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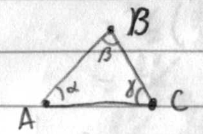
Chapter 4: The Kinematics of Rigid Body Motion

Section 1. The Independent Coordinates of a Rigid Body

A rigid body is some object (or collection of point particles) with the constraint that the distance between any two points is constant. For example, consider a triangle as shown below



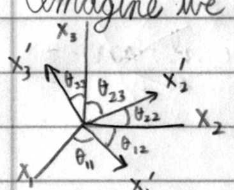
where the sides are constant. We can imagine we can act on this triangle (rotate it or translate it (flip it, stick it, and see you later bye)) and it won't change shape, i.e., fix the lengths and the angles are also fixed.

A common question to ask is how many degrees of freedom do we need to define this problem. Normally for N points, we would need 3N coordinates (say (x, y, z) for each point). It turns out, we only need six degrees of freedom.

1. Define one point (the pivot point, 3 dof), then use fixed distances (3 more dof) to determine the rest of the rigid body
2. Define the center of mass (3 dof) and specify the orientation (3 dof).

Section 2. Orthogonal transformations

Imagine we define our coordinate system



If we define

$$a_{ij} = \cos\theta_{ij} \tag{4.11}$$

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \tag{4.12}$$

Following Goldstein's convention,

$$x_i' = a_{ij}x_j$$

i.e., there is an implied summation over j.

From experience, we know that if we change the coordinate system, the magnitude of the vector should not change

$$x_i x_i = x_i' x_i' \tag{4.13}$$

$$= a_{ij}a_{ik}x_j x_k$$

$$\Rightarrow a_{ij}a_{ik} = \delta_{jk} \tag{4.15}$$

Combining (4.12) and (4.15) allows us to create orthogonal transformation

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{4.16}$$

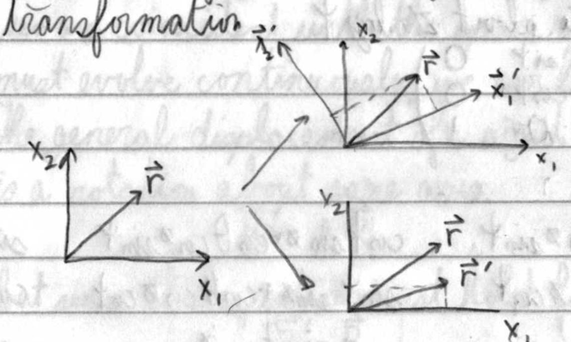
which can be thought of as an operator. If we write

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

then we can rewrite (4.12) as

$$\vec{x}' = A\vec{x} \tag{4.19}$$

If we, for example, define A as rotation in the counterclockwise direction, then $A^{-1} = A^T$ is rotation in the clockwise direction. Another thing to note is that if we act A on the coordinate system, we call that a passive transformation. If instead we act A on the vector, that is an active transformation. A counterclockwise passive transformation is equivalent to a clockwise active transformation.



Passive transformation

Active transformation

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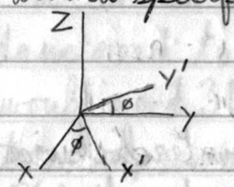
Section 3 *Formal Properties of the Transformation Matrix*

Most of this section is better explained in the first chapter of Shankar. It goes over the properties of matrices, so I will be skipping this section.

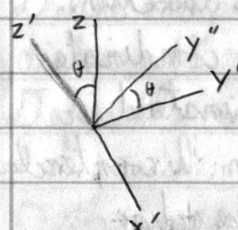
Section 4. *The Euler Angles*

Note that Euler angles are defined differently for QM and CM. In order to describe the motion of a rigid body, we need three independent coordinates. That is, we can define any arbitrary rotation using three rotations in a specific order around specific axes.



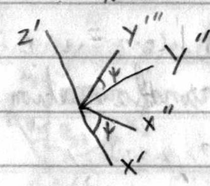
1. Rotate about z by ϕ

$$D = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.43)$$



2. Rotate about x' by θ

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad (4.44)$$



3. Rotate about z' by ψ

$$B = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.45)$$

$$A = BCD = \begin{bmatrix} \cos\psi\cos\phi - \cos\phi\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix} \quad (4.46)$$

For those who try to defy God's will and go swanning about in the sky, the three Euler angles are known as heading, pitch, and banking (vertical, transverse, longitudinal axes).

Section 5. *The Cayley-Klein Parameters and Related Quantities*

The Cayley-Klein Parameters are another way to write A (the rotation in terms of Euler angles).

$$A = \begin{bmatrix} \frac{1}{2}(\alpha^2-\gamma^2+\delta^2-\beta^2) & \frac{i}{2}(\gamma^2-\alpha^2+\delta^2-\beta^2) & \gamma\delta-\alpha\beta \\ \frac{i}{2}(\alpha^2+\gamma^2-\beta^2-\delta^2) & \frac{1}{2}(\alpha^2+\gamma^2+\beta^2+\delta^2) & -i(\alpha\beta+\gamma\delta) \\ \beta\delta-\alpha\gamma & i(\alpha\gamma+\beta\delta) & \alpha\delta+\beta\gamma \end{bmatrix}$$

$$\beta = -\gamma^* \; ; \; \delta = \alpha^*$$

$$\alpha = e_0 + ie_3 \, , \; \beta = e_2 + ie_1 \; , \; e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

$$A = \begin{bmatrix} \frac{1}{2}(\alpha^2-\beta^{*2}+\alpha^{*2}-\beta^2) & \frac{i}{2}(\beta^{*2}-\alpha^2+\alpha^{*2}-\beta^2) & -\beta^*\alpha^*-\alpha\beta \\ \frac{i}{2}(\alpha^2+\beta^{*2}-\beta^2-\alpha^{*2}) & \frac{1}{2}(\alpha^2+\beta^{*2}+\beta^2+\alpha^{*2}) & -i(\alpha\beta-\beta^*\alpha^*) \\ \beta\alpha^*+\alpha\beta^* & i(\alpha\beta^*+\beta\alpha^*) & \alpha\alpha^*-\beta\beta^* \end{bmatrix}$$

$$= \begin{bmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2(e_1e_2+e_0e_3) & 2(e_1e_3-e_0e_2) \\ 2(e_1e_2-e_0e_3) & e_0^2-e_1^2+e_2^2-e_3^2 & 2(e_2e_3+e_0e_1) \\ 2(e_1e_3+e_0e_2) & 2(e_2e_3-e_0e_1) & e_0^2-e_1^2-e_2^2+e_3^2 \end{bmatrix} \quad (4.47)$$

Section 6. *Euler's theorem on the motion of a rigid body*

We can imagine that if we rotate an object over time, the transformation matrix will evolve as a function of time, $A(t)$. We can also imagine that initially, $A(0) = I$, i.e., the body axes are coincident with the space axes. Since the transformation must evolve continuously, we are left with Euler's Theorem:

The general displacement of a rigid body with one point fixed is a rotation about some axis.

$$\vec{R}' = A\vec{R}$$

but since this equation must hold for all t, and since $A(0) = I$,

$$\vec{R}' = \vec{R} \quad (4.48)$$

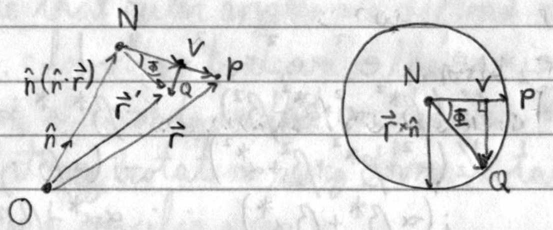
$$A\vec{R} = \lambda\vec{R} \quad (4.49)$$

Another way to write Euler's Theorem: the real orthogonal matrix specifying the physical motion of a rigid body with one point fixed always has the eigenvalue $+1$.

A corollary to Euler's Theorem is Chasles' Theorem: The most general displacement of a rigid body is a translation plus a rotation.

Section 7. Finite Rotations

Imagine we have



$$\vec r' = \hat n(\hat n\cdot\vec r) + [\vec r - \hat n(\hat n\cdot\vec r)]\cos\Phi + (\vec r\times\hat n)\sin\Phi$$
$$= \vec r\cos\Phi + (1-\cos\Phi)\,\hat n(\hat n\cdot\vec r) + (\vec r\times\hat n)\sin\Phi \qquad (4.62)$$

(4.62) is the rotation formula, and it holds for any rotation.

$$\mathrm{Tr}\,A = 1+2\cos\Phi \qquad (4.61)$$

where A is defined according to (4.46).

$$\mathrm{Tr}\,A = \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi - \sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi + \cos\theta$$
$$= \cos\psi\cos\phi - \sin\psi\sin\phi + \cos\theta(\cos\phi\cos\psi - \sin\phi\sin\psi) + \cos\theta$$
$$= \cos(\psi+\phi) + \cos\theta\cos(\psi+\phi) + \cos\theta$$
$$\mathrm{Tr}\,A + 1 = 2 + 2\cos\Phi$$
$$(\cos(\psi+\phi)+1)(\cos\theta+1) = 2(1+\cos\Phi)$$
$$2\cos\left(\tfrac{\psi+\phi}{2}\right)\cdot 2\cos\tfrac{\theta}{2} = 4\cdot\cos\tfrac{\Phi}{2}$$
$$\cos\left(\tfrac{\Phi}{2}\right) = \cos\left(\tfrac{\psi+\phi}{2}\right)\cos\left(\tfrac{\theta}{2}\right) \qquad (4.63)$$

Section 8. Infinitesimal Rotations

See section in Sakurai about rotations

Section 9. Rate of change of a vector

Imagine we have a vector $\vec G$. The change in dt

$$(d\vec G)_{space} = (d\vec G)_{body} + (d\vec G)_{rot}$$
$$(d\vec G)_{rot} = d\vec\Omega\times\vec G$$
$$(d\vec G)_{space} = (d\vec G)_{body} + d\vec\Omega\times\vec G \qquad (4.81)$$
$$\left(\frac{d\vec G}{dt}\right)_{space} = \left(\frac{d\vec G}{dt}\right)_{body} + \vec\omega\times\vec G \qquad (4.82)$$

We can then write (4.82) as an operator equation.

$$\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_r + \vec\omega\times \qquad (4.86)$$

Expressed in terms of Euler angles:

$$\omega_{x'} = \dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi$$
$$\omega_{y'} = \dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi$$
$$\omega_{z'} = \dot\phi\cos\theta + \dot\psi \qquad (4.87).$$

Section 10. The Coriolis Effect

For most situations, we can treat the earth as an inertial frame (not accelerating or rotating), but for certain situations, such as the Paris Gun or Foucault Pendulum, the rotation of the earth must be taken into account.

Say we have some particle with radius vector $\vec r$ and

$$\vec v_s = \vec v_r + \vec\omega\times\vec r \qquad (4.88)$$

Applying the rotation equation (4.86) to (4.88),

$$\left(\frac{dv_s}{dt}\right)_s = \left(\frac{dv_s}{dt}\right)_r + \vec\omega\times\vec v_s$$
$$\vec a_s = \frac{d(\vec v_r + \vec\omega\times\vec r)}{dt} + \vec\omega\times\vec v_r + \vec\omega\times(\vec\omega\times\vec r)$$
$$= \vec a_r + \vec\omega\times\vec v_r + \vec\omega\times\vec v_r + \vec\omega\times(\vec\omega\times\vec r)$$
$$= \vec a_r + 2(\vec\omega\times\vec v_r) + \vec\omega\times(\vec\omega\times\vec r) \qquad (4.89)$$

$$m\vec{a}_s = m\vec{a}_r + 2m(\vec{\omega}\times\vec{v}_r) + m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$
$$\vec{F} = m\vec{a}_r + 2m(\vec{\omega}\times\vec{v}_r) + m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$
$$m\vec{a}_r = \vec{F} - 2m(\vec{\omega}\times\vec{v}_r) - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) \qquad (4.90)$$
$$\vec{F}_{eff} = \vec{F} - 2m(\vec{\omega}\times\vec{v}_r) - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) \qquad (4.91)$$

We recognize the first term as the force we normally see. The last term has a magnitude $m\omega^2 r \sin\theta$, which we recognize centrifugal force. The middle term comes from the Coriolis effect. We see that this only takes place if the particle is moving

Derivation

1. $A(BC) = (AB)C$ ← If this isn't true, I'm in trouble

$$A^T A = I \quad \text{orthogonal condition} \qquad (4.36)$$

Say A & B are orthogonal, prove

$$(AB)(AB)^T = I$$
$$(AB)B^T A^T = I$$
$$AA^T = I$$
$$I = I$$

2. $(AB)^T = B^T A^T$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$AB = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}$$

$$(AB)^T = \begin{pmatrix} a\alpha + b\gamma & c\alpha + d\gamma \\ a\beta + b\delta & c\beta + d\delta \end{pmatrix}$$

$$B^T A^T = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$
$$= \begin{pmatrix} a\alpha + b\gamma & c\alpha + d\gamma \\ a\beta + b\delta & c\beta + d\delta \end{pmatrix}$$

$$(AB)^\dagger = \begin{pmatrix} c\beta + d\delta & -a\beta - b\delta \\ -c\alpha - d\gamma & a\alpha + b\gamma \end{pmatrix}$$

$$B^\dagger A^\dagger = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
$$= \begin{pmatrix} d\delta + c\beta & -b\delta - a\beta \\ -d\gamma - c\alpha & b\gamma + a\alpha \end{pmatrix}$$

3. $A' = BAB^{-1} \qquad (4.41)$

$$\mathrm{Tr}\,A' = \mathrm{Tr}\,B \cdot \mathrm{Tr}\,A\,\mathrm{Tr}\,B^{-1}$$
$$= \mathrm{Tr}\,A \cdot \mathrm{Tr}\,BB^{-1}$$
$$= \mathrm{Tr}\,A \cdot \mathrm{Tr}\,I = \mathrm{Tr}\,A$$

$$A = -A^T \quad \text{antisymmetric matrix} \qquad (4.39)$$

If A is antisymmetric and B is orthogonal

$$A' = BAB^{-1} = -A'^T$$
$$= (bab^{-1})_{ij} = b_{ik}a_{kl}b^{-1}_{lj}$$
$$= -b^{-1}_{ki}a_{lk}b_{jl}$$
$$= -b_{jl}a_{lk}b^{-1}_{ki}$$
$$= -(BAB^{-1}) = -A'^T$$

4.

a. $A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}$

$$\det(A-\lambda) = -\lambda(\lambda^2 + a_{23}^2) - a_{12}(-a_{12}\lambda + a_{13}a_{23}) + a_{13}(a_{12}a_{23} - \lambda a_{13})$$

let $\lambda = \pm 1$, we can convince ourselves that $\det(A-\lambda) \neq 0$

$\Rightarrow 1 \pm A$ is nonsingular

b. $(1+A)(1-A)^{-1} = (1-A^T)(1+A^T)^{-1}$

$$BB^T = (1+A)(1-A)^{-1}(1+A^T)(1-A^T)^{-1}$$
$$= (1-A^T)(1+A^T)^{-1}(1+A^T)(1-A^T)^{-1}$$
$$= (1-A^T)(1-A^T)^{-1} = I$$

5. $$D = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.43)$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad (4.44)$$

$$B = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.45)$$

$A = BCD$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \quad (4.46)$$

$$A^TA = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & -\sin\theta\cos\phi \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{pmatrix}\Big(\quad\Big)$$

$$B_{11} = (\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi)(\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi)$$
$$+(-\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi)(-\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi)$$
$$+\sin^2\theta\sin^2\phi$$

7. $P_\pm = \frac{1}{2}(1 \pm A)$

$$P_+^2 = P_+P_+ = \frac{1}{4}(1+A)(1+A)$$
$$= \frac{1}{4}(1+2A+A^2)$$

Since A is a rotation by 180°, we would expect A^2 to be a rotation by 360°, i.e., identity

$$P_+^2 = \frac{1}{4}(2+2A)$$
$$= \frac{1}{2}(1+A) = P_+$$

$A = 180^\circ$ rotation about z-axis

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.43)$$

$$P_+ = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad P_- = \begin{bmatrix} -1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

8

a. $$A = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 + e_0e_3) & 2(e_1e_3 - e_0e_2) \\ 2(e_1e_2 - e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 + e_0e_1) \\ 2(e_1e_3 + e_0e_2) & 2(e_2e_3 - e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix} \quad (4.47')$$

$$S = \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$$

$$\Rightarrow e_0^2 + e_1^2 - e_2^2 - e_3^2 = e_0^2 - e_1^2 + e_2^2 - e_3^2 = e_0^2 - e_1^2 - e_2^2 + e_3^2 = -1$$

$2e_1^2 - 2e_2^2 = 0 \qquad 2e_2^2 - 2e_3^2 = 0$

$e_1^2 = e_2^2 \qquad e_2 = \pm e_3$

$\pm e_1 = e_2$

$2e_1^2 - 2e_3^2 = 0$

$e_1 = \pm e_3$

Looking at off-diagonal terms,

$e_0e_3 = 0$

$e_0e_2 = 0$

$e_0e_1 = 0$

b. $AA^T = \begin{bmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2(e_1e_2+e_0e_3) & 2(e_1e_3-e_0e_2) \\ 2(e_1e_2-e_0e_3) & e_0^2-e_1^2+e_2^2-e_3^2 & 2(e_2e_3+e_0e_1) \\ 2(e_1e_3+e_0e_2) & 2(e_2e_3-e_0e_1) & e_0^2-e_1^2-e_2^2+e_3^2 \end{bmatrix}$

$\cdot \begin{bmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2(e_1e_2-e_0e_3) & 2(e_1e_3+e_0e_2) \\ 2(e_1e_2+e_0e_3) & e_0^2-e_1^2+e_2^2-e_3^2 & 2(e_2e_3-e_0e_1) \\ 2(e_1e_3-e_0e_2) & 2(e_2e_3+e_0e_1) & e_0^2-e_1^2-e_2^2+e_3^2 \end{bmatrix}$

$= \begin{bmatrix} (e_0^2+e_1^2-e_2^2-e_3^2)^2+4(e_1e_2+e_0e_3)^2+4(e_1e_3-e_0e_2)^2 & & \\ & \ddots & \\ & & \ddots \end{bmatrix}$

$e_0^4+2e_0^2e_1^2-2e_0^2e_2^2-2e_0^2e_3^2+e_1^4-2e_1^2e_2^2-2e_1^2e_3^2+e_2^4+2e_2^2e_3^2+e_3^4$
$+4e_0^2e_2^2+4e_0^2e_3^2 \quad +4e_1^2e_2^2+4e_1^2e_3^2$

$= e_0^4+e_1^4+e_2^4+e_3^4+2e_0^2e_1^2+2e_0^2e_2^2+2e_1^2e_2^2+2e_1^2e_3^2+2e_2^2e_3^2$

$= (e_0^2+e_1^2)^2 +$

10.

a. $e^Be^C = (1+B+B^2/2)(1+C+C^2/2)$

$= 1+B+C+C^2/2+BC+B^2/2+\dots$

$= 1+(B+C)+\frac{1}{2}(B+C)^2 = \exp(B+C)$

b. $AA^{-1}=1$

$(1+B+B^2/2)(1-B+B^2/2)$

$= 1-B+B^2/2+B-B^2+B^3/2+B^2/2-B^3/2+B^4/4$

$= 1+B^2-B^2+\dots = 1$

c. $\exp(CBC^{-1}) = \exp(C)\exp(B)\exp(C^{-1})$

$= (1+C+\frac{1}{2}C^2)A(1+C^{-1}+\frac{1}{2}C^{-2})$

$= A+AC^{-1}+\frac{1}{2}AC^{-2}+CA+CAC^{-1}+\frac{1}{2}CAC^{-2}$
$+\frac{1}{2}C^2A+\frac{1}{2}C^2AC^{-1}+\frac{1}{4}C^2AC^{-2}$

$= A+AC^{-1}+CA+CAC^{-1}+\dots$

$= CAC^{-1}$

d. $AA^T = (1+B)(1-B)^{-1}$

see 4b

11. $\det(-B) = \det(-I\cdot B) = \det(-I)\det B$

$= (-1)^n \det(B)$

12.

Problems

21. $\vec{F}_{eff} = \vec{F} - 2m(\vec{\omega}\times\vec{v}_r) - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$ (4.91)

Since we are looking at the deflection due to Coriolis effect

$\vec{F}_{eff} = -2m(\vec{\omega}\times\vec{v}_r)$

~~We can look at the problem in two parts going up going down~~

$\vec{\omega} = \omega\sin\theta\,\hat{y} + \omega\cos\theta\,\hat{z}$

$\vec{v}_r = v_z\hat{z}$

$\vec{\omega}\times\vec{v}_r = v_z\sin\theta\cos\hat{x}$

$\frac{d^2x}{dt^2} = -2v_z\omega\sin\theta$ (4.93)

$\vec{v}_z = -gt+v_0$

time total $= 2v_0/g$

$d^2x = -2(v_0-gt)\omega\sin\theta\, dt^2$

$dx = [(-2v_0t+gt^2)\omega\sin\theta\, dt]$

$x = (-v_0t^2+gt^3/3)\omega\sin\theta$

$x(2v_0/g) = \left(-v_0\cdot\frac{4v_0^2}{g^2}+\frac{g\cdot 8v_0^3}{3g^2}\right)\omega\sin\theta$

Going down

$= (-4+8/3)\cdot v_0^3/g^2\ \omega\sin\theta$

$v_z = -gt$

$= -4v_0^3/3g^2\ \omega\sin\theta$

Going down, $v_z = -gt$

$t = v_0/g$

$d^2x = 2gt\,\omega\sin\theta\,dt^2$

$dx = gt^2\omega\sin\theta\,dt$

$x = gt^3/3\cdot\omega\sin\theta$

$x(v_0/g) = v_0^3/3g^2\cdot\omega\sin\theta$

A few words, I believe there should be some deviation in v_z due to Coriolis effect (from the additional v_x); however, this is ignored because it is very small.

22. $\vec\omega = \omega\hat z$

$\vec v_r = v_0\hat\phi$

$a_{cor} = -2(\vec\omega\times\vec v_r)$

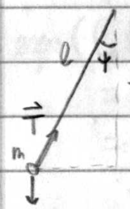
$= -2\omega v_0\cos\theta\,\hat\theta$

The particle travels $v_0 t$ (no Coriolis)

It moves $-\omega v_0\cos\theta\cdot t^2$ in the same time in the horizontal direction

$$\frac{-\omega v_0\cos\theta\, t^2}{v_0 t} = -\omega\cos\theta\, t$$

$\Delta/t = \omega\cos\theta$

23. 

$T = T\cdot\frac{x}{l}\hat x + T\cdot\frac{y}{l}\hat y + 0$

Since we say ψ is a small angle

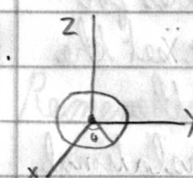
$\vec F = -mg\hat z + \vec T - 2m(\vec\omega\times\vec v)$

$$\vec\omega\times\vec v = \begin{vmatrix}\hat i & \hat j & \hat k\\ 0 & \omega\sin\theta & \omega\cos\theta\\ v_x & v_y & v_z\end{vmatrix} = (\omega v_z\sin\theta - \omega v_y\cos\theta)\hat x + \omega v_x\cos\theta\,\hat y - \omega x_x\sin\theta\,\hat z$$

We only care about $x + y$ directions since we know that the pendulum should not move much in the z-direction

$\ddot x = Tx/l - 2(\omega v_z\sin\theta - \omega v_y\cos\theta)$

$\ddot y = Ty/l - 2\omega v_x\cos\theta$

24. 

$f = F_N\cdot\mu$

$= -2m(\vec\omega\times\vec v_r)\cdot\mu$

$= 2m\mu\omega v$

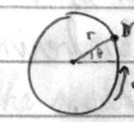
The normal force comes from the Coriolis effect.

Needs to be balanced against centrifugal force

$F = m\omega^2 r$

$m\omega^2 r = 2m\mu\omega v$

$r = 2\mu v/\omega = 0.1\,cm$

25. 

$\omega(t) = .12\ rev/s = 2\pi(.12)\ rad/s$

$F_{eff} = \vec F - 2m(\vec\omega\times\vec v_r) - m\vec\omega\times(\vec\omega\times\vec r)$ (4.91)

ignore $\dot r + \ddot r$ terms since the ball is stationary in $\hat r$

$F_{eff} = -mr\dot\theta^2\hat r + mr\ddot\theta\hat\theta$

$= -3\cdot7\cdot((2\pi)(.12))^2\hat r + 3\cdot7\cdot2\pi\cdot(.02)$

$= -11.938\hat r + 2.639\hat\theta$

$|\vec F| = 12.226N$

in $\tan^{-1}(2.639/11.938) = 12.465°$