p_y\sin(\theta)$$

$$\bar{p}_y = p_x\sin(\theta) + p_y\cos(\theta)$$

is a canonical transformation.

$$\{\bar{x}, \bar{y}\} = 0 \quad (1)$$

$$\{\bar{p}_x, \bar{p}_y\} = 0 \quad (2)$$

$$\{\bar{x}, \bar{p}_x\} = \cos^2(\theta) + \sin^2(\theta) = 1 \quad (3)$$

$$\{\bar{y}, \bar{p}_y\} = \sin^2(\theta) + \cos^2(\theta) = 1 \quad (4)$$

$$\{\bar{x}, \bar{p}_y\} = \cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta) = 0 \quad (5)$$

$$\{\bar{y}, \bar{p}_x\} = \sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta) = 0 \quad (6)$$

2.7.5 Canonical Transformation Example

Show that the polar variables $\rho = (x^2 + y^2)^{1/2}$**,** $\phi = \tan^{-1}(y/x)$

$$p_\rho = \hat{e}_\rho \cdot \vec{p} = \frac{xp_x + yp_y}{(x^2 + y^2)^{1/2}}, \quad p_\phi = xp_y - yp_x (= l_z)$$

are canonical. ($\hat{e}_\rho$ **is the unit vector in the radial direction.)**

Since there are no momentum terms,

$$\{\rho, \phi\} = 0 \tag{1}$$

$$\{p_\rho, p_\phi\} = 0 \tag{2}$$

$$\{\rho, p_\rho\} = \frac{(x^2 + y^2)(x^2 + y^2)^{1/2}}{(x^2 + y^2)^{1/2}(x^2 + y^2)} = 1 \tag{3}$$

$$\{\phi, p_\phi\} = \frac{x^2 + y^2}{x^2 + y^2} = 1 \tag{4}$$

$$\{\rho, p_\phi\} = \{\phi, p_\rho\} = 0 \tag{5}$$

2.7.6 Canonical Transformation Example

Verify that the change from the variables $\vec{r}_1$, $\vec{r}_2$, $\vec{p}_1$, $\vec{p}_2$ to $\vec{r}_{cm}$, $\vec{p}_{cm}$, $\vec{r}$, $\vec{p}$ is a canonical transformation. (See Exercise 2.5.4)

For reference, removing arrows because I don't want to type them out,

$$r = r_1 - r_2, \quad p = \frac{m_1 m_2}{m_1 + m_2}(\dot{r}_1 - \dot{r}_2) = \frac{m_2 p_1 - m_1 p_2}{m_1 + m_2} \tag{1}$$

$$r_{cm} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}, \quad p_{cm} = (m_1 + m_2)\dot{r}_{cm} = p_1 + p_2 \tag{2}$$

Because they don't have any momentum (or position terms),

$$\{r, r_{cm}\} = 0 \tag{3}$$

$$\{p, p_{cm}\} = 0 \tag{4}$$

$$\{r, p\} = \frac{m_1 + m_2}{m_1 + m_2} = 1 \tag{5}$$

$$\{r, p_{cm}\} = 1 - 1 = 0 \tag{6}$$

$$\{r_{cm}, p\} = \frac{m_1}{m_1 + m_2}\frac{m_2}{m_1 + m_2} + \frac{m_2}{m_1 + m_2}\frac{-m_1}{m_1 + m_2} = 0 \tag{7}$$

$$\{r_{cm}, p_{cm}\} = \frac{m_1 + m_2}{m_1 + m_2} = 1 \tag{8}$$

2.7.7 Canonical Transformation Example

Verify that

$$\bar{q} = \ln(q^{-1}\sin(p))$$

$$\bar{p} = q\cot(p)$$

is a canonical transformation

Because we only have one set of variables, we already know $\{\bar{q},\bar{q}\} = \{\bar{p},\bar{p}\} = 0$, so we only need to show,

$$\{\bar{q},\bar{p}\} = \frac{1-\cos^2(p)}{\sin^2(p)} = 1 \tag{1}$$

2.7.8 Canonical Transformation Momentum

We would like to derive here Eq.(2.7.9), which gives the transformation of the momenta under a coordinate transformation in configuration space:

$$q_i \to \bar{q}_i(q_1, ..., q_n)$$

Argue that if we invert the above equation to get $q = q(\bar{q})$, we can derive the following counterpart of Eq. (2.7.7):

$$\dot{q}_i = \sum_j \frac{\partial q_i}{\partial \bar{q}_j} \dot{\bar{q}}_j$$

We start by looking at Eq.(2.7.7),

$$\dot{\bar{q}}_i = \sum_j \left(\frac{\partial \bar{q}_i}{\partial q_j} \right) \dot{q}_j$$

$$\dot{\bar{q}}_i \left(\frac{\partial \bar{q}_i}{\partial q_j} \right)^{-1} = \dot{q}_j \tag{1}$$

$$\dot{q}_i = \sum_j \frac{\partial q_i}{\partial \bar{q}_j} \dot{\bar{q}}_j \tag{2}$$

Show from the above that

$$\left(\frac{\partial \dot{q}_i}{\partial \dot{\bar{q}}_j} \right)_{\bar{q}} = \frac{\partial q_i}{\partial \bar{q}_j}$$

Inserting q_i,

$$\left(\frac{\partial \dot{q}_i}{\partial \dot{\bar{q}}_j} \right)_{\bar{q}} = \frac{\partial}{\partial \dot{\bar{q}}_j} \left(\sum_j \frac{\partial q_i}{\partial \bar{q}_j} \dot{\bar{q}}_j \right) \tag{3}$$

Taking the derivative is fairly straightforward,

$$= \sum_j \frac{\partial q_i}{\partial \bar{q}_j} \tag{4}$$

Now calculate

$$\bar{p}_i = \left[\frac{\partial \mathscr{L}(\bar{q}, \dot{\bar{q}})}{\partial \dot{\bar{q}}_i}\right]_{\bar{q}} = \left[\frac{\partial \mathscr{L}(q, \dot{q})}{\partial \dot{\bar{q}}_i}\right]_{\bar{q}}$$

Use the chain rule and the fact that $q = q(\bar{q})$ and not $q(\bar{q}, \dot{\bar{q}})$ to derive Eq.(2.7.9).
Using the chain rule,

$$\bar{p}_i = \frac{\partial \mathscr{L}(\bar{q}, \dot{\bar{q}}}{\partial \dot{\bar{q}}_i} = \frac{\partial \mathscr{L}}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial \dot{\bar{q}}_i} \tag{5}$$

Now using the result from (2),

$$= \frac{\partial \mathscr{L}}{\partial \dot{q}} \frac{\partial q}{\partial \bar{q}} \tag{6}$$

And (2.5.1),

$$= p \frac{\partial q}{\partial \bar{q}} = \sum_j \left(\frac{\partial q_j}{\partial \bar{q}_i}\right) p_j \tag{7}$$

Verify, by calculating the PB in Eq. (2.7.18), that the point transformation is canonical

$$\{\bar{q}, \bar{p}\} = \frac{\partial \bar{q}}{\partial q} \frac{\partial \bar{p}}{\partial p} - \frac{\partial \bar{q}}{\partial p} \frac{\partial \bar{p}}{\partial q} = \frac{\partial \bar{q}}{\partial q} \frac{\partial q}{\partial \bar{q}} - 0 = 1 \tag{8}$$

2.7.9 Invariant Poisson Bracket

Verify Eq.(2.7.19) by direct computation. Use the chain rule to go from q, p derivatives to $\bar{q}$, $\bar{p}$ derivatives. Collect terms that represent PB of the latter.

We want to show,

$$\{\omega, \sigma\}_{q,p} = \{\omega, \sigma\}_{\bar{q},\bar{p}}$$

We start by writing out the left side,

$$\{\omega, \sigma\}_{q,p} = \frac{\partial \omega}{\partial q}\frac{\partial \sigma}{\partial p} - \frac{\partial \omega}{\partial p}\frac{\partial \sigma}{\partial q} \tag{1}$$

Using the chain rule,

$$= \frac{\partial \omega}{\partial \bar{q}}\frac{\partial \sigma}{\partial \bar{p}}\frac{\partial \bar{q}}{\partial q}\frac{\partial \bar{p}}{\partial p} - \frac{\partial \omega}{\partial \bar{p}}\frac{\partial \sigma}{\partial \bar{q}}\frac{\partial \bar{p}}{\partial p}\frac{\partial \bar{q}}{\partial q} = \{\omega, \sigma\}_{\bar{q},\bar{p}} \cdot \{\bar{q}, \bar{p}\} \tag{2}$$

which using (2.7.18), gives us what we want,

$$= \{\omega, \sigma\}_{\bar{q},\bar{p}} \tag{3}$$

2.8 Symmetries and their Consequences

2.8.1 Infinitesimal Translation

Show that $p = p_1 + p_2$, the total momentum, is the generator of infinitesimal translations for a two-particle systems.

Looking at (2.8.3), let's insert p as the generator,

$$\bar{x} = x + \epsilon \frac{\partial p}{\partial p_x} = x + \epsilon \tag{1}$$

$$\bar{y} = y + \epsilon \frac{\partial p}{\partial p_y} = y + \epsilon \tag{2}$$

$$\bar{p}_x = p_x - \epsilon \frac{\partial p}{\partial x} = p_x \tag{3}$$

$$\bar{p}_y = p_y - \epsilon \frac{\partial p}{\partial y} = p_y \tag{4}$$

2.8.2 Infinitesimal Transformation and Canonical Transformation

Verify that the infinitesimal transformation generated by any dynamical variable g is a canonical transformation. (Hint: Work, as usual, to first order in ϵ)

Let's insert (2.8.3) into (2.7.18),

$$\{\bar{q}, \bar{p}\} = \left(\frac{\partial q}{\partial q} + \epsilon \frac{\partial^2 g}{\partial p \partial q}\right)\left(\frac{\partial p}{\partial p} - \epsilon \frac{\partial^2 g}{\partial p \partial q}\right) - \left(\frac{\partial q}{\partial p} + \epsilon \frac{\partial^2 g}{\partial^2 p}\right)\left(\frac{\partial p}{\partial q} - \epsilon \frac{\partial^2 g}{\partial q^2}\right) = 1 \tag{1}$$

2.8.3 Non-canonical Transformation

Consider

$$\mathscr{H} = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2)$$

whose invariance under the rotation of the coordinates and momenta leads to the conservation of l_z. But $\mathscr{H}$ is also invariant under the rotation of just the coordinates. Verify that this is a noncanonical transformation. Convince yourself that in this case it is not possible to write $\delta\mathscr{H}$ as $\epsilon\{\mathscr{H}, g\}$ for any g, i.e., that no conservation law follows.

Under rotation of just the coordinates,

$$\begin{cases} \bar{x} = x\cos(\theta) - y\sin(\theta) \\ \bar{y} = x\sin(\theta) + y\cos(\theta) \end{cases} \tag{1}$$

$$\begin{cases} \bar{p}_x = p_x \\ \bar{p}_y = p_y \end{cases} \tag{2}$$

Let's now look at how they behave with Poisson Brackets,

$$\{\bar{x}, \bar{p}_x\} = \cos(\theta) \tag{3}$$

$$\{\bar{y}, \bar{p}_y\} = \cos(\theta) \tag{4}$$

$$\{\bar{x}, \bar{p}_y\} = -\sin(\theta) \tag{5}$$

$$\{\bar{y}, \bar{p}_x\} = \sin(\theta) \tag{6}$$

Which do not follow the canonical transformation rules. Because of this, $\{g, \mathscr{H}\}$ is also not conserved.

2.8.4 Generator of Infinitesimal Rotation in Phase Space

Consider $\mathscr{H} = {}^1\!/\!{}_2 p^2 + {}^1\!/\!{}_2 x^2$, which is invariant under infinitesimal rotations in phase space(the $x - p$ plane). Find the generator of this transformation(after verifying that is is canonical). (You could have guessed the answer based on Exercise 2.5.2).

Here, our transformation in phase space looks like,

$$\begin{cases} \bar{x} = x + \epsilon p \\ \bar{p} = p - \epsilon x \end{cases} \tag{1}$$

To find the generator, we want something that satisfies,

$$\begin{cases} \dfrac{\partial g}{\partial p} = p \\ \\ \dfrac{\partial g}{\partial x} = x \end{cases} \tag{2}$$

The solution to this is,

$$g = \frac{1}{2}(p^2 + x^2) \tag{3}$$

We can easily convince ourselves that $\{g, \mathscr{H}\} = 0$.

2.8.5 Noncanonical Transformation

Why is it that a noncanonical transformation that leaves $\mathscr{H}$ invariant does not map a solution into another? Or, in view of the discussions on consequence II, why is it that an experiment and its transformed version do not give the same result when the transformation that leaves $\mathscr{H}$ invariant is not canonical? It is best to consider an example. Consider the potential given in Exercise 2.8.3. Suppose I release a particle at $(x = a, y = 0)$ with $(p_x = b, p_y = 0)$ and you release one in the transformed state in which $(x = 0, y = a)$ and $(p_x = b, p_y = 0)$, i.e., you rotate the coordinates but not the momenta. This is a noncanonical transformation that leaves $\mathscr{H}$ invariant. Convince yourself that at later times the states of the two particles are not related by the same transformation. Try to understand what goes wrong in the general case.

In the transformed case,

$$\begin{cases} \bar{x} = x\cos(\theta) - y\sin(\theta) \\ \bar{y} = x\sin(\theta) - x\cos(\theta) \end{cases} \tag{1}$$

If we then look at how these equations evolve in time,

$$\begin{cases} \ddot{x} = m\omega^2 x \\ \ddot{y} = m\omega^2 y \end{cases} \tag{2}$$

$$\begin{cases} \ddot{\bar{x}} = m\omega^2(x\cos(\theta) - y\sin(\theta)) \\ \ddot{\bar{y}} = m\omega^2(x\sin(\theta) + y\cos(\theta)) \end{cases} \tag{3}$$

I think this question is asking why we care about canonical transformations. Canonical transformations tell us the equations of motion don't care about how coordinates are defined in space. In this example, because the transformation is not canonical, the starting position does matter, which means that even though the Hamiltonian looks the same, it acts differently on the two cases.

2.8.6 Action of the Classical Path

Show that $\partial S_{cl}/\partial x_f = p(t_f)$**.**

We start by multiplying by unity,

$$\frac{\partial S_{cl}}{\partial x_f}\frac{\partial t_f}{\partial t_f} = \frac{\partial S_{cl}}{\partial t_f}\frac{\partial t_f}{\partial x_f} \tag{1}$$

Then using (2.8.18),

$$= -\mathscr{H}(t_f)\frac{\partial t_f}{\partial x_f} \tag{2}$$

Using (2.5.12),

$$= \dot{p}(x_f)\partial t_f = p(t_f) \tag{3}$$

2.8.7 Harmonic Oscillator, Classical Path

Consider the harmonic oscillator, for which the general solution is

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

Express the energy in terms of A and B and note that it does not depend on time. Now choose A and B such that $x(0) = x_1$ and $x(T) = x_2$. Write down the energy in terms of x_1, x_2, and T. Show that the action for the trajectory connecting x_1 and x_2 is

$$S_{cl}(x_1, x_2, T) = \frac{m\omega}{2\sin(\omega T)}[(x_1^2 + x_2^2)\cos(\omega T) - 2x_1x_2]$$

Verify that $\partial S_{cl}/\partial T = -E$

We know the energy of the harmonic oscillator is,

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2, \quad k = m\omega^2 \tag{1}$$

Inserting $x(t)$ into this,

$$E = \frac{1}{2}m\omega^2(A^2 + B^2) \tag{2}$$

At $t = 0$,

$$x(0) = A = x_1 \tag{3}$$

At $t = T$,

$$x(T) = A\cos(\omega T) + B\sin(\omega T) = x_2 \tag{4}$$

$$B = \frac{x_2 - x_1\cos(\omega T)}{\sin(\omega T)} \tag{5}$$

$$E = \frac{1}{2}m\omega^2\left(x_1^2 + \left(\frac{x_2 - x_1\cos(\omega T)}{\sin(\omega T)}\right)\right) \tag{6}$$

To find the action of the classical path, we need the Lagrangian,

$$\mathscr{L} = \frac{1}{2}m\omega^2[A^2(\sin^2(\omega t) - \cos^2(\omega t)) + B^2(\cos^2(\omega t) - \sin^2(\omega t)) - 4AB\sin(\omega t)\cos(\omega t)] \tag{7}$$

We want to rewrite this in terms of $\cos(2\omega t)$ for easier integration,

$$= \frac{1}{2}m\omega^2[-A^2\cos(2\omega t) + B^2\cos(2\omega t) - 4AB\sin(\omega t)\cos(\omega t)] \tag{8}$$

Integrating, we get the solution we want,

$$\int_0^T \mathscr{L} = \frac{m\omega}{2\sin(\omega t)}[(x_1^2 + x_2^2)\cos(\omega T) - 2x_1x_2] \tag{9}$$

Of we take the derivative according to T,

$$\frac{\partial S_{cl}}{\partial T} = -\frac{m\omega^2}{2}\left(\frac{x_1^2 + x_2^2 - 2x_1x_2\cos(\omega T)}{\sin^2(\omega T)}\right) \tag{10}$$

which simplifies to our expression for the total energy.

Chapter 3

All Is Not Well with Classical Mechanics

Chapter 4

The Postulates-a General Discussion

4.1 The Postulates

4.2 Discussion of Postulates I-III

4.2.1 Angular Momentum Operator Example

Consider the following operators on a Hilbert space $\mathbb{V}^3$(C):

$$L_x = \frac{1}{2^{1/2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \quad L_y = \frac{1}{2^{1/2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}; \quad L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

What are the possible values one can obtain if L_z is measured?

According to Postulate III, if we make a measurement of L_z, we expect to get one of the eigenvalues. Solving the eigenvalue problem for L_z gives, $\omega = -1, 0, 1$.

Take the state in which $L_z = 1$. In this state what are $\langle L_x \rangle$, $\langle L_x^2 \rangle$, and ΔL_x?

According to (4.2.6),

$$\langle L_x \rangle = \langle L_z = 1 | L_x | L_z = 1 \rangle \tag{1}$$

The first thing we should do is find the eigenvector corresponding to $\omega = 1$,

$$|L_z = 1\rangle = |\psi\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

Inserting,

$$\langle L_x \rangle = \frac{1}{2^{1/2}} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \tag{3}$$

$$\langle L_x^2 \rangle = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \tag{4}$$

The standard deviation is given by (4.2.7),

$$\Delta L_x = \left\langle (L_x - \langle L_x \rangle)^2 \right\rangle^{1/2} \tag{5}$$

However, since we know that $\langle L_x \rangle = 0$, this simplifies to,

$$= \left\langle L_x^2 \right\rangle^{1/2} = \frac{1}{2^{1/2}} \tag{6}$$

Find the normalized eigenstates and the eigenvalues of L_x in the L_z basis.

We can assume (or easily show) the eigenvectors of L_z are the usual suspects,

$$|L_z = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad |L_z = 0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad |L_z = -1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{7}$$

The eigenvalues of L_x, $\lambda = -1, 0, 1$. The corresponding eigenstates,

$$|L_x = -1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -1/\sqrt{2} \\ 1 \\ -1/\sqrt{2} \end{bmatrix}; \quad |L_x = 0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}; \quad |L_x = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{bmatrix} \tag{8}$$

If the particle is in the state with $L_z = -1$, and L_x is measured, what are the possible outcomes and their probabilities

We know from Postulate III, a measurement of L_x will give us an eigenvalue of L_x. (4.2.2) gives us the probability for landing in that state,

$$P(L_x = -1) = |\langle L_x = -1 | L_z = 1\rangle|^2 = \frac{1}{4} \tag{9}$$

$$P(L_x = 0) = |\langle L_x = 0 | L_z = 1\rangle|^2 = \frac{1}{2} \tag{10}$$

$$P(L_x = 1) = |\langle L_x = 1 | L_z = 1\rangle|^2 = \frac{1}{4} \tag{11}$$

As a quick check, we can see that the probabilities add up to unity. When we make a measurement, we get something out.

Consider the state

$$|\psi\rangle = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2^{1/2} \end{bmatrix}$$

in the L_z basis. If L_z^2 is measured in this state and a result $+1$ is obtained, what is the state after the measurement? How probable was this result? If L_z is measured immediately afterwards, what are the outcomes and respective probabilities?

Let's apply the L_z^2 measurement and see what happens,

$$L_z^2 |\psi\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2^{1/2} \end{bmatrix} = \frac{1}{(3/4)^{1/2}} \begin{bmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \tag{12}$$

The probability going from $|\psi\rangle$ to $L_z^2 |\psi\rangle$,

$$|\langle \psi | L_z^2 | \psi \rangle|^2 = \left| \frac{1}{(3/4)^{1/2}} \begin{bmatrix} 1/2 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2^{1/2} \end{bmatrix} \right|^2 = \frac{3}{4} \tag{13}$$

If L_z is measured immediately after, we would normally be locked in the $L_z^2 = 1$ state. However, because of the squared, we can get $L_z = \pm 1$. We know the resulting states, so the probability of landing in those states,

$$P(L_z = 1) = \frac{1}{3/4} \left| \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right|^2 = \frac{1}{3} \tag{14}$$

$$P(L_z = -1) = \frac{1}{3/4} \left| \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right|^2 = \frac{2}{3} \tag{15}$$

If we wanted to, we could also check $L_z = 0$ and show that the probability of landing in that state is 0.

A particle is in a state for which the probabilities are $P(L_z = 1) = 1/4$, $P(L_z = 0) = 1/2$, and $P(L_z = -1) = 1/4$. Convince yourself that the most general, normalized state with this property is

$$|\psi\rangle = \frac{\exp(i\delta_1)}{2} |L_z = 1\rangle + \frac{\exp(i\delta_2)}{2^{1/2}} |L_z = 0\rangle + \frac{\exp(i\delta_3)}{2} |L_z = -1\rangle$$

It was stated earlier on that if $|\psi\rangle$ is a normalized state then the state $\exp(i\theta)|\psi\rangle$ is a physically equivalent normalized state. Does this mean that the factors $\exp(i\delta_i)$ multiplying the L_z eigenstates are irrelevant? [Calculate for example $P(L_z = 0)$]

Measuring each of the probabilities using (4.2.2), for example,

$$P(L_z = 1) = |\langle L_z = 1|\psi\rangle|^2 \tag{16}$$

And continuing for the other states, we see that the simplest state is,

$$|\psi'\rangle = \frac{1}{2} |L_z = 1\rangle + \frac{1}{\sqrt{2}} |L_z = 0\rangle + \frac{1}{2} |L_z = -1\rangle \tag{17}$$

We can then multiply by a phase factor, $\exp(i\delta_i)$ to get the generalized state.

However, the factors multiplying the L_z eigenstates are relevant since we can change the state based on our choice of δ_i. For example, if we chose $\delta_1 = \pi$, $\delta_2 = 0$, $\delta_3 = \pi$, we get $|\psi\rangle = |L_x = -1\rangle$ and if we chose $\delta_1 = \delta_2 = \delta_3 = 0$, we get $|\psi\rangle = |L_x = 1\rangle$. This is because,

$$\exp(i\theta) \neq \exp(i\delta_1) + \exp(i\delta_2) + \exp(i\delta_3) \tag{18}$$

This only holds if the relative phase difference is kept the same.

As a concrete example, if we calculate,

$$P(L_x = 0) = |\langle L_x = 0|\psi\rangle|^2 = \frac{1}{8}(\exp(i\delta_1) - \exp(i\delta_3)) \tag{19}$$

4.2.2 Momentum of a Real Wave Function

Show that for a real wave function $\psi(x)$, the expectation value of momentum $\langle P\rangle = 0$. (Hint: Show that the probabilities for the momenta $\pm p$ are equal.) Generalize this result to the case $\psi = c\psi_r$, where ψ_r is real and c an arbitrary (real or complex) constant. (Recall that $|\psi\rangle$ and $\alpha\,|\psi\rangle$ are physically equivalent.)

From (4.2.6), we get the expectation value. Because our wave function is continuous, we must work in integrals,

$$\langle P\rangle = \langle\psi|P|\psi\rangle = \int_{-\infty}^{\infty} \langle\psi|x\rangle\,\langle x|P|\psi\rangle\;\,dx \tag{1}$$

$$= \int_{-\infty}^{\infty} \psi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x)\;dx \tag{2}$$

Since we are working with a real wave function, $\psi^*(x) = \psi(x)$,

$$= -i\hbar\int_{-\infty}^{\infty} \psi(x)\frac{d\psi(x)}{dx}\;dx \tag{3}$$

We then notice

$$\frac{d\psi^2(x)}{dx} = 2\psi(x)\frac{d\psi(x)}{dx} \tag{4}$$

Inserting this into Equation 3,

$$\langle P\rangle = -\frac{i\hbar}{2}\int_{-\infty}^{\infty} \frac{d\psi^2(x)}{dx}\;dx = -\frac{i\hbar}{2}\psi^2(x)\Big|_{-\infty}^{\infty} = 0 \tag{5}$$

Which goes to zero since we expect the wave function to have a finite integral, which implies that it goes to zero as x goes to infinity.

If we instead have $\psi = c\psi_r$, our expectation value,

$$\langle P\rangle = \langle\psi|P|\psi\rangle = \int_{-\infty}^{\infty} c^*\psi^*(x)\left(-i\hbar\frac{d}{dx}\right)c\psi(x)\;dx \tag{6}$$

$$= -i\hbar c^2\int_{-\infty}^{\infty} \psi(x)\frac{d\psi(x)}{dx}\;dx \tag{7}$$

We can convince ourselves that this is the same as Equation 3 just with a multiplicative constant. Since the integral is the same and that went to 0, this integral should also go to 0.

4.2.3 Shifted Momentum

Show that if $\psi(x)$ has mean momentum $\langle P\rangle$, $\exp(ip_0x/\hbar)\psi(x)$ has mean momentum $\langle P\rangle + p_0$.

From (4.2.6) and the previous question, we know the mean momentum of $\psi(x)$,

$$\langle P\rangle = \langle\psi|P|\psi\rangle = -i\hbar\int_{-\infty}^{\infty}\psi^*(x)\frac{d\psi(x)}{dx}\ dx \tag{1}$$

Let's look at the expectation value of the shifted momentum of $\phi = \exp(ip_0x/\hbar)\psi(x)$,

$$\langle\phi|P|\phi\rangle = \int_{-\infty}^{\infty}\exp\left(-\frac{ip_0x}{\hbar}\right)\psi^*(x)\left[-i\hbar\frac{d}{dx}\left(\exp\left(\frac{ip_0x}{\hbar}\right)\psi(x)\right)\right]\ dx \tag{2}$$

$$= -i\hbar\int_{-\infty}^{\infty}\exp\left(-\frac{ip_0x}{\hbar}\right)\psi^*(x)\left[\frac{ip_0}{\hbar}\exp\left(\frac{ip_0x}{\hbar}\right)\psi(x) + \exp\left(\frac{ip_0x}{\hbar}\right)\frac{d\psi(x)}{dx}\right]\ dx \tag{3}$$

We can then split this into two integrals and kill the exponential,

$$= -i\hbar\frac{ip_0}{\hbar}\int_{-\infty}^{\infty}\psi^*(x)\psi(x)\ dx - i\hbar\int_{-\infty}^{\infty}\psi^*(x)\frac{d\psi(x)}{dx}\ dx \tag{4}$$

The first integral goes to unity since the wave function is normalized, and we know the result of the second integral,

$$= p_0 + \langle P\rangle \tag{5}$$

4.3 The Schrodinger Equation (Dotting Your i's and Crossing Your $\hbar$'s)