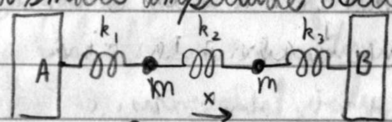
Thus did this poor soul struggle in its anguish. Eighteen hundred years before this ill-fated man, the mysterious being in whom are concentrated all the saintliness and all the sufferings of humanity had also refused for a long time the terrible chalice, streaming with darkness and brimming with shadows, that appeared to him in the star-filled depths while the olive trees shook in the fierce blast of the infinite.

-Victor Hugo (The Wretched)

2012-2013

1.* Two objects of mass m are connected by springs of force constants k_1, k_2, and k_3 as shown. They can move only in the $\hat{x}$ direction. Points A and B are fixed. Calculate the normal mode frequencies for small amplitude oscillations.



- $\mathcal{L} = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}(k_1 x_1^2 + k_2(x_2 - x_1)^2 + k_3(x_2)^2)$

$= \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}(k_1 x_1^2 + k_2 x_2^2 - 2k_2 x_1 x_2 + k_2 x_1^2 + k_3 x_2^2)$

$= \frac{1}{2}(m\dot{x}_1^2 + m\dot{x}_2^2) - \frac{1}{2}[(k_1+k_2)x_1^2 - 2k_2 x_1 x_2 + (k_2+k_3)x_2^2]$

$T = \begin{bmatrix} m & \\ & m \end{bmatrix}$ $\quad V = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$

$\det(V - \lambda T) = \begin{bmatrix} k_1+k_2-\lambda m & -k_2 \\ -k_2 & k_2+k_3-\lambda m \end{bmatrix}$

$= (k_1+k_2-\lambda m)(k_2+k_3-\lambda m) - k_2^2$

$= k_1k_2 + k_1k_3 - k_1\lambda m + k_2^2 + k_2k_3 - k_2\lambda m - k_2\lambda m - k_3\lambda m + \lambda^2 m^2 - k_2^2$

$= \lambda^2 m^2 - \lambda m(k_1 + 2k_2 + k_3) + (k_1k_2 + k_1k_3 + k_2k_3)$

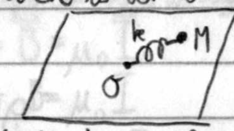
$$\lambda = \frac{m(k_1+2k_2+k_3) \pm \sqrt{m^2(k_1+2k_2+k_3)^2 - 4m^2(k_1k_2+k_1k_3+k_2k_3)}}{2m^2}$$

$$\omega^2 = \frac{(k_1+2k_2+k_3) \pm \sqrt{k_1^2+4k_2^2+k_3^2-2k_1k_3}}{2m}$$

$$\omega_1 = \left[\frac{(k_1+2k_2+k_3) + [(k_1-k_3)^2+4k_2^2]^{1/2}}{2m}\right]^{1/2}$$

$$\omega_2 = \left[\frac{(k_1+2k_2+k_3) - [(k_1-k_3)^2+4k_2^2]^{1/2}}{2m}\right]^{1/2}$$

2. A block of mass M which can slide without friction on a horizontal tabletop (assumed to be infinitely large) is attached to one end of a spring with spring constant k, whose unstretched length is l_0. The other end of the spring is fixed at the origin, so the spring can rotate freely about the origin in the plane of the tabletop. For times $t < 0$, the spring is executing uniform circular motion and the period for one circular orbit is $2\pi/\omega_0$. The block's position is described by cylindrical polar coordinates (r, θ). At time $t = 0$, when the block is at $\theta = 0$, it is given a small radially outward impulse J.



a. Write down the Lagrangian for the system in terms of θ and r.

- $x = r\cos(2\pi\theta/\omega_0)$ $\quad \dot{x} = \dot{r}\cos(2\pi\theta/\omega_0) - r \cdot 2\pi\dot{\theta}/\omega_0 \cdot \sin(2\pi\theta/\omega_0)$

$y = r\sin(2\pi\theta/\omega_0)$ $\quad \dot{y} = \dot{r}\sin(2\pi\theta/\omega_0) + r \cdot 2\pi\dot{\theta}/\omega_0 \cdot \cos(2\pi\theta/\omega_0)$

$T = \frac{M}{2}(\dot{x}^2 + \dot{y}^2)$

$V = \frac{k}{2}[(x^2+y^2)^{1/2} - l_0]^2$

$\mathcal{L} = T - V = \frac{M}{2}(\dot{r}^2 + r^2\dot{\theta}^2 \cdot 4\pi^2/\omega_0^2) - \frac{k}{2}(r - l_0)^2$

b. Find the frequency $\omega_r = 2\pi/T_r$, where T_r is the period of small radial oscillations, in terms of ω_0 and other parameters given in the problem.

- $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0$

$\frac{d}{dt}(M\dot{r}) - (Mr\dot{\theta}^2 \cdot 4\pi^2/\omega_0^2 - k(r - l_0)) = 0$

$M\ddot{r} = \frac{Mr\dot{\theta}^2 \cdot 4\pi^2}{\omega_0^2} - kr + kl_0$

$\ddot{r} = \left(\frac{\dot{\theta}^2 \cdot 4\pi^2}{\omega_0^2} - k\right) r + kl_0$

$\omega_r = \sqrt{(4\pi^2 - k)}$

3. A star of mass M and radius R is moved through a static medium of particles ~~of~~ with density ρ at a constant velocity $\vec{v}$. The star interacts gravitationally with the particles in the medium. Calculate the force required to keep $\vec{v}$ unchanged if the particles which touch the star stick to it without recoiling.

- $p_1 = p_0$

  $\Phi = 2\pi R^2$

  $\Delta p = 2\pi R^2 \cdot \rho \cdot \vec{v}$

~~In some time, will impact with~~ $\frac{2\pi R^2 \rho}{v}$

$$F = kg \cdot m/s^2 = \frac{kg}{m^3} \cdot \frac{m^2}{s^2} \cdot 2\pi R^2$$

$$= \rho v^2 \cdot 2\pi R^2$$

4*. Consider a uniformly electrically polarized sphere of radius R with polarization $\vec{P} = P_0 \hat{z}$ surrounded by an infinite dielectric medium with dielectric constant ε. Find the electrostatic potential $\Phi(\vec{x})$ for all points inside and outside the sphere.

- $\vec{P} = P_0(\cos\theta\, \hat{r} - \sin\theta\, \hat{\theta})$

$\sigma_b = \vec{P} \cdot \hat{r} = P_0 \cos\theta$

$\rho_b = \frac{1}{r^2} \cdot (2r P_0 \cos\theta) + \frac{1}{r\sin\theta} \cdot \frac{\partial}{\partial\theta}(-P_0 \sin^2\theta)$

$= \frac{2}{r} \cdot P_0 \cos\theta + \frac{1}{r\sin\theta} \cdot + P_0 \cdot \frac{1}{r} \cdot 2\cos\theta \sin\theta$

$= \frac{4}{r} P_0 \cos\theta$

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{4/3 \cdot \pi r^3 \rho_b}{r^2} = \frac{4 P_0 \cos\theta\, \hat{r}}{3}$$

$E = -\nabla\Phi$

$\Phi_{out} = 0$

$$\Phi_{in} = \frac{4 P_0 \cos\theta \cdot r}{3}$$

5*. A large sample of superconductor occupies the whole left half-space with $x < 0$. In the right half-space there is an infinitely long and thin wire, positioned at $x = L > 0$, $z = 0$ and running parallel to the y-axis. A current I is applied through the wire pointing in the positive $\hat{y}$ direction. Only take into account two properties of a superconductor for this problem: that the magnetic field on its surface must be tangent to the surface, and there is no magnetic field inside its bulk.

a. Compute the magnetic field $\vec{B}(x, y, z)$ for the $x > 0$ half-space.

- $\vec{\nabla} \times \vec{B} = \mu_0 \vec{I}$

$2\pi\rho B = \mu_0 I$

$$|B| = \frac{\mu_0 I}{2\pi\rho} = \frac{\mu_0 I}{2\pi(x^2+y^2+z^2)^{1/2}}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi(x^2+y^2+z^2)^{1/2}} (\sin\theta\, \hat{x} - \cos\theta\, \hat{z})$$

b. Compute the current density $\vec{j}_s(x = 0, y, z)$ on the superconductor surface at $x = 0$.

- $\vec{B} = \frac{\mu_0 I}{2\pi(y^2+z^2)^{1/2}} (\sin\theta\, \hat{x} - \cos\theta\, \hat{z})$

6*. Consider an Ising model for magnetism with

$$\mathcal{H} = -J\sum_{\langle ij\rangle} S_i S_j - h\sum_{i=1}^{N} S_i$$

where each spin S_i can take the values ± 1, N is the number of spins, $\sum_{\langle ij\rangle}$ denotes a sum over nearest neighbor pairs, and z is the number of nearest neighbors for each spin. Use the mean-field approximation with $\sigma = \langle S_i \rangle$ to calculate

a. The single site partition function

$$Z = \sum \exp(-\beta E_r)$$

$$= \sum_{S_i=\pm 1} \exp(\beta(J z \sigma S_i + h S_i))$$

b. For $h=0$, use the result of part (a) above to determine the self-consistent equation for σ, and

$$Z_1 = \exp(-\beta z J\sigma) + \exp(\beta z J \sigma)$$

$$= 2\cosh(\beta z J\sigma)$$

$$\sigma = \langle S_i \rangle = \frac{\sum_{S_i=\pm 1} S_i \exp(\beta J \sigma z S_i)}{2\cosh(\beta z J\sigma)} = \frac{\sinh(\beta z\sigma J)}{\cosh(\beta z \sigma J)}$$

c. the critical temperature

$$\sigma = \tanh(\beta_c z \sigma J)$$

$$1 = \beta_c z J$$

$$k_B T_c = z J$$

$$T_c = zJ/k_B$$

7. Imagine a collection of massless, spin-$\frac{1}{2}$ fermions confined to move in a one-dimensional cavity. Assume that the fermion number is not conserved and that the fermions are in thermal equilibrium at a temperature T. Show that the density of states per unit energy is constant, and use this to calculate the number of particles per unit length in the cavity.

8. A fluid of particles with a repulsive interparticle interaction can be modeled as a "lattice gas" as follows. Consider the container to be divided into N cells, each of volume v, comparable with the volume of a particle. An unoccupied cell and a cell occupied by one particle have zero energy. A cell occupied by two particles has an energy of ε, and no cell may be occupied by more than two particles. Use the grand canonical ensemble to find, in terms of the temperature and the chemical potential

a. the average energy per cell

$Z_G = \sum z \exp(-\beta E_r)$

$= [1 + \exp(\beta\mu) + \exp(2\beta\mu - \beta\varepsilon)]^N$

$$\langle E\rangle = -\frac{1}{Z_G}\cdot\frac{\partial Z_G}{\partial\beta}$$

$$= \frac{-1}{[1+\exp(\beta\mu)+\exp(2\beta\mu-\beta\varepsilon)]^N}\cdot N[1+\exp(\beta\mu)+\exp(2\beta\mu-\beta\varepsilon)]^{N-1}$$

$\cdot\, \mu\exp(\beta\mu) + (2\mu-\varepsilon)\exp(2\beta\mu-\beta\varepsilon)$

$$= \frac{N(\varepsilon\exp(\beta\mu)\exp(-\beta\varepsilon) - \mu(2\exp(\beta\mu)+1))\exp(\beta\mu)}{1+\exp(\beta\mu)+\exp(2\beta\mu-\beta\varepsilon)}$$

$$\frac{\langle E\rangle}{N} = \frac{\varepsilon z\exp(-\beta\varepsilon) - \mu(2z+1)}{z^{-1}+1+z\exp(-\beta\varepsilon)}$$

b. the concentration c of particles (c is the total number of particles divided by N)

$$\langle N\rangle = -\frac{\partial \ln Z_G}{\partial(-\beta\mu)} = \frac{-\partial}{\partial(-\beta\mu)} N\ln(1+\exp(\beta\mu)+\exp(2\beta\mu-\beta\varepsilon))$$

$$= \frac{-N}{1+\exp(\beta\mu)+\exp(2\beta\mu-\beta\varepsilon)}\cdot -\exp(\beta\mu) - 2\exp(2\beta\mu-\beta\varepsilon)$$

$$= \frac{N\exp(\beta\mu)(1+\exp(\beta\mu)\exp(-\beta\varepsilon))}{1+\exp(\beta\mu)+\exp(2\beta\mu-\beta\varepsilon)}$$

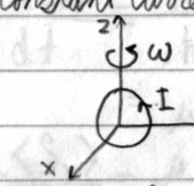
c. the pressure P.

$PV = k_B T\ln Z_G$

$= k_B T N\ln(1+z+z^2\exp(-\beta\varepsilon))$

$P = \frac{k_B T}{v}\ln(1+z+z^2\exp(-\beta\varepsilon))$

9. A circular ring of radius a is rotating about the z axis with angular frequency ω as shown in the figure. The ring carries a constant current I.



a. What is the complex magnetic dipole moment of the ring?

$m = I\cdot \text{Area}$

$= I\cdot\pi a^2$

$\vec{m} = m_0(\cos(\omega t)\hat{x} + i\sin(\omega t)\hat{y})$

$= m_0(\hat{x} + i\hat{y})$

b. $\hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$

$\hat{y} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$

$\hat{n} = \hat{r}$

$$\hat{n}\times\vec{m} = \begin{vmatrix}\hat{r} & \hat{\theta} & \hat{\phi}\\ 1 & 0 & 0\\ m_r & m_\theta & m_\phi\end{vmatrix} = (0, -m_\phi, m_\theta)$$

$|\hat{n}\times\vec{m}|^2 = m_\phi^2 + m_\theta^2 = [\cos\theta\cos\phi + i\cos\theta\sin\phi]^2 + [-\sin\phi + i\cos\phi]^2$

$= \cos^2\theta[\cos^2\phi + \sin^2\phi] + 1 = 1 + \cos^2\theta$

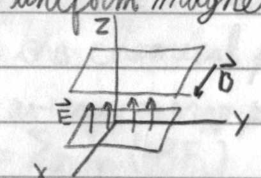
$$\frac{dP}{d\Omega} \propto 1 + \cos^2\theta$$

$P \propto \int_{-1}^{1}(1+\cos^2\theta)\,d\cos\theta = 2 + 2/3 = 8/3$

c. $(\hat{n}\times\vec{p})\times\hat{n} = \begin{vmatrix} r & \theta & \phi \\ 0 & -m_\phi & m_\theta \\ 1 & 0 & 0 \end{vmatrix} = (0,\ m_\theta,\ m_\phi)$

$= m_0(0, \cos\theta(\cos\phi + i\sin\phi), -\sin\phi + \cos\phi)$

10. A charged parallel-plate capacitor, with uniform electric field between the plates, taken to be $\vec{E} = E\hat{z}$, is placed in a uniform magnetic field, $\vec{B} = B\hat{x}$.



a. Find the electromagnetic momentum in the space between the plates.

- $g = \frac{1}{c^2}\vec{E}\times\vec{B}$
  $= \frac{1}{c^2}\cdot(EB)\hat{y}$
  $\vec{p} = \frac{1}{4\pi c}(EB)\cdot V\hat{y}$

b. A resistive wire is connected between the plates, along the z-axis, so that the capacitor slowly discharges. The current in the wire will experience a magnetic force. Find the total impulse delivered to the system, and relate it to (a).

- $V = E/d = IR$
  $F = IBd\hat{y} = \frac{EBd}{dR} = \frac{BE\hat{y}}{R} = \frac{4\pi c\vec{p}}{VR}$

c. Instead of turning off the electric field, suppose we slowly reduce the magnetic field. Find the total impulse, and again, relate it to (a).

- Should be the same as part b.

11*. A homogeneous magnetic field $\vec{B} = B\hat{z}$ is applied along the $\hat{z}$-axis. Consider a spin-$\frac{1}{2}$ quantum particle, e.g. an electron, in the magnetic field. At time t_0 the electron spin is measured to be along the positive $\hat{y}$-axis. At a given later time $t > t_0$, compute

a. The expectation value of spin $\hat{x}$-component $\langle \hat{S}_x(t)\rangle$

- $\hat{H} = -\mu B\hat{S}_z$

$$\frac{d\langle \hat{S}_x\rangle}{dt} = \frac{1}{i\hbar}[S_x, -\mu BS_z] = \frac{-\mu B}{i\hbar}\cdot -i\hbar\langle S_y\rangle$$
$$= -\mu B\langle S_y\rangle$$
$$\frac{d\langle S_y\rangle}{dt} = \mu B\langle S_x\rangle$$
$$\langle S_x\rangle = a\cos(\mu Bt) + b\sin(\mu Bt) = -\hbar/2\ \sin(\mu B(t-t_0))$$
$$\langle S_y\rangle = a\cos(\mu Bt) - b\sin(\mu Bt) = \hbar/2\ \cos(\mu B(t-t_0))$$

b. The probability $P_z(t)$ to find its spin along the positive $\hat{z}$-axis. (You may assume electron magnetic moment to be μ_E)

$$U = \exp(-iHt/\hbar) = \exp(i\mu B\hat{S}_z t/\hbar)$$
$$= \cos(i\mu B\hat{S}_z t/\hbar) + i\sin(\mu B\hat{S}_z t/\hbar)$$
$$= \begin{pmatrix} \cos(i\mu Bt/\hbar) + i\sin(\mu Bt/\hbar) & \\ & -\cos(\mu Bt/\hbar) - i\sin(\mu Bt/\hbar) \end{pmatrix}$$
$$U\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos(\mu Bt/\hbar) + i\sin(\mu Bt/\hbar) \\ -i\cos(\mu Bt/\hbar) + \sin(\mu Bt/\hbar)\end{pmatrix}$$
$$\left|\frac{1}{\sqrt{2}}(1\ \ 0)\begin{pmatrix} \cos(\mu Bt/\hbar) + i\sin(\mu Bt/\hbar) \\ -i\cos(\mu Bt/\hbar) + \sin(\mu Bt/\hbar)\end{pmatrix}\right|^2 = \frac{1}{2}$$

12*. Consider an electron confined to move in one dimension in the interval $0 \le x \le a$, in which

$$V = \begin{cases} 0; & 0 \le x \le a \\ \infty; & \text{elsewhere} \end{cases}$$

a. Solve for the energy eigenvalues and eigenfunctions

$|\psi\rangle = \sqrt{\tfrac{2}{a}}\,\sin(n\pi x/a)$

$$-\frac{\hbar^2}{2m}\psi = E\psi$$

$$E = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

b. Assume that an electric field is applied in the $\hat{x}$ direction to give an additional potential

$$V' = \begin{cases} -|e|Ex; & 0 \le x \le a \\ 0; & \text{elsewhere} \end{cases}$$

Using first order perturbation theory, calculate the correction to the ground-state energy due to V'

- $\langle\psi|V'|\psi\rangle = \frac{2}{a}\int_0^a -eEx\sin^2(n\pi x/a)\,dx \quad n=1$

$$= 2/a(-eE)\cdot\left[\frac{\sin(n\pi x/a)}{(n\pi/a)^2} - \frac{x\cos(n\pi x/a)}{(n\pi/a)}\right]\Bigg|_0^a$$

$$= -2eE/a\left(-a\cos(\pi)\cdot a/n\pi\right)$$

$$= \frac{-2eEa}{\pi}$$

c. Write an expression for the groundstate wavefunction correct to first order in V'. You may express this as a sum involving matrix elements whose integral forms should be specified, but you need not evaluate those integrals explicitly.

- $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - eE'x\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\cdot\frac{d^2\psi}{dx^2} = (E + eE'x)\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{-2m}{\hbar^2}\left(\frac{\hbar^2\pi^2}{2ma^2} - \frac{2eEa}{\pi} + eEx\right)\psi$$

$$\frac{d^2\psi}{\psi} = \frac{-2m}{\hbar^2}\left(\frac{\hbar^2\pi^2}{2ma^2} - \frac{2eEa}{\pi} + eEx\right)dx^2$$

13*. Consider a spinless particle constrained to move on a two-dimensional plane, with a perpendicular magnetic field $\vec{B} = B_0\hat{z}$. The particle has mass m and charge q.

a. Using the vector potential $\vec{A} = -B_0 y\hat{x}$, write down the Schrodinger equation for the particle

$\vec{p} \to \vec{p} - q\vec{A}/c$

$$H = \frac{(\vec{p} - q\vec{A}/c)^2}{2m}$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{q}{2mc}\left(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}\right)\psi + \frac{q^2\vec{A}^2}{2mc^2}\psi = E\psi$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{i\hbar q B_0 y}{2mc}\frac{\partial}{\partial x} + \frac{q^2 B_0^2 y^2}{2mc^2}\right]\psi = E\psi$$

b. Show that the wavefunctions have a good quantum number $\hbar k_x$ corresponding to momentum along the $\hat{x}$ direction.

- let $\psi \propto \exp(k\cdot x)$, then

$$-\frac{\hbar^2 k^2}{2m} + \frac{iqB_0 y}{2mc}\cdot\hbar k + \frac{q^2 B_0^2 y^2}{2mc^2} = E$$

where we have factor of $(\hbar k)^n$ in each term.

zero-probability for part c

14. Consider two spin-1 particles in a state $|m_1=1; m_2=0\rangle = |10\rangle_m$.
What is the probability of finding the system in an eigenstate of total spin S^2:

a. with quantum number $S=1$.

· Only state allowed is $|1,1; 0,0\rangle$

15.* A particle of mass m moving at 12/13 of the speed of light strikes and is absorbed by another particle of identical mass at rest.

a. What is the mass M of the resulting particle

• $P_i = (E_i, \vec{p}_i)$ $\qquad \gamma = (1+v^2)^{-1/2}$

$P_t = (E_t, 0)$ $\qquad E_i^2 = m^2 + \gamma_i^2 m^2 v_i^2$

$P_f = (E_f, \vec{p}_f)$ $\qquad E_t = m$

$E_f^2 = M^2 + \gamma_f^2 M^2 v_f^2$

$P_i + P_t = P_f$

$E_i + E_t = E_f$ $\qquad \gamma_i m v_i = \gamma_f M v_f$

$(m^2 + \gamma_i^2 m^2 v_i^2)^{1/2} + m = (M^2 + \gamma_f^2 M^2 v_f^2)^{1/2}$

$= (M^2 + \gamma_i^2 m^2 v_i^2)^{1/2}$

$m^2 + \cancel{\gamma_i^2 m^2 v_i^2} + 2m(m^2 + \gamma_i^2 m^2 v_i^2)^{1/2} + m^2 = M^2 + \cancel{\gamma_i^2 m^2 v_i^2}$

$M^2 = 2m^2 + 2m^2(1 + \gamma_i^2 v_i^2)$

$= 2m^2 + 2m^2(457/313)$

$M^2 = 2m^2(770/313)$

$M = 2.2\,m$

b. What is the speed v of the resulting particle in terms of c, the speed of light?

• $\gamma_i m v_i = \gamma_f M v_f$

$\frac{m v_i}{(1+v_i^2)^{1/2}} = \frac{M v_f}{(1+v_f^2)^{1/2}}$

$\frac{m^2 v_i^2}{1+v_i^2} = \frac{M v_f^2}{1+v_f^2}$

$m v_i^2 (1+v_f^2) = M v_f^2 (1+v_i^2)$

$m v_i^2 + m v_i^2 v_f^2 = M v_f^2 (1+v_i^2)$

$m v_i^2 = v_f^2 [M(1+v_i^2) - m v_i^2]$

$v_f^2 = \frac{\cancel{m} v_i^2}{2.2\cancel{m} + 0.2\cancel{m} v_i^2} = \frac{144}{169} \cdot \frac{1}{2.2 + 0.2 \cdot \frac{144}{169}}$

$v_f = 0.6c$