Shankar Solutions

Benjamin D. Suh

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Chapter 1

Mathematical Introduction

1.1 Linear Vector Spaces: Basics

1.1.1 Vector Space Axioms

Verify these claims. For the first consider $|0\rangle + |0'\rangle$ and use the advertised properties of the two null vectors in turn. For the second start with $|0\rangle = (0+1)|V\rangle + |-V\rangle$. For the third, begin with $|V\rangle + (-|V\rangle) = 0|V\rangle = |0\rangle$. For the last, let $|W\rangle$ also satisfy $|V\rangle + |W\rangle = |0\rangle$. Since $|0\rangle$ is unique, this means $|V\rangle + |W\rangle = |V\rangle + |-V\rangle$. Take it from here.

Null Vector is Unique

We first assume that $|0\rangle$ and $|0'\rangle$ are unique null vectors. Thus, if we add them,

$$|0\rangle + |0'\rangle = |0\rangle$$

$$|0\rangle + |0'\rangle = |0'\rangle$$

Our two results have to be the same, so $|0\rangle = |0'\rangle$.

Scalar Product of 0

As Shankar suggests,

$$|0\rangle = (0+1)|V\rangle + |-V\rangle$$

$$= 0|V\rangle + (|V\rangle + |-V\rangle)$$

Now, using the inverse property,

$$|V\rangle + |-V\rangle = |0\rangle$$

we get

$$0|V\rangle + |0\rangle = 0|V\rangle$$

Inverse Vector

We start as Shankar suggests,

$$|V\rangle + (-|V\rangle) = |0\rangle$$

To both sides, we add $|-V\rangle$,

$$|V\rangle + |-V\rangle + (-|V\rangle) = |0\rangle + |-V\rangle$$

$$|0\rangle + (-|V\rangle) = |0\rangle + |-V\rangle$$

$$-|V\rangle = |-V\rangle$$

Unique Additive Inverse

As usual, we start by assuming the additive inverse is not unique,

$$|V\rangle + |W\rangle = |0\rangle$$

We then break up the right side using the normal inverse,

$$|V\rangle + |W\rangle = |V\rangle + |-V\rangle$$

Matching terms,

$$|W\rangle = |-V\rangle$$

1.1.2 Vector Example

Consider the set of all entities of the form (a, b, c) where the entries are real numbers. Addition and scalar multiplication are defined as follows:

$$(a, b, c) + (d, e, f) = (a + d, b + e, c + f)$$

$$\alpha(a, b, c) = (\alpha a, \alpha b, \alpha c)$$

Write down the null vector and inverse of (a, b, c). Show that vectors of the form $(a, b, 1)$ do not form a vector space.

We first write a sample vector,

$$|V\rangle = (a, b, c)$$

By inspection, we can see the null vector and inverse,

$$|0\rangle = (0, 0, 0)$$

$$|-V\rangle = (-a, -b, -c)$$

To show that vectors $(a, b, 1)$ do not form a vector space, it is easiest to show that it does not follow closure. To do this,

$$|V\rangle = (a, b, 1)$$

$$|W\rangle = (d, e, 1)$$

$$|V\rangle + |W\rangle = (a + d, b + e, 2)$$

which does not follow closure since it does not take the form $(x, y, 1)$. We could show that it does not follow certain other conditions (no null vector, no inverse).

1.1.3 More Vector Spaces

Do functions that vanish at the end points $x = 0$ and $x = L$ form a vector space? How about periodic functions obeying $f(0) = f(L)$? How about functions that obey $f(0) = 4$? If the functions do not qualify, list the things that go wrong.

Only the last is not a vector space since it does not follow closure, there is no null vector, and there is no inverse in the vector space.

1.1.4 Linear Independent Vectors

Consider three elements from the vector space of real 2x2 matrices:

$$|1\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad |2\rangle = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \quad |3\rangle = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}$$

Are they linearly independent? Support your answer with details. (Notice we are calling these matrices vectors and using kets to represent them to emphasize their role as elements of a vector space.)

We can determine if these elements are linearly independent by,

$$a\,|1\rangle + b\,|2\rangle + c\,|3\rangle = 0$$

If we can find (a, b, c) that are not all 0, the elements are linearly dependant. We get the equations,

$$\begin{cases} 0 + b - 2c = 0 \\ \\ a + b - c = 0 \end{cases}$$

Solving, we get

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Since we found elements that are not the trivial solution, these elements are linearly dependant.

1.1.5 Linear Dependent

Show that the following row vectors are linearly dependent: $(1, 1, 0)$**,** $(1, 0, 1)$**, and** $(3, 2, 1)$**. Show the opposite for** $(1, 1, 0)$**,** $(1, 0, 1)$**, and** $(0, 1, 1)$**.**

For the first, we need to solve

$$a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We can find a non-trivial solution,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

For the second set,

$$a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

There is no solution other than the trivial solution, so we have a linear independent set.

1.2 Inner Product Spaces

1.3 Dual Spaces and the Dirac notation

1.3.1 Orthonormal Basis

Form an orthonormal basis in two dimensions starting with $\vec{A} = 3\vec{i} + 4\vec{j}$ and $\vec{B} = 2\vec{i} - 6\vec{j}$. Can you generate another orthonormal basis starting with these two vectors? If so, produce another.

We can get one orthonormal basis by using Gram-Schmidt on $|A\rangle$.

$$|1\rangle = \frac{1}{5}(3,4)$$

$$|2'\rangle = (1,-3) - \frac{1}{25}(3,4)\begin{pmatrix}3\\4\end{pmatrix}(1,-3)$$

$$= (1,-3) + \frac{9}{25}(3,4)$$

$$|2\rangle = \frac{1}{5}(4,-3)$$

We can get a second orthonormal basis by starting with $|B\rangle$,

$$|1\rangle = \frac{1}{\sqrt{10}}(1,-3)$$

$$|2'\rangle = (3,4) - \frac{1}{10}(1,-3)\begin{pmatrix}1\\-3\end{pmatrix}(3,4)$$

$$= (3,4) + \frac{9}{10}(1,-3)$$

$$|2\rangle = \frac{1}{\sqrt{10}}(3,1)$$

1.3.2 Gram-Schmidt

Show how to go from the basis

$$|I\rangle = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}; \quad |II\rangle = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}; \quad |III\rangle = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

to the orthonormal basis

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad |2\rangle = \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}; \quad |3\rangle = \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

We start by normalizing $|I\rangle$,

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The second ket,

$$|2'\rangle = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - 0$$

Normalizing,

$$|2\rangle = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

The third vector,

$$|3'\rangle = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} - \frac{12}{5} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - 0$$

$$|3'\rangle = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Normalizing,

$$|3\rangle = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

1.3.3 Schwarz Inequality

When will this equality be satisfied? Does this agree with your experience with arrows?

The equality we are looking at is the Schwarz inequality,

$$\langle V|V\rangle \geq \frac{\langle W|V\rangle \langle V|W\rangle}{|W|^2}$$

We see that the two sides are equal if $|W\rangle = a\,|V\rangle$, i.e., if $|W\rangle$ is a re-scaling of $|V\rangle$.

$$\langle V|V\rangle = |V|^2 = \frac{a^* \langle V|V\rangle\, a \langle V|V\rangle}{a^2|V|^2}$$

1.3.4 Triangle Inequality

Prove the triangle inequality starting with $|V+W|^2$. You must use $\Re\langle V|W\rangle \leq |\langle V|W\rangle$ and the Schwarz inequality. Show that the final inequality becomes an equality only if $|V\rangle = a|W\rangle$ where a is a real positive scalar.

We start by finding the length of $|V\rangle + |W\rangle$,

$$|V+W|^2 = \langle V+W|V+W\rangle$$

$$= \langle V|V\rangle + \langle W|V\rangle + \langle V|W\rangle + \langle W|W\rangle$$

$$= \langle V|V\rangle + \langle V|W\rangle^* + \langle V|W\rangle + \langle W|W\rangle$$

Using the Schwarz inequality,

$$\leq |V|^2 + 2\langle V|W\rangle + |W|^2$$

$$\leq |V|^2 + 2|V||W| + |W|^2$$

$$= (|V| + |W|)^2$$

Taking the square root of both sides,

$$|V+W| \leq |V| + |W|$$

To show the equality, we go back to

$$|V+W|^2 = \langle V|V\rangle + \langle W|V\rangle + \langle V|W\rangle + \langle W|W\rangle$$

To make this an equality, we want the two terms to be equal, and we want to get

$$= (|V| + |W|)^2$$

This holds if $\langle W|V\rangle = \langle V|W\rangle$, which implies $|V\rangle = a|W\rangle$

1.4 Subspaces

1.4.1 Orthogonal Vector Subspace

In a space $\mathscr{V}^n$, prove that the set of all vectors $\{|V_\perp^1\rangle, |V_\perp^2\rangle, ...\}$, orthogonal to any $|V\rangle \neq |0\rangle$, form a subspace $\mathscr{V}^{n-1}$.

One way to think about this is imagining a three-dimensional vector space. If we choose some vector $|V\rangle$, which we'll say is $(0, 0, 1)$, we can convince ourselves that the orthogonal vectors form a plane. We then have a two-dimensional vector space whose orthonormal basis is given by $(1, 0)$ and $(0, 1)$. We can further convince ourselves that this two-dimensional vector space is a subspace of the three-dimensional vector space.

1.4.2 Adding Vector Spaces

Suppose $\mathcal{V}_1^{n_1}$ and $\mathcal{V}_2^{n_2}$ are two subspaces such that any element of $\mathcal{V}_1$ is orthogonal to any element of $\mathcal{V}_2$. Show that the dimensionality of $\mathcal{V}_1 \bigoplus \mathcal{V}_2$ is $n_1 + n_2$. (Hint: Theorem 4).

The aforementioned theorem 4: The dimensionality of a space equals $n_\perp$, the maximum number of mutually orthogonal vectors in it.

At first glance, $\dim(V_1 + V_2) = n_1 + n_2 + n_{else}$, where n_{else} is made up of vectors formed by combining V_1 and V_2. However, there are no new vectors since all elements of V_1 are orthogonal to a vector in V_2 and vice-versa. What this means is that these new vectors can be described as a linear composition of the basis of V_1 or V_2.

1.5 Linear Operators

1.6 Matrix Elements of Linear Operators

1.6.1 Sample Operator

An operator Ω is given by the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

What is its action?

Let's act our sample operator on a basis,

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We get,

$$\begin{cases} \Omega\,|1\rangle = |2\rangle \\ \\ \Omega\,|2\rangle = |3\rangle \\ \\ \Omega\,|3\rangle = |1\rangle \end{cases}$$

which we recognize as permutation.

1.6.2 Hermitian Operators

Given Ω and Λ are Hermitian what can you say about

$\Omega\Lambda$

Taking the adjoint of $\Omega\Lambda$,

$$(\Omega\Lambda)^\dagger = \Lambda^\dagger\Omega^\dagger$$

Using the Hermitian condition (1.8),

$$= \Lambda\Omega$$

which is not Hermitian.

$\Omega\Lambda + \Lambda\Omega$

Using the previous part,

$$\Omega\Lambda + \Lambda\Omega = (\Lambda\Omega)^\dagger + (\Omega\Lambda)^\dagger$$

We can rearrange the order of sums (but not products),

$$(\Omega\Lambda)^\dagger + (\Lambda\Omega)^\dagger = (\Omega\Lambda + \Lambda\Omega)^\dagger$$

which is Hermitian.

$[\Omega, \Lambda]$

Taking the adjoint,

$$[\Omega, \Lambda]^\dagger = (\Omega\Lambda)^\dagger - (\Lambda\Omega)^\dagger$$

$$= \Lambda^\dagger\Omega^\dagger - \Omega^\dagger\Lambda^\dagger = -[\Omega^\dagger, \Lambda^\dagger]$$

which is anti-Hermitian.

$i[\Omega, \Lambda]$

Taking the adjoint,

$$(i[\Omega, \Lambda])^\dagger = -i(\Omega\Lambda - \Lambda\Omega)^\dagger$$

Using part c,

$$= i[\Omega, \Lambda]$$

which is Hermitian.

1.6.3 Unitary Operators

Show that a product of unitary operators is unitary

Say we have two unitary operators, U and V. Now, let's perform the operation,

$$(UV)(UV)^\dagger$$

$$= UVV^\dagger U^\dagger$$

Since V is unitary, the middle part goes to identity,

$$= UIU^\dagger = UU^\dagger = I$$

1.6.4 Unitary Operator Determinant

It is assumed that you know (1) what a determinant is, (2) that $\det \Omega^T = \det \Omega$ **(**T **denotes transpose), (3) that the determinant of a product of matrices is the product of the determinants. [If you do not, verify these properties for a two-dimensional case**

$$\Omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\det \Omega = (\alpha\delta - \beta\gamma)$**.] Prove that the determinant of a unitary matrix is a complex number of unit modulus.**

We'll start with the unitary condition (1.9),

$$\Omega\Omega^\dagger = I$$

We take the determinant of both sides and use the third property,

$$\det \Omega \cdot \det \Omega^\dagger = \det I$$

We know that the determinant of identity is 1,

$$\det \Omega \cdot \det \Omega^\dagger = 1$$

Using the second condition,

$$|\det \Omega|^2 = 1$$

1.6.5 Unitary Operator

Verify that $R\left(\frac{1}{2}\pi\hat{i}\right)$ is unitary (orthogonal) by examining its matrix.

As a reminder, we're going to call this operator R_x,

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Not only is the rotation matrix unitary (as we will show), it is also Hermitian,

$$R_x R_x^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

1.6.6 Unitary Matrices

Verify that the following matrices are unitary. Verify that the determinant is of the form $\exp(i\theta)$ in each case. Are any of the above matrices Hermitian?

$$\Omega = \frac{1}{2^{1/2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

We can verify by brute force,

$$\Omega\Omega^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & \frac{-i}{2} + \frac{i}{2} \\ \frac{i}{2} - \frac{i}{2} & \frac{1}{2} + \frac{1}{2} \end{bmatrix} = I$$

$$\det\Omega = \frac{1}{2} - \frac{i^2}{2} = \frac{1}{2} + \frac{1}{2} = 1 = \exp(0)$$

This matrix is not Hermitian.

$$\Lambda = \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

Again, using brute force,

$$\Lambda\Lambda^\dagger = \frac{1}{4} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} = I$$

$$\det\Lambda = \frac{1}{4}[(1+i)^2 - (1-i)^2] = \frac{1}{4}[1 + 2i - 1 - 1 + 2i + 1] = i = \exp\left(i\frac{\pi}{2}\right)$$

1.7 Active and Passive Transformations

1.7.1 Trace

The trace of a matrix is defined to be the sum of its diagonal matrix elements

$$Tr(\Omega) = \sum_i \Omega_{ii}$$

Show that

Tr($\Omega\Lambda$)=Tr($\Lambda\Omega$)

We start by expanding out the trace. Then, we know that we only care about the diagonal elements,

$$Tr(\Omega\Lambda) = \sum_i (\Omega\Lambda)_{ii} = \sum_i \Omega_{ii}\Lambda_{ii}$$

Since we can rearrange scalar terms for free, we get the result we want,

$$= \sum_i \Lambda_{ii}\Omega_{ii} = \sum_i (\Lambda\Omega)_{ii} = Tr(\Lambda\Omega)$$

Tr($\Omega\Lambda\theta$)=Tr($\Lambda\theta\Omega$)=Tr($\theta\Omega\Lambda$) (The permutations are cyclic)

We want to group operators and then use the property found in the previous part,

$$Tr(\Omega\Lambda\theta) = Tr(\Omega(\Lambda\theta)) = Tr(\Lambda\theta\Omega)$$

$$Tr(\Lambda\theta\Omega) = Tr(\Lambda(\theta\Omega)) = Tr(\theta\Omega\Lambda)$$

The trace of an operator is unaffected by a unitary change of basis $|i\rangle \to U|i\rangle$. [Equivalently, show $Tr(\Omega) = Tr(U^\dagger \Omega U)$.]

We'll start with the right-hand side of that equation. Using the permutation rule,

$$Tr(U^\dagger \Omega U) = Tr(\Omega U^\dagger U) = Tr(\Omega I) = Tr(\Omega)$$

1.7.2 Unitary Change of Basis, Determinant

Show that the determinant of a matrix is unaffected by a unitary change of basis. [Equivalently show $\det\Omega = \det(U^\dagger \Omega U)$]

We start with the right side of the equation. We can break up the individual parts of the determinant,

$$\det(U^\dagger \Omega U) = \det(U^\dagger)\det(\Omega)\det(U)$$

Since each of these are scalars, we can rearrange them freely and then recombine them,

$$= \det(\Omega)\det(U^\dagger)\det(U) = \det(\Omega)\det(I) = \det(\Omega)$$

1.8 The Eigenvalue Problem

1.8.1 Eigenvalues

Find the eigenvalues and normalized eigenvectors of the matrix

$$\Omega = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

The characteristic equation,

$$\det(\Omega - \omega I) = \det \begin{bmatrix} 1-\omega & 3 & 1 \\ 0 & 2-\omega & 0 \\ 0 & 1 & 4-\omega \end{bmatrix}$$

$$= (1-\omega)(2-\omega)(4-\omega)$$

Our eigenvalues are $\omega = 1, 2, 4$.
For $|\omega = 1\rangle$,

$$\begin{bmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|\omega = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $|\omega = 2\rangle$,

$$\begin{bmatrix} -1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|\omega = 2\rangle = \frac{1}{\sqrt{30}} \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

For $|\omega = 4\rangle$,

$$\begin{bmatrix} -3 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|\omega = 4\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Is the matrix Hermitian? Are the eigenvectors orthogonal?

By inspection, the matrix is not Hermitian nor are the eigenvectors orthogonal. Note that we can't use Gram-Schmidt here since that changes the vector.

1.8.2 Eigenvectors

Consider the matrix

$$\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Is it Hermitian?

Yes.

Find its eigenvalues and eigenvectors

The characteristic equation,

$$\det(\Omega - \omega I) = \det \begin{bmatrix} -\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & -\omega \end{bmatrix}$$

$$= -\omega^3 + 1(\omega) = \omega(\omega^2 - 1)$$

The eigenvalues are $\omega = -1, 0, 1$.
For $|\omega = -1\rangle$,

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|\omega = -1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For $|\omega = 0\rangle$,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|\omega = 0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $|\omega = 1\rangle$,

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|\omega = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Verify that $U^\dagger \Omega U$ is diagonal, U being the matrix of eigenvectors of Ω

We'll start by writing out U,

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U^\dagger \Omega U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.8.3 Hermitian Matrix Eigenvalues

Consider the Hermitian matrix

$$\Omega = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Show that $\omega_1 = \omega_2 = 1$**;** $\omega_3 = 2$

The characteristic equation,

$$\det \Omega - \omega I = \det \begin{bmatrix} 1-\omega & 0 & 0 \\ 0 & \frac{3}{2}-\omega & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2}-\omega \end{bmatrix}$$

$$= (1-\omega)\left(\left(\frac{3}{2}-\omega\right)^2 - \frac{1}{4}\right) = (1-\omega)(2-\omega)(1-\omega)$$

We see that our eigenvalues are $\omega = 1, 2$.

Show that $|\omega = 2\rangle$ **is any vector of the form**

$$\frac{1}{(2a^2)^{1/2}}\begin{bmatrix} 0 \\ a \\ -a \end{bmatrix}$$

We can find the eigenvectors,

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The eigenvector is

$$|\omega = 2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

which follows the form given.

Show that the $\omega = 1$ eigenspace contains all vectors of the form

$$\frac{1}{(b^2 + 2c^2)^{1/2}} \begin{bmatrix} b \\ c \\ c \end{bmatrix}$$

either by feeding $\omega = 1$ into the equation or by requiring that the $\omega = 1$ eigenspace be orthogonal to $|\omega = 2\rangle$.

We solve

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The simplest solution is

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

which is of the form given.

1.8.4 Eigenvectors

An arbitrary $n \times n$ matrix need not have n eigenvectors. Consider as an example

$$\Omega = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

Show that $\omega_1 = \omega_2 = 3$

The characteristic equation,

$$\det \Omega - \omega I = \det \begin{bmatrix} 4-\omega & 1 \\ -1 & 2-\omega \end{bmatrix}$$

$$= (4-\omega)(2-\omega) + 1 = \omega^2 - 6\omega + 9$$

The solution is $\omega = 3$.

By feeding in this value show we get only one eigenvector of the form

$$\frac{1}{(2a^2)^{1/2}} \begin{bmatrix} +a \\ -a \end{bmatrix}$$

We cannot find another one that is LI

We solve

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$|\omega = 3\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

1.8.5 Eigenvectors

Consider the matrix

$$\Omega = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Show that it is unitary.

We can show this by brute force,

$$\Omega\Omega^\dagger = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Show that its eigenvalues are $\exp(i\theta)$ **and** $\exp(-i\theta)$

Solving the characteristic equation,

$$\det(\Omega - \omega I) = \det \begin{bmatrix} \cos(\theta) - \omega & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) - \omega \end{bmatrix}$$

$$= (\cos(\theta) - \omega)^2 + \sin^2(\theta) = \omega^2 - 2\omega\cos(\theta) + 1$$

$$\omega = \cos(\theta) \pm i\sin(\theta) = \exp(\pm i\theta)$$

Find the corresponding eigenvectors; show that they are orthogonal

---

For $\omega = \exp(i\theta)$,

$$\begin{bmatrix} -i\sin(\theta) & \sin(\theta) \\ -\sin(\theta) & -i\sin(\theta) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can actually simplify by factoring out a $\sin(\theta)$, so we just need to solve,

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$|\exp(i\theta)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ -1 \end{bmatrix}$$

For $\omega = \exp(-i\theta)$, we can perform the same simplification,

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$|\exp(-i\theta)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

To show that these are orthogonal,

$$\langle\exp(i\theta)|\exp(-i\theta)\rangle = \frac{1}{2}\begin{bmatrix} -i & -1 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}(1-1) = 0$$

Verify that $U^\dagger \Omega U =$(diagonal matrix), where U is the matrix of eigenvectors of Ω

We start with

$$U = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Now multiplying through,

$$U^\dagger \Omega U = \frac{1}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \exp(i\theta) & 0 \\ 0 & \exp(-i\theta) \end{bmatrix}$$

1.8.6 Determinant and Trace

We have seen that the determinant of a matrix is unchanged under a unitary change of basis. Argue now that

$$\det(\Omega) = product\ of\ eigenvalues\ of\ \Omega = \prod_{i=1}^{n} \omega_i$$

for a Hermitian or unitary Ω.

Remember Shankar 1.7.2, where we showed,

$$\det(\Omega) = \det(U^\dagger \Omega U)$$

Looking at Shankar 1.8.5, we know that $U^\dagger \Omega U$ = (diagonal matrix) whose elements are the eigenvalues. Thus,

$$\det(\Omega) = \prod_{i=1}^{n} \omega_i$$

Using the invariance of the trace under the same transformation, show that

$$Tr(\Omega) = \sum_{i=1}^{n} \omega_i$$

By the same logic, a unitary change of basis results in a diagonal matrix whose elements are the eigenvalues.

$$Tr(\Omega) = Tr(U^\dagger \Omega U) = \sum_{i=1}^{n} \omega_i$$

1.8.7 Trace and Determinant

By using the results on the trace and determinant from the last problem, show that the eigenvalues of the matrix

$$\Omega = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

are 3 and -1. Verify this by explicit computation. Note that the Hermitian nature of the matrix is an essential ingredient.

We calculate that the determinant is -3 and the trace is 2. The only combination of eigenvalues that give these results are $\omega = 3, -1$. To compute explicitly, we solve the characteristic equation,

$$\det(\Omega - \omega I) = \det \begin{bmatrix} 1-\omega & 2 \\ 2 & 1-\omega \end{bmatrix}$$

$$= \omega^2 - 2\omega - 3$$

which gives $\omega = 3, -1$.

1.8.8 Eigenvalues

Consider Hermitian matrices M^1, M^2, M^3, M^4 that obey

$$M^i M^j + M^j M^i = 2\delta^{ij} I, \quad i,j = 1, ..., 4$$

Show that the eigenvalues of M^i are ± 1. (Hint: go to the eigenbasis of M^i, and use the equation for $i = j$.

Using the suggestion,

$$M^i M^i = I$$

Taking the determinant and trace,

$$\begin{cases} \det(M^i M^i) = \det(I) = 1 \\ Tr(M^i M^i) = n \end{cases}$$

where n is the dimension of the matrix. The determinant implies that $\det(M^i) = 1$ since we can split the determinant. Furthermore, we know that the eigenvalues of Hermitian operators are real, so the only possible eigenvalues are $\omega = \pm 1$.

By considering the relation

$$M^i M^j = -M^j M^i \quad for\ i \neq j$$

show that M^i are traceless. [Hint: $Tr(ACB) = Tr(CBA)$.

We multiply both sides by M^i,

$$M^i M^j M^i = -M^j M^i M^i$$

From the previous part, we showed that $M^i M^i = I$,

$$M^i M^j M^i = -M^j$$

Taking the trace and using the given hint,

$$Tr(M^i M^j M^i = -Tr(M^j)$$

$$Tr(M^i M^i M^j) = -Tr(M^j)$$

$$Tr(M^j) = -Tr(M^j)$$

which implies that M^j is traceless.

Show that they cannot be odd-dimensional matrices.

We showed that

$$\begin{cases} \det(M^i) = \prod_{i=1}^n \omega_i = \pm 1 \\ Tr(\Omega) = \sum_{i=1}^n \omega_i = 0 \end{cases}$$

The only way for this to be true is if we have an equal number of $\omega = 1$ and $\omega = -1$, which implies that our matrices are even-dimensional.

1.8.9 Angular Momentum

A collection of masses m_α, located at $\vec{r}_\alpha$ and rotating with angular velocity $\vec{\omega}$ around a common axis has an angular momentum

$$\vec{l} = \sum_\alpha m_\alpha (\vec{r}_\alpha \times \vec{v}_\alpha)$$

where $\vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha$ is the velocity of m_α. By using the identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

show that each Cartesian component l_i of $\vec{l}$ is given by

$$l_i = \sum_j M_{ij} \omega_j$$

where

$$M_{ij} = \sum_\alpha m_\alpha [r_\alpha^2 \delta_{ij} - (\vec{r}_\alpha)_i (\vec{r}_\alpha)_j]$$

or in Dirac notation

$$|l\rangle = M\,|\omega\rangle$$

Will the angular momentum and angular velocity always be parallel

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Only if $|\omega\rangle$ is an eigenvector of M

Show that the moment of inertia matrix M_{ij} is Hermitian

We want to show

$$M_{ij} = M_{ji}$$

Let's look at each component individually. We know that $\delta_{ij} = \delta_{ji}$, so that part is pretty easy. In addition,

$$(\vec{r}_\alpha)_i (\vec{r}_\alpha)_j = (\vec{r}_\alpha)_j (\vec{r}_\alpha)_i$$

since we are just multiplying two scalar values.

Consider the moment of inertia matrix of a sphere. Due to the complete symmetry of the sphere, it is clear that every direction is its eigendirection for rotation. What does this say about the three eigenvalues of the matrix M

Every direction is an eigendirection, which means that all eigenvalues are equivalent, so the eigenvalues are degenerate.

1.8.10 Simultaneous Diagonalization

By considering the commutator, show that the following Hermitian matrices may be simultaneously diagonalized. Find the eigenvectors common to both and verify that under a unitary transformation to this basis, both matrices are diagonalized.

$$\Omega = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Since Ω is degenerate and Λ is not, you must be prudent in deciding which matrix dictates the choice of basis.

If $[\Omega, \Lambda] = 0$, there exists a common basis.

$$\Omega\Lambda = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

$$\Lambda\Omega = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

which shows that Ω and Λ commute. Let's work with Λ,

$$\det(\Lambda - \omega I) = \det \begin{bmatrix} 2-\omega & 1 & 1 \\ 1 & -\omega & -1 \\ 1 & -1 & 2-\omega \end{bmatrix}$$

$$= (\omega - 2)(\omega - 3)(\omega + 1)$$

Our eigenvalues are $\omega = 2, 3, -1$.
For $\omega = 2$,

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|2\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Similarly,

$$|3\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$|-1\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\end{bmatrix}$$

Our unitary matrix is

$$U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$U^\dagger \Omega U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$U^\dagger \Omega U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$