"It means, that though the Witch knew the Deep Magic, there is a magic deeper still which she did not know: Her knowledge goes back only to the dawn of time. But, if she could have looked a little further back, into the stillness and the darkness before Time dawned, she would have read there a different incantation. She would have known that when a willing victim who had committed no treachery was killed in a traitor's stead, the Table would crack and Death itself would start working backwards. - Aslan (The Lion, the Witch, and the Wardrobe)

Chapter 2: Variational Principles and Lagrange's Equations

Section 1 Hamilton's Principle

Say you have some system that can be described by n generalized coordinates $q_1, \ldots, q_n$. This is known as configuration space.

- Hamilton's principle describes the motion of a monogenic system i.e., a system for which all the forces can be derived from a generalized scalar potential.

The motion from time t_1 to time t_2 is such that

$$I = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

$\mathcal{L} = T - V$ is stationary for the actual path of the motion

$$\delta I = \delta \int_{t_1}^{t_2} \mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)\, dt = 0$$

Section 2 Some techniques of the calculus of variations

- In one-dimension, take the function $f(y, \dot{y}, x)$ defined on a path $y(x)$ between x_1 & x_2.

$$\dot{y} = \frac{dy}{dx}$$

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x)\, dx$$

Want J to be constant for some function $y(x)$ and paths infinitesimally close to $y(x)$.

We also want to define configuration space $\mathcal{C} = \{\{y_r(x)\}_{r=1,\ldots,n}\}$.

Each point in $\mathcal{C}$ is denoted by $X(t) = \{y_1(x), y_2(x), \ldots y_n(x)\}$

- We can then define each path as $y(x, \alpha)$ with $y(x, 0)$ as the original function.

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x,\alpha), \dot{y}(x,\alpha), x)\, dx$$

$$\left(\frac{dJ}{d\alpha}\right)_{\alpha=0} = 0$$

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}} \cdot \frac{\partial \dot{y}}{\partial \alpha}\right) dx$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}} \cdot \frac{\partial \dot{y}}{\partial \alpha}\, dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}} \cdot \frac{\partial^2 y}{\partial x \partial \alpha}\, dx$$

$$u = \frac{\partial f}{\partial \dot{y}} \qquad v = \frac{\partial y}{\partial \alpha}$$

$$\frac{du}{dx} = \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} \qquad \frac{dv}{dx} = \frac{\partial^2 y}{\partial x \partial \alpha}$$

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx \quad \text{Integration by parts}$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}} \cdot \frac{\partial^2 y}{\partial x \partial \alpha}\, dx = \cancel{\frac{\partial f}{\partial \dot{y}} \cdot \frac{\partial y}{\partial \alpha}\Big|_{x_1}^{x_2}}^{\,0} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) \frac{\partial y}{\partial \alpha}\, dx$$

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \cdot \frac{\partial f}{\partial \dot{y}}\right) \frac{\partial y}{\partial \alpha}\, dx$$

$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \cdot \frac{\partial f}{\partial \dot{y}}\right) \left(\frac{\partial y}{\partial \alpha}\right)_0 dx = 0 \quad \text{Condition for stationary value}$$

- $\int_{x_1}^{x_2} M(x)\, \eta(x)\, dx = 0$ Fundamental lemma of calculus of variations

If above is true for all arbitrary functions $\eta(x)$ continuous through the second derivative, $M(x) = 0$ in the interval (x_1, x_2).

$$M(x) = \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}$$

$$\eta(x) = \left(\frac{\partial y}{\partial \alpha}\right)_0$$

$$\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx} \cdot \frac{\partial f}{\partial \dot{y}} = 0$$

What follows are some examples of calculus of variations. Let's work through one of them.

- Shortest distance between two points in a plane.

$ds = \sqrt{dx^2 + dy^2}$ element of length

$I = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ total length of curve

$$f = \sqrt{1 + \dot{y}^2}$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial \dot{y}} = (1 + \dot{y}^2)^{-1/2} \cdot \tfrac{1}{2} \cdot 2\dot{y} = \dot{y}(1 + \dot{y}^2)^{-1/2}$$

$$\frac{d}{dx}\left[\dot{y}(1 + \dot{y}^2)^{-1/2}\right] = 0$$

$$\Rightarrow \dot{y}(1 + \dot{y}^2)^{-1/2} = c$$

$$\dot{y}^2 = c^2(1 + \dot{y}^2)$$

$$\dot{y}^2(1 - c^2) = c^2$$

$$\dot{y} = \frac{c}{\sqrt{1 - c^2}}$$

$$y = \frac{c}{\sqrt{1 - c^2}} x + b$$

$= ax + b$ which is the equation of a straight line

Before going on to derive Lagrange's equation, let's first go over functionals and functional derivatives

Let's define a functional F as $F[f(x)]$ or a function of a function.

$$\delta F = \int \frac{\delta F}{\delta f(x)} \delta f(x)\, dx$$

There is much more math and rigour that could go into this, but I'm much more interested in how to use them.

$$\frac{\delta(\lambda F + \mu G)[\rho]}{\delta\rho(x)} = \lambda \frac{\delta F[\rho]}{\delta\rho(x)} + \mu \frac{\delta G[\rho]}{\delta\rho(x)}$$

$$\frac{\delta(FG)[\rho]}{\delta\rho(x)} = \frac{\delta F[\rho]}{\delta\rho(x)} G[\rho] + F[\rho] \frac{\delta G[\rho]}{\delta\rho(x)}$$

$$\frac{\delta F[G[\rho]]}{\delta\rho(y)} = \int \frac{\delta F[G]}{\delta G(x)}\Big|_{G=G[\rho]} dx \cdot \frac{\delta G[\rho](x)}{\delta\rho(y)}$$

- Example: Thomas-Fermi kinetic energy functional

$$T_{TF}[\rho] = C_F \int \rho^{5/3}(\vec{r})\, d\vec{r}$$

$$\frac{\delta T_{TF}}{\delta\rho(\vec{r})} = C_F \cdot \frac{\partial\rho^{5/3}(\vec{r})}{\partial\rho(\vec{r})}$$

$$= C_F \cdot \tfrac{5}{3}\rho^{2/3}(\vec{r})$$

Section 3. Derivation of Lagrange's Equations from Hamilton's Principle

- $S[q(t)] = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, dt$

$$S[q+\delta q] = \int_{t_1}^{t_2} \mathcal{L}(q+\delta q, \dot{q}+\delta\dot{q}, t)\, dt$$

$$S[q+\delta q] - S[q] \simeq \delta S[q] = \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta\dot{q}\right] dt$$

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial\mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right]\delta q\, dt = 0 \quad \text{stationary}$$

$$\frac{\partial\mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) = 0 \quad \text{Euler Lagrange equation}$$

Section 4. Extending Hamilton's Principle to Systems with Constraints

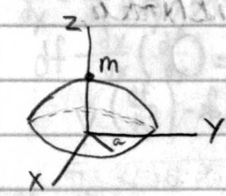
- Must use the Lagrange multipliers

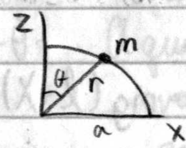
$$I = \int_1^2 \left(\mathcal{L} + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha\right) dt$$

$$\delta I = \int_1^2 \left(\sum_{i=1}^{n}\left(\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} - \frac{\partial\mathcal{L}}{\partial q_i} + \sum_{\alpha=1}^{m}\lambda_\alpha \frac{\partial f_\alpha}{\partial q_i}\right)\delta q_i\right) dt = 0$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_k} - \frac{\partial\mathcal{L}}{\partial q_k} + \sum_{\alpha=1}^{m}\lambda_\alpha\frac{\partial f_\alpha}{\partial q_k} = 0$$

Example of Lagrangian multipliers: mass rolling on a semi-sphere





$$\mathcal{L} = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

Constraints:

$$y = 0$$

$$f(r) = a - r$$

$$r^2 = x^2 + z^2 = r^2\sin^2\theta + r^2\cos^2\theta \qquad a - r = 0$$

$$\mathcal{L} = \tfrac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta \qquad r = a$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} + \lambda\frac{\partial f}{\partial r} = 0$$

$$\left.\begin{aligned} & m\ddot{r} - (mr\dot{\theta}^2 - mg\cos\theta) + \lambda(-1) = 0 \\ & mr^2\ddot{\theta} - (mgr\sin\theta) = 0 \end{aligned}\right\} \text{Lagrange equations}$$

Next, solve for λ, which gives the equation for the constraint force

$$\lambda = mg\cos\theta - ma\dot{\theta}^2$$

$$mr^2\ddot{\theta} = mgr\sin\theta$$

$$\frac{d\dot{\theta}}{dt} = \tfrac{g}{a}\sin\theta$$

$$\frac{d\dot{\theta}}{d\theta}\cdot\frac{d\theta}{dt} = \tfrac{g}{a}\sin\theta$$

$$\dot{\theta}\, d\dot{\theta} = \tfrac{g}{a}\sin\theta\, d\theta$$

$$\tfrac{1}{2}\dot{\theta}^2 = \tfrac{g}{a}(1-\cos\theta)$$

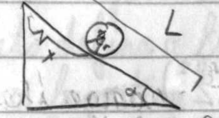
$$\dot{\theta}^2 = \tfrac{2g}{a}(1-\cos\theta)$$

$$\lambda = mg\cos\theta - 2mg(1-\cos\theta)$$

$$= mg(3\cos\theta - 2)$$

$\lambda = 0$ when $\cos\theta = 2/3$, which gives the angle when the ball leaves the sphere.

Example: Hoop of mass m rolling down a ramp



Rolling constraint: $f = r\,d\theta - dX = 0$

$r\dot{\theta} - \dot{X} = 0$

$\mathcal{L} = \frac{1}{2} m \dot{X}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 - mg(l - X)\sin\alpha$

$m\ddot{X} - mg\sin\alpha + \lambda = 0$

$mr^2\ddot{\phi} - \lambda r = 0$

$r\ddot{\phi} = \ddot{X} \Rightarrow m\ddot{X} = \lambda$

$\ddot{X} = \frac{g\sin\alpha}{2}$, $\lambda = \frac{mg\sin\alpha}{2}$, $\ddot{\phi} = \frac{g\sin\alpha}{2r}$

2.5. Advantages of a variational principle formulation

2.6. Conservation Theorems and Symmetry Properties

If we have $f(q_1, q_2, \dots \dot{q}_1, \dot{q}_2, \dots, t) = \text{constant}$, first order differential equations, as equations of motion, this gives us the conservation laws

$\Pi_r = \frac{\partial \mathcal{L}}{\partial \dot{q}_r}$ Canonical momentum

$\frac{\partial \mathcal{L}}{\partial q_r} = 0 \Rightarrow q_r$ is a cyclic coordinate ($\dot{q}_r$ can still show up)

$\frac{d\Pi_r}{dt} = 0 \Rightarrow \Pi_r$ is conserved

The generalized momentum conjugate to a cyclic coordinate is conserved

2.7. Energy function and the conservation of energy

If $\mathcal{L} = \mathcal{L}(q_r, \dot{q}_r, t)$

$\frac{d\mathcal{L}}{dt} = \sum_r \frac{\partial \mathcal{L}}{\partial q_r}\dot{q}_r + \frac{\partial \mathcal{L}}{\partial \dot{q}_r}\ddot{q}_r + \frac{\partial \mathcal{L}}{\partial t}$

$h(q_r, \dot{q}_r, t) = \sum_r \dot{q}_r \Pi_r - \mathcal{L}$ Energy function

$\frac{dh}{dt} = 0 \Rightarrow h$ is conserved

For velocity-independent potentials, $h = T + V$

If constraints are rheonomous and $\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow h$ is conserved but $h \neq T + V$

Derivations

1. $f = \sqrt{\frac{1+\dot{y}^2}{2gy}} = \sqrt{1+\dot{y}^2}\,(2gy)^{-1/2}$

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$

$\partial f - \frac{dy}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = a$

$\dot{y}\left(\frac{\partial f}{\partial \dot{y}}\right) - f = a$

A quick note on a: a will change the value of a throughout this integration rather than creating a new variable every time it changes. I know this is not mathematically rigorous

$\frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{(2gy)(1+\dot{y}^2)}}$

$\frac{\dot{y}^2}{\sqrt{2gy(1+\dot{y}^2)}} - \sqrt{\frac{1+\dot{y}^2}{2gy}} = a$

$\frac{\dot{y}^2 - (1+\dot{y}^2)}{\sqrt{2gy(1+\dot{y}^2)}} = a$

$\frac{-1}{\sqrt{2gy(1+\dot{y}^2)}} = a$

$a = 2gy(1+\dot{y}^2)$

$\dot{y}^2 = \frac{a}{y} - 1$

$\frac{dy}{dx} = \sqrt{\frac{a-y}{y}}$

$dx = \int \sqrt{\frac{y}{a-y}}\,dy$ $\quad y = a\sin^2(\phi/2)$

$= \int \sqrt{\frac{a\sin^2(\phi/2)}{a - a\sin^2(\phi/2)}}\,a\sin(\phi/2)\cos(\phi/2)\,d\phi$ $\quad dy = a \cdot 2 \cdot \frac{1}{2}\sin(\phi/2)\cos(\phi/2)\,d\phi$

$= \int \sqrt{\frac{\sin^2(\phi/2)}{\cos^2(\phi/2)}}\,a\sin(\phi/2)\cos(\phi/2)\,d\phi$ $\quad = a\sin(\phi/2)\cos(\phi/2)\,d\phi$

$= a\int \sin^2(\phi/2)\,d\phi$

$x = a(\phi - \sin(\phi/2))$

$y = a\sin^2(\phi/2) = a(1 - \cos\phi)$

If $\frac{1}{2}mv^2 = mgy + \frac{1}{2}mv_0^2$

$v^2 = 2gy + v_0^2$

So $f = \sqrt{\frac{1+\dot{y}^2}{2gy + v_0^2}}$, then follow the same integration, which gives

$x = a(\phi - \sin\phi)$

$y = a(1 - \cos\phi) - v_0^2/2g$

$y_{\ldots} = 0$

$a = \frac{v_0^2}{2g(1-\cos\theta)}$

$y_{max} = \frac{v_0^2}{2g}$

2. Nearest I can tell, this is closest to (2.50).

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = \frac{\partial T}{\partial \dot\theta} - \frac{\partial U}{\partial \dot\theta}$$

$\frac{\partial T}{\partial \dot\theta} = \mathcal{L}_\theta$, and the proof for this is provided in the book, so let's look at $\frac{\partial U}{\partial \dot\theta}$

$$\frac{\partial U}{\partial \dot\theta} = \frac{\partial U}{\partial \vec{r}_i} \cdot \frac{\partial \vec{r}_i}{\partial \dot\theta} + \frac{\partial U}{\partial \dot{\vec{r}}_i} \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot\theta}$$

$$\frac{\partial \vec{r}_i}{\partial \dot\theta} = 0$$

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot\theta} = \hat{n} \times \vec{r}_i$$

$$p_\theta = \mathcal{L}_\theta - \sum_i \frac{\partial U}{\partial \dot{\vec{r}}_i} \cdot (\hat{n} \times \vec{r}_i)$$

$$= \mathcal{L}_\theta - \sum_i \hat{n} \cdot \vec{r}_i \times \vec{\nabla}_{\vec{v}_i} U$$

For electromagnetic forces, use the same argument.

$$U = q\phi - \frac{q}{c}\vec{A} \cdot \vec{v} \qquad (1.62)$$

$$\frac{\partial U}{\partial \dot{\vec{r}}_i} = -\frac{q}{c}\vec{A}$$

$$p_\theta = \mathcal{L}_\theta + \sum_i \hat{n} \times \vec{r}_i \times {}^{q_i}/_{c} \vec{A}_i$$

3. $ds = \sqrt{dx^2 + dy^2 + dz^2}$

$$I = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx$$

$$f = \sqrt{1 + \dot{y}^2 + \dot{z}^2}$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot y}\right) = 0 \qquad (2.11)$$

$$-\frac{d}{dx}\left(\frac{\dot y}{\sqrt{1+\dot y^2 + \dot z^2}}\right) = 0$$

$$\dot y / \sqrt{1+\dot y^2+\dot z^2} = a \qquad \dot y = a\sqrt{1+\dot y^2 + \dot z^2}$$

$$\dot z / \sqrt{1+\dot y^2+\dot z^2} = b$$

$$\dot z = c\dot y$$

Now plug this back in to get

$$\dot y / \sqrt{1+(1+c^2)\dot y^2} = a$$

$$y = \alpha x + \beta$$

We can do the same for z, giving $z = \gamma x + \varepsilon$

These two define planes, and the intersection forms a line

4. $ds = \sqrt{d\theta^2 + d\phi^2}$ because we don't want to deal with the radius, let's set $r = 1$

$$I = \int_{\phi_1}^{\phi_2} \sqrt{1+\dot\theta^2}\, d\phi$$

$$f = \sqrt{1+\dot\theta^2}$$

$$\frac{d}{d\phi}\left(\tfrac{1}{2}(1+\dot\theta^2)^{-1/2} \cdot 2\dot\theta\right) = 0$$

$$\Rightarrow \dot\theta = \text{constant}$$

N.B. return to this

5. $\mathcal{L} = \frac{m}{2} \cdot \dot x^2 + Fx$

Let's look at the bounds first

$$x(0) = 0 = A$$

$$x(t_0) = a = Bt_0 + Ct_0^2$$

Now let's look at the Euler-Lagrange equation

$$\frac{d}{dt}(m\dot x) - F = 0$$

$$m\ddot x = F$$

$$2C = F/m$$

$$C = F/2m$$

$$a = Bt_0 + \frac{F}{2m}t_0^2$$

$$B = a/t_0 - Ft_0/2m$$

$$x(t) = (a/t_0 - Ft_0/2m)t + Ft^2/2m$$

6. $M_E = 4\pi \int_0^R r^2 \rho\, dr$

$$= \frac{4\pi}{3}R^3\rho$$

$$\rho = \frac{3M_E}{4\pi R^3}$$

$$M(r) = \frac{M_E r^3}{R^3}$$

$$V = -\int_0^r F\, dr' = \int_0^r \frac{GM_E r'^3 \cdot m}{R^3 r'^2}\, dr'$$

$$= \int_0^r \frac{GM_E m}{R^3} r'\, dr = \frac{GM_E m r'^2}{2R^3}$$

$$E = T + V = \frac{mv^2}{2} + \frac{GM_E m r^2}{2R^3}$$

If we say $v = 0$ at $r = R$

$$\frac{GM_E m R^2}{2R^3} = \frac{mv^2}{2} + \frac{GM_E m r^2}{2R^3}$$

$$v = \sqrt{\frac{GM}{R^3}(R^2 - r^2)}$$

$$t = \int_A^B \frac{ds}{v} = \int_A^B \sqrt{\frac{(x'^2 + y'^2)R}{g(R^2 - (x^2 + y^2))}}$$

$$f - x'\frac{df}{dx} = 0$$

Exercises 7. $x_1 = x_2$

$y_2 = -y_1$

Start with the solution found in example 2 of section 2.2

$$x = a\cosh\left(\frac{y-b}{a}\right)$$

$$x_1 = a\cosh\left(\frac{y_1 - b}{a}\right) = a\cosh\left(\frac{y_2 + b}{a}\right)$$

$$\cosh\left(\frac{y_1 - b}{a}\right) = \cosh\left(\frac{-y_1 - b}{a}\right)$$

$$\cosh\left(\frac{y_1 - b}{a}\right) = \cosh\left(\frac{y_1 + b}{a}\right) \qquad \cosh(x) = \cosh(-x)$$

$$y_1 - b = y_1 + b$$

$$-b = b$$

$$b = 0$$

$$x = a\cosh(y/a)$$

$$x_2 = a\cosh\left(\frac{y_2}{a}\right)$$

$$x_2/a = \cosh\left(\frac{y_2}{x_2} \cdot \frac{x_2}{a}\right)$$

$$k = \cosh(\alpha \cdot k)$$

Taking the derivative according to k

$$1 = \sinh(k\alpha_0) \cdot \alpha_0$$

$$1 = \alpha_0^2 \sinh^2(k\alpha_0)$$

$$\cosh^2(k\alpha_0) - \sinh^2(k\alpha_0) = \alpha_0^2 \sinh^2(k\alpha_0) \qquad \sinh^2(x) + 1 = \cosh^2(x)$$

$$k^2 - 1/\alpha_0^2 = 1$$

$$\alpha_0 = 1/\sqrt{k^2 - 1}$$

$$k = \cosh\left(\frac{k}{\sqrt{k^2 - 1}}\right)$$

$$k \approx 1.81$$

$$\alpha_0 = .66$$

$\alpha = \cosh(k)/k$. Graph this

8. $\pi(y_1^2+y_2^2)$ Goldschmidt solution

$x_1 = a\cosh(y_1/a)$

$x_2 = a\cosh(y_2/a)$

$k = x_2/a$

$\alpha = y_2/x_2$

For the symmetric case, $A_g = \pi(2y_2^2)$ from the Goldschmidt solution
$= 2\pi y_2^2$

Area from $x_2 = a\cosh(y_2/a)$ gives $A_c = 2\cdot 2\pi\int_0^{y_2} x\sqrt{1+\dot{y}^2}\,dx$

$$= \frac{4\pi}{a}\int_0^{y_2} x^2\, d\cancel{y}$$

$$= 4\pi/a \int_0^{x_2} a^2\cosh^2(y/a)\,dy$$

$$= \pi a^2\left[\sinh(2y_2/a) + 2y_2/a\right]$$

$$\frac{A_c}{A_g} = \frac{\pi a^2\left[\sinh(2y_2/a) + 2y_2/a\right]}{2\pi y_2^2} = \frac{a^2}{2y_2^2}\cdot\frac{x_2^2}{x_2^2}\left[\sinh\left(\frac{2}{a}\cdot y_2\cdot\frac{x_2}{x_2}\right) + \frac{2y_2}{a}\cdot\frac{x_2}{x_2}\right]$$

$$= \frac{1}{2k^2\alpha^2}\left[\sinh(2\alpha k) + 2\alpha k\right]$$

$$= \sinh(2\alpha k)/2\alpha^2k^2 + 1/2\alpha k$$

For large α:

$$\sinh(2\alpha k) = \exp(2\alpha k) - \exp(-2\alpha k)/2 \approx \tfrac{1}{2}\left[(1+2\alpha k+2\alpha^2k^2) - (1-2\alpha k+2\alpha^2k^2)\right]$$

$$= \tfrac{1}{2}(4\alpha k) = 2\alpha k$$

$$\frac{A_c}{A_g} \approx \sinh(2\alpha k)/2\alpha^2k^2$$

$$\approx 2\alpha k/2\alpha^2k^2 = 1/\alpha k < 1$$

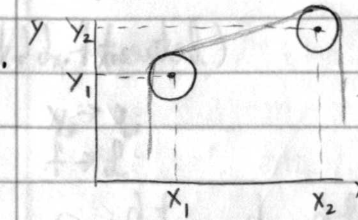
$\Rightarrow A_c < A_g$ for a sufficiently large α

For $\alpha = \alpha_0$

$\alpha_0 = 1/\sqrt{k^2-1}$

$$\frac{A_c}{A_g} \approx \frac{1}{\alpha k} + \frac{1}{2\alpha k} = \frac{3}{2\alpha k} = \frac{3}{2}\cdot\frac{\sqrt{k^2-1}}{k} > 1$$

Assuming $k \gg 1$

9. 

The chain will assume whatever shape will produce the least amount of potential energy

$$E = \int_0^L g\cdot y(s)\cdot\lambda\, ds$$

$$ds = \sqrt{1+\dot{y}^2}\,dy$$

$$E = \lambda g\int_{x_1}^{x_2} y\sqrt{1+\dot{y}^2}\,dx$$

Since y is a function of x, this reduces to the problem of minimum surface of revolution

10. $y = at + bt^2$

$y_0 = a\sqrt{2y_0/g} + 2by_0/g$

$\mathcal{L} = \tfrac{1}{2}m\dot{y}^2 - mgy$

$\int_0^{t_0}\mathcal{L}\,dt$

$\frac{\partial\mathcal{L}}{\partial y} = -mg \qquad \frac{\partial\mathcal{L}}{\partial\dot{y}} = m\dot{y}$

$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) = m\ddot{y}$

$-mg - m\ddot{y} = 0$

$\ddot{y} = -g$

$\dot{y}(t) = -gt + a \qquad \dot{y}(0) = 0 \Rightarrow a = 0$

$y(t) = -\tfrac{g}{2}t^2 + c$

Here, we run into a bit of notation. I originally defined the ground as $y=0$, but it looks like the problem should actually have been defined such that the starting position is $y=0$. This leads to

$y(t) = \tfrac{g}{2}t^2$ being the extremum

11. $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i$ (between 1.66 & 1.67)

$= F$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} \cdot dt = F\,dt$$

$$\int_{\Delta t} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \int_{\Delta t} \frac{\partial \mathcal{L}}{\partial q_i}\,dt = \int F\,dt$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\partial}{\partial q_i}(T-V) = -\frac{\partial V}{\partial q_i}$$ which is independent of time

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)_f - \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)_i = S_j$$

12. $$\frac{\partial J}{\partial \alpha}\,d\alpha = \int_1^2 \left(\frac{\partial f}{\partial y_i}\cdot\frac{\partial y_i}{\partial \alpha}\,d\alpha + \frac{\partial f}{\partial \dot{y}_i}\cdot\frac{\partial \dot{y}_i}{\partial \alpha}\,d\alpha + \frac{\partial f}{\partial \ddot{y}_i}\cdot\frac{\partial \ddot{y}_i}{\partial \alpha}\,d\alpha\right)dx$$

$$\int \frac{\partial f}{\partial \dot{y}}\cdot\frac{\partial^2 y}{\partial \alpha\, \partial x}\,dx = \frac{\partial f}{\partial \dot{y}}\cdot\frac{\partial y}{\partial \alpha}\Big|_1^2 - \int_1^2 \frac{\partial y}{\partial \alpha}\cdot\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)dx$$

$u = \frac{\partial f}{\partial \dot{y}}$ $\quad v = \frac{\partial y}{\partial \alpha}$

$du = \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)$ $\quad dv = \frac{\partial^2 y}{\partial \alpha\, \partial x}$

$$= -\int_1^2 \frac{\partial y}{\partial \alpha}\cdot\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)dx$$

$$\int \frac{\partial f}{\partial \ddot{y}}\cdot\frac{\partial^2 \dot{y}}{\partial \alpha\, \partial x}\,dx = \frac{\partial f}{\partial \ddot{y}}\cdot\frac{\partial \dot{y}}{\partial \alpha}\Big|_1^2 - \int_1^2 \frac{\partial \dot{y}}{\partial \alpha}\cdot\frac{d}{dx}\left(\frac{\partial f}{\partial \ddot{y}}\right)dx$$

$u = \frac{\partial f}{\partial \ddot{y}}$ $\quad v = \frac{\partial \dot{y}}{\partial \alpha}$

$du = \frac{d}{dx}\left(\frac{\partial f}{\partial \ddot{y}}\right)$ $\quad dv = \frac{\partial^2 \dot{y}}{\partial \alpha\, \partial x}$

$$= -\int_1^2 \frac{\partial \dot{y}}{\partial \alpha}\cdot\frac{d}{dx}\left(\frac{\partial f}{\partial \ddot{y}}\right)dx$$

$$-\int_1^2 \frac{\partial^2 y}{\partial \alpha\, \partial x}\cdot\frac{d}{dx}\left(\frac{\partial f}{\partial \ddot{y}}\right) = -\frac{\partial y}{\partial \alpha}\cdot\frac{d}{dx}\left(\frac{\partial f}{\partial \ddot{y}}\right)\Big|_1^2 + \int_1^2 \frac{\partial y}{\partial \alpha}\cdot\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial \ddot{y}}\right)dx$$

$u = \frac{d}{dx}\left(\frac{\partial f}{\partial \ddot{y}}\right)$ $\quad v = \frac{\partial y}{\partial \alpha}$

$du = \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial \ddot{y}}\right)$ $\quad dv = \frac{\partial^2 y}{\partial \alpha\, \partial x}$

$$= +\int_1^2 \frac{\partial \dot{y}}{\partial \alpha}\cdot\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial \ddot{y}}\right)dx$$

$x \to t$

$y_i \to q_i$

$f \to \mathcal{L}$

$$0 = \frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{d^2}{dt^2}\cdot\frac{\partial \mathcal{L}}{\partial \ddot{q}_i}$$

For $\mathcal{L} = -\frac{m}{2} q\ddot{q} - \frac{k}{2} q^2$

$$\frac{\partial \mathcal{L}}{\partial \ddot{q}} = -\frac{m}{2} q$$

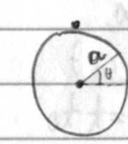
$$\frac{\partial \mathcal{L}}{\partial q} = -\frac{m}{2}\ddot{q} - kq$$

$$0 = -m\ddot{q}/2 - kq - m\ddot{q}/2$$

$$= -m\ddot{q} - kq$$

$\ddot{q} = -k/m\, q$ Equation of motion for a simple harmonic oscillator

13.



$\mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\sin\theta$

Constraints: $f = r - a = 0$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} + \sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f}{\partial q_k} = 0 \qquad (2.22)$$

$q_k = r$: $\frac{d}{dt}(m\dot{r}) - (mr\dot{\theta}^2 - mg\sin\theta) + \lambda = 0$

$$m\ddot{r} - mr\dot{\theta}^2 + mg\sin\theta + \lambda = 0$$

Applying the condition $r = a$, $\dot{r} = \ddot{r} = 0$,

$$-ma\dot{\theta}^2 + mg\sin\theta + \lambda = 0$$

$q_k = \theta$: $\frac{d}{dt}(mr^2\dot{\theta}) - (mgr\cos\theta) = 0$

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} - mgr\cos\theta = 0$$

Apply constraint

$$ma^2\ddot{\theta} - mga\cos\theta = 0$$

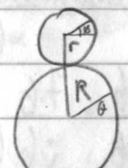
$$ma\dot{\theta}^2 - mg\sin\theta - \lambda = 0$$
$$a\ddot{\theta} = g\cos\theta$$
$$a\dot{\theta}\ddot{\theta} - g\dot{\theta}\cos\theta = 0$$
$$\tfrac{1}{2}a\dot{\theta}^2 - g\sin\theta = c \quad \text{integrate}$$

Plugging in initial condition: $\dot{\theta} = 0$, $\theta = \pi/2$

$$-g = c$$
$$\tfrac{1}{2}a\dot{\theta}^2 = g(\sin\theta - 1)$$
$$a\dot{\theta}^2 = 2g(\sin\theta - 1)$$
$$2mg(\sin\theta - 1) - mg\sin\theta - \lambda = 0$$
$$mg\sin\theta - 2mg = \lambda$$
$$\lambda = mg(\sin\theta - \tfrac{1}{2})$$

The particle falls off when $\lambda = 0$, so $\theta = 30^\circ$

$$h = r + r/2 = 3r/2$$

14. 

$$\rho = r + R$$
$$\mathcal{L} = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2) + \tfrac{1}{2}mr^2\dot{\phi}^2 - mg\rho\sin\theta$$

Constraints:

- Hoops are touching: $f_1 = \rho - a = 0$
- Not slipping: $f_2 = (r+R)\theta - r\phi = 0$

I'm not sure why this isn't written as $\rho\theta - r\phi$, but the solutions I looked at had it this way. It also seems to eliminate a multiplier in one of the equations of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} + \sum_{\alpha=1}^{m} \lambda_\alpha \cdot \frac{\partial f_\alpha}{\partial q_k} = 0$$

$q_k = \rho$: $\frac{d}{dt}(m\dot{\rho}) - (m\rho\dot{\theta}^2 - mg\sin\theta) + \lambda_1 = 0$

$$m\ddot{\rho} - m\rho\dot{\theta}^2 + mg\sin\theta + \lambda_1 = 0$$
$$ma\dot{\theta}^2 - mg\sin\theta - \lambda_1 = 0$$

$q_k = \theta$: $\frac{d}{dt}(m\rho^2\dot{\theta}) - (-mg\rho\cos\theta) + (r+R)\lambda_2 = 0$

$$ma^2\ddot{\theta} + mga\cos\theta + a\lambda_2 = 0$$

$q_k = \phi$: $\frac{d}{dt}(mr^2\dot{\phi}) + (-r)\lambda_2 = 0$

$$mr^2\ddot{\phi} - r\lambda_2 = 0$$

Since $\rho\theta = r\phi$, $\rho\dot{\theta} = r\dot{\phi}$, $\rho\ddot{\theta} = r\ddot{\phi}$

$$mr^2\ddot{\phi} - r\lambda_2 = 0$$
$$mr\ddot{\phi} = \lambda_2$$
$$m\rho\ddot{\theta} = \lambda_2$$

$$ma^2\ddot{\theta} + mga\cos\theta + ma^2\ddot{\theta} = 0$$
$$a\ddot{\theta} + g\cos\theta = 0$$
$$a\dot{\theta}\ddot{\theta} + g\dot{\theta}\cos\theta = 0$$
$$\tfrac{1}{2}a\dot{\theta}^2 + g\sin\theta = c$$

at $t = 0$, $\dot{\theta} = 0$, $\theta = \pi/2$

$$c = g$$
$$\tfrac{1}{2}a\dot{\theta}^2 = g(1 - \sin\theta)$$
$$a\dot{\theta}^2 = 2g(1 - \sin\theta)$$
$$ma\dot{\theta}^2 - mg\sin\theta - \lambda_1 = 0$$
$$2mg(1 - \sin\theta) - mg\sin\theta = \lambda_1$$
$$\lambda_1 = 2mg - 3mg\sin\theta$$
$$= mg(2 - 3\sin\theta)$$

$\lambda_1 = 0$ when the particle falls off

$$\theta = \sin^{-1}(2/3) \approx 41.8^\circ$$
$$h = 2(R+r)/3$$

15. Pass

16. $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$

$\frac{d}{dt}(\exp(\gamma t) m\dot{q}) - (\exp(\gamma t)(-kq)) = 0$

$m\gamma \exp(\gamma t)\dot{q} + m\exp(\gamma t)\ddot{q} + k\exp(\gamma t)q = 0$

$m\ddot{q} + m\gamma\dot{q} + kq = 0$

which we recognize as damped harmonic motion

To determine if a coordinate is cyclic (constant of motion)

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0$ We see this is not true, thus there are no constants of motion

$s = \exp(\gamma t/2)\, q$

$\dot{s} = \frac{\gamma}{2}\exp(\gamma t/2)\, q + \exp(\gamma t/2)\,\dot{q}$

$q^2 = s^2/\exp(\gamma t)$

$\dot{q}^2 = 1/\exp(\gamma t)\,(\dot{s} - \frac{\gamma}{2}s)^2$

$\mathcal{L} = m/2\,(\dot{s} - \gamma s/2)^2 - k/2\,(s)^2$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{s}}\right) - \frac{\partial \mathcal{L}}{\partial s} = 0$

$\frac{d}{dt}(m(\dot{s} - \gamma s/2)) - (m(\dot{s} - \gamma s/2)\cdot(-\gamma/2) - ks) = 0$

$m\ddot{s} - m\gamma\dot{s}/2 + m\gamma\dot{s}/2 - m\gamma^2 s/4 + ks = 0$

$m\ddot{s} + (k - m\gamma^2/4)s = 0$

Harmonic oscillator

17. $\mathcal{L} = T - V$

$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}}\right) - \frac{\partial (T-V)}{\partial q} = 0$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial (T-V)}{\partial q} = 0$

$\frac{d}{dt}(2\dot{q} f(q)) - \dot{q}^2 \frac{\partial}{\partial q} f(q) + \frac{\partial}{\partial q} V(q) = 0$

$2\ddot{q} f(q) + 2\dot{q}\frac{df(q)}{dt} - \dot{q}^2\frac{\partial}{\partial q} f(q) + \frac{\partial}{\partial q} V(q) = 0$

$2\ddot{q} f(q) + 2\dot{q}\frac{df(q)}{dq}\cdot\frac{dq}{dt} - \dot{q}^2\frac{\partial f(q)}{\partial q} + \frac{\partial}{\partial q} V(q) = 0$

$2\ddot{q} f(q) + \dot{q}^2\frac{\partial f(q)}{\partial q} + \frac{\partial V(q)}{\partial q} = 0$

$\dot{q}\ddot{q}\, dt = 1/2\, d\dot{q}^2$

$\frac{\partial f}{\partial q}\dot{q}\, dt = \frac{\partial f}{\partial q}\cdot dq = df$

$2\dot{q}\ddot{q} f(q)\, dt + \dot{q}^2 \frac{\partial f(q)}{\partial q}\dot{q}\, dt + \frac{\partial V(q)}{\partial q}\dot{q}\, dt = 0$

$2\cdot 1/2\cdot f(q)\, d\dot{q}^2 + \dot{q}^2\, df + dV = 0$

$f\, d\dot{q}^2 + \dot{q}^2\, df + dV = 0$

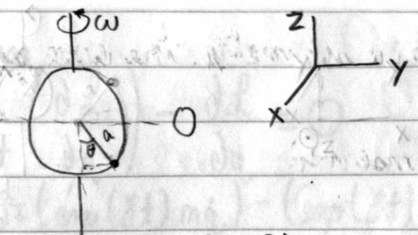
$f\dot{q}^2 + V = T + V$

$= E$

$\dot{q} = \pm\sqrt{\frac{E-V}{f(q)}}$

$dt = \pm\sqrt{\frac{f(q)}{E-V}}\, dq$

$t - t_0 = \pm\int_{q_0}^{q}\sqrt{\frac{f(q)}{E-V}}\, dq$

18. 

$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$V = -mgz$

$x = a \sin\theta \cos(\omega t)$ $\qquad \dot{x} = a\dot{\theta}\cos\theta\cos(\omega t) - a\omega\sin\theta\sin(\omega t)$

$y = a\sin\theta\sin(\omega t)$ $\qquad \dot{y} = a\dot{\theta}\cos\theta\sin(\omega t) + a\omega\sin\theta\cos(\omega t)$

$z = a\cos\theta$ $\qquad \dot{z} = -a\dot{\theta}\sin\theta$

$$T = \frac{m}{2}\big(a^2\dot{\theta}^2\cos^2\theta\cos^2(\omega t) - 2a^2\dot{\theta}\omega\cos\theta\sin\theta\cos(\omega t)\sin(\omega t) + a^2\omega^2\sin^2\theta\sin^2(\omega t)$$
$$+ a^2\dot{\theta}^2\cos^2\theta\sin^2(\omega t) + 2a^2\dot{\theta}\omega\cos\theta\sin\theta\cos(\omega t)\sin(\omega t) + a^2\omega^2\sin^2\theta\cos^2(\omega t)$$
$$+ a^2\dot{\theta}^2\sin^2\theta)$$

$\mathcal{L} = \frac{m}{2}(a^2\dot{\theta}^2 + a^2\omega^2\sin^2\theta) + mga\cos\theta$

a remains constant as well as $\phi (= \omega t)$, but we declare those as constants, so I'm not sure if they count.

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

$$\frac{d}{dt}(ma^2\dot{\theta}) - (ma^2\omega^2\sin\theta\cos\theta - mga\sin\theta) = 0$$

$$ma^2\ddot{\theta} - ma^2\omega^2\sin\theta\cos\theta + mga\sin\theta = 0$$

$$\ddot{\theta} - \omega^2\sin\theta\cos\theta + \frac{g}{a}\sin\theta = 0$$

For the particle to be stationary, $\dot{\theta} = 0 = \ddot{\theta}$

$$-\omega^2\sin\theta\cos\theta + \frac{g}{a}\sin\theta = 0$$

$$\omega^2\cos\theta = g/a$$

$$\omega^2 = g/a\cos\theta$$

$$\omega = \sqrt{g/a\cos\theta}$$

Below $\omega = \sqrt{g/a}$, the particle remains at the bottom. Above that it can have another stationary point.

19. pg. 60, if the mass distribution has a symmetry, the corresponding variables will be conserved

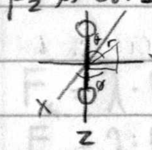
a. Symmetric in x + y ⇒ p_x and p_y are conserved

b. Symmetric in x ⇒ p_x is conserved

c. In cylindrical coordinates, z and ϕ are symmetric ⇒ p_z + p_ϕ are conserved

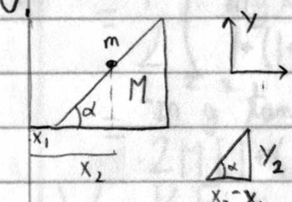
d. p_ϕ is conserved

e. p_z is conserved

f.  p_ϕ is conserved

g. Say the distance between each coil is h. Then, for p to remain constant, we need to translate and rotate

$p_z + hp_\phi/2\pi$

20. 

$$\mathcal{L} = \frac{m}{2}(\dot{x}_2^2 + \dot{y}_2^2) + \frac{M}{2}(\dot{x}_1^2 + \dot{y}_1^2) - mgy_2$$
$$= \frac{m}{2}(\dot{x}_2^2 + \dot{y}_2^2) + \frac{M}{2}(\dot{x}_1^2) - mgy_2$$

$\tan\alpha = \frac{y_2}{x_2 - x_1}$ $\qquad f = y_2 - (x_2 - x_1)\tan\alpha = 0$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) - \frac{\partial\mathcal{L}}{\partial q} + \lambda\cdot\frac{\partial f}{\partial q} = 0 \qquad (2.22)$$

$$\frac{d}{dt}(m\dot{x}_2) - 0 + \lambda(-\tan\alpha) = 0$$

$$m\ddot{x}_2 - \lambda\tan\alpha = 0$$

$$\frac{d}{dt}(m\dot{y}_2) - (-mg) + \lambda = 0$$

$$m\ddot{y}_2 + mg + \lambda = 0$$

$$\frac{d}{dt}(M\dot{x}_1) + \lambda\tan\alpha = 0$$

$$M\ddot{x}_1 + \lambda\tan\alpha = 0$$

$m\ddot{x}_2 - \lambda\tan\alpha = 0$ $\quad m\ddot{x}_2 = \lambda\tan\alpha$

$m\ddot{y}_2 + mg + \lambda = 0$ $\quad \lambda = -m\ddot{y}_2 - mg$

$M\ddot{x}_1 + \lambda\tan\alpha = 0$ $\quad M\ddot{x}_1 = -\lambda_1\tan\alpha$

$$y_2 - (x_2 - x_1)\tan\alpha = 0$$

$$\ddot{y}_2 - (\ddot{x}_2 - \ddot{x}_1)\tan\alpha = 0$$

$$m\ddot{y}_2 - (m\ddot{x}_2 - m\ddot{x}_1)\tan\alpha = 0$$

$$\ddot{x}_2 = \frac{\lambda}{m}\tan\alpha$$

$$\ddot{x}_1 = -\frac{\lambda}{M}\tan\alpha$$

$$\ddot{y}_2 = -\frac{\lambda}{m} - g = (\ddot{x}_2 - \ddot{x}_1)\tan\alpha$$

$$= (\lambda/m + \lambda/M)\tan^2\alpha$$

$$-\lambda - mg = (\lambda + \lambda m/M)\tan^2\alpha$$

$$-mg = \lambda + \lambda(1 + m/M)\tan^2\alpha$$

$$\lambda = \frac{-mg}{1 + (1 + \frac{m}{M})\tan^2\alpha}$$

$$\ddot{x}_1 = \frac{mg\tan\alpha}{M + (M+m)\tan^2\alpha}$$

$$\ddot{x}_2 = \frac{-g\tan\alpha}{1 + (1 + m/M)\tan^2\alpha}$$

$$\ddot{y}_2 = \frac{g}{1 + (1 + m/M)\tan^2\alpha} - g$$

$$= \frac{-g(1 + m/M)\tan^2\alpha}{1 + (1 + m/M)\tan^2\alpha}$$

The total momentum in the x-direction is constant as shown by

$$m\ddot{x}_2 + M\ddot{x}_1 = 0$$

$$m\dot{x}_2 + M\dot{x}_1 = c$$

If the wedge is fixed, $m/M = 0$

$$\ddot{x}_1 = \frac{m}{M}\cdot\frac{g\tan\alpha}{1 + (1 + \frac{m}{M})\tan^2\alpha} = 0$$ wedge doesn't move

$$\ddot{x}_2 = \frac{-g\tan\alpha}{1 + \tan^2\alpha} = \frac{-g\cdot\frac{\sin\alpha}{\cos\alpha}}{\frac{\cos^2\alpha + \sin^2\alpha}{\cos^2\alpha}} = -g\frac{\sin\alpha}{\cos\alpha}\cdot\cos^2\alpha = -g\sin\alpha\cos\alpha$$

$$\ddot{y}_2 = \frac{-g\tan^2\alpha}{1 + \tan^2\alpha} = -g\sin^2\alpha$$

$$F_{x_1} = \lambda\cdot\frac{\partial G}{\partial x_1} = \lambda\tan\alpha$$

$$F_{x_2} = \lambda\cdot\frac{\partial G}{\partial x_2} = -\lambda\tan\alpha$$

$$F_{y_2} = \lambda\cdot\frac{\partial G}{\partial y_2} = \lambda$$

$$W_1 = \int F_{x_1}\,dx_1 = \int F_{x_1}\dot{x}_1\,dt$$

$$= \tfrac{1}{2}F_{x_1}\ddot{x}_1 t^2$$

$$= \frac{1}{2}\left(\frac{-mg\tan\alpha}{1 + (1 + \frac{m}{M})\tan^2\alpha}\right)\left(\frac{-g\tan\alpha}{1 + (1 + \frac{m}{M})\tan^2\alpha}\cdot\frac{m}{M}\right)t^2$$

$$= \frac{-m^2g^2\tan^2\alpha\; t^2}{2M[1 + (1 + m/M)\tan^2\alpha]^2}$$

$$W_2 = \tfrac{1}{2}F_{x_2}\ddot{x}_2 t^2 + \tfrac{1}{2}F_{y_2}\ddot{y}_2 t^2$$

$$= \frac{1}{2}\left(\frac{mg\tan\alpha}{1 + (1 + \frac{m}{M})\tan^2\alpha}\right)\left(\frac{-g\tan\alpha}{1 + (1 + m/M)\tan^2\alpha}\right)t^2$$

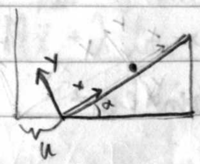
$$+ \frac{1}{2}\left(\frac{-mg}{1 + (1 + m/M)\tan^2\alpha}\right)\left(\frac{-g(1 + m/M)\tan^2\alpha}{1 + (1 + m/M)\tan^2\alpha}\right)t^2$$

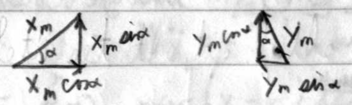
$$= \frac{-mg^2\tan^2\alpha\, t^2}{2[1 + (1 + m/M)\tan^2\alpha]^2} + \frac{mg^2(1 + m/M)\tan^2\alpha\, t^2}{2[1 + (1 + m/M)\tan^2\alpha]^2}$$

$$= \frac{-g^2m^2\tan^2\alpha\, t^2}{2M[1 + (1 + m/M)\tan^2\alpha]^2}$$

$$W_1 + W_2 = 0$$

Now, let's solve with the y-axis normal to the wedge





$$x_m = u + x\cos\alpha - y\sin\alpha \qquad f = y = 0$$
$$y_m = x\sin\alpha + y\cos\alpha \qquad \dot y = 0$$
$$\mathcal{L} = \tfrac{m}{2}(\dot x_m^2 + \dot y_m^2) + \tfrac{M}{2}\dot u^2 - mgy_m \qquad \ddot y = 0$$
$$= \tfrac{m}{2}[(\dot u + \dot x\cos\alpha - \dot y\sin\alpha)^2 + (\dot x\sin\alpha + \dot y\cos\alpha)^2] + \tfrac{M\dot u^2}{2} - mg(x\sin\alpha + y\cos\alpha)$$
$$= \tfrac{m}{2}[\dot u^2 + \dot u\dot x\cos\alpha - \dot u\dot y\sin\alpha + \dot u\dot x\cos\alpha + \dot x^2\cos^2\alpha - \cancel{\dot x\dot y\cos\alpha\sin\alpha}$$
$$- \dot u\dot y\sin\alpha - \cancel{\dot x\dot y\cos\alpha\sin\alpha} + \dot y^2\sin^2\alpha + \dot x^2\sin^2\alpha + \cancel{2\dot x\dot y\sin\alpha\cos\alpha} + \dot y^2\cos^2\alpha]$$
$$+ \tfrac{M\dot u^2}{2} - mg(x\sin\alpha + y\cos\alpha)$$
$$= \tfrac{m}{2}[\dot u^2 + 2\dot u\dot x\cos\alpha - 2\dot u\dot y\sin\alpha + \dot x^2 + \dot y^2] + \tfrac{M\dot u^2}{2} - mg(x\sin\alpha + y\cos\alpha)$$

$$\frac{d}{dt}(m\dot u + m\dot x\cos\alpha - m\dot y\sin\alpha + M\dot u) = 0$$
$$m\ddot u + m\ddot x\cos\alpha - m\ddot y\sin\alpha + M\ddot u = 0$$ conservation of momentum
$$\frac{d}{dt}(m\dot u\cos\alpha + m\dot x) - (-mg\sin\alpha) = 0$$
$$m\ddot u\cos\alpha + m\ddot x + mg\sin\alpha = 0$$
$$\frac{d}{dt}(-m\dot u\sin\alpha + m\dot y) - (-mg\cos\alpha) + \lambda = 0$$
$$-m\ddot u\sin\alpha + m\ddot y + mg\cos\alpha + \lambda = 0$$

$$(m+M)\ddot u + m\ddot x\cos\alpha = 0 \qquad m\ddot x = -(m+M)\ddot u/\cos\alpha$$
$$m\ddot u\cos\alpha + m\ddot x + mg\sin\alpha = 0$$
$$-m\ddot u\sin\alpha + mg\cos\alpha + \lambda = 0$$

$$m\ddot u\cos\alpha - (m+M)\ddot u/\cos\alpha + mg\sin\alpha = 0$$
$$m\ddot u\cos^2\alpha - m\ddot u - M\ddot u + mg\cos\alpha\sin\alpha = 0$$
$$-m\ddot u\sin^2\alpha - M\ddot u = -mg\cos\alpha\sin\alpha$$
$$\ddot u = \frac{mg\cos\alpha\sin\alpha}{M + m\sin^2\alpha}$$

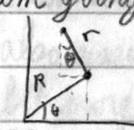
$$m\ddot x = -\frac{(m+M)}{\cos\alpha}\cdot\frac{mg\cos\alpha\sin\alpha}{M+m\sin^2\alpha} =$$
$$\ddot x = -\frac{(m+M)g\sin\alpha}{M+m\sin^2\alpha}$$

$$-\frac{m^2g\cos\alpha\sin^2\alpha}{M+m\sin^2\alpha} + mg\cos\alpha + \lambda = 0$$
$$\lambda = \frac{m^2g\cos\alpha\sin^2\alpha - (M+m\sin^2\alpha)mg\cos\alpha}{M+m\sin^2\alpha}$$
$$= \frac{-mMg\cos\alpha}{M+m\sin^2\alpha}$$

If the wedge is fixed. $m/M = 0$
$$\ddot u = \frac{g\cos\alpha\sin\alpha}{1+\frac{m}{M}\sin^2\alpha}\cdot\frac{m}{M} = 0$$
$$\ddot x = -g\sin\alpha$$

21.

a. I'm going to draw this without springs



$$x = R\cos\theta - r\sin\theta \qquad \theta = \omega t$$
$$y = R\sin\theta + r\cos\theta$$
$$x = R\cos(\omega t) - r\sin(\omega t)$$
$$\dot x = \dot R\cos(\omega t) - R\omega\sin(\omega t) - \dot r\sin(\omega t) - r\omega\cos(\omega t)$$
$$y = R\sin(\omega t) + r\cos(\omega t)$$
$$\dot y = \dot R\sin(\omega t) + R\omega\cos(\omega t) + \dot r\cos(\omega t) - r\omega\sin(\omega t)$$
$$\dot x^2 + \dot y^2 = \dot R^2\cos^2(\omega t) - 2R\dot R\omega\cos(\omega t)\sin(\omega t) - 2\dot R\dot r\cos(\omega t)\sin(\omega t) - 2\dot R r\omega\cos^2(\omega t)$$
$$+ R^2\omega^2\sin^2(\omega t) + 2R\dot r\omega\sin^2(\omega t) + \cancel{2Rr\omega^2\sin(\omega t)\cos(\omega t)} + \dot r^2\sin^2(\omega t) + \cancel{2r\dot r\omega\sin(\omega t)\cos(\omega t)}$$
$$+ r^2\omega^2\cos^2(\omega t)$$
$$+ \dot R^2\sin^2(\omega t) + 2R\dot R\omega\cos(\omega t)\sin(\omega t) + 2\dot R\dot r\cos(\omega t)\sin(\omega t) - 2\dot R r\omega\sin^2(\omega t)$$
$$+ R^2\omega^2\cos^2(\omega t) + 2R\dot r\omega\cos^2(\omega t) - 2Rr\omega^2\sin(\omega t)\cos(\omega t) + \dot r^2\cos^2(\omega t)$$
$$\cancel{-2r\dot r\omega\sin(\omega t)\cos(\omega t)} + r^2\omega^2\sin^2(\omega t)$$

$= \dot{R}^2 - 2\dot{R}r\omega + R^2\omega^2 + 2R\dot{r}\omega + \dot{r}^2 + r^2\omega^2$

$= \dot{R}^2 - 2\dot{R}r\omega + r^2\omega^2 + \dot{r}^2 + 2R\dot{r}\omega + R^2\omega^2$

$= (\dot{R} - r\omega)^2 + (\dot{r} + R\omega)^2$

$T = \frac{m}{2}[(\dot{R} - r\omega)^2 + (\dot{r} + R\omega)^2]$

$V = \frac{k}{2} r^2 + \frac{K}{2}(R - r_0)^2$ I'm going by the diagram, not the words

$x = R\cos\theta - r\sin\theta$ $\quad R = x\cos(\omega t) + y\sin(\omega t)$

$y = R\sin\theta + r\cos\theta$ $\quad r = -x\sin(\omega t) + y\cos(\omega t)$

Alternatively,

$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$

$V = \frac{k}{2}(-x\sin(\omega t) + y\cos(\omega t))^2 + \frac{K}{2}(x\cos(\omega t) + y\sin(\omega t) - r_0)^2$

b. $h = \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} + \dot{y}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \mathcal{L}$ (2.53)

$\mathcal{L} = T - V$

$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$

$\frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y}$

$h = T + V$, which is not conserved since there is a time-dependant term. Also, since V does not depend on the generalized velocities, $h = E$ (2.58), energy is not conserved

c. Use the first $T + V$ that were derived in a

$h = \dot{R}\frac{\partial \mathcal{L}}{\partial \dot{R}} + \dot{r}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \mathcal{L}$ $\quad \mathcal{L} = T - V$

$\frac{\partial \mathcal{L}}{\partial \dot{R}} = m(\dot{R} - r\omega)$

$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m(\dot{r} - R\omega)$

$h = m\dot{R}^2 - \cancel{mr\dot{R}\omega} + m\dot{r}^2 - mR\dot{r}\omega - \frac{m\dot{R}^2}{2} + \cancel{mr\dot{R}\omega} - \frac{mr^2\omega^2}{2}$

$- \frac{m\dot{r}^2}{2} - mR\dot{r}\omega - \frac{mR^2\omega^2}{2} + V$

$= \frac{m\dot{R}^2}{2} - \frac{m\dot{r}^2}{2} - 2mR\dot{r}\omega - \frac{mr^2\omega^2}{2} - \frac{mR^2\omega^2}{2} + V$

$\frac{dh}{dt} \neq 0$ Energy is still not conserved

22. $E = T + V$

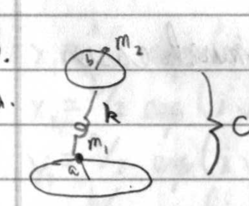
$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dV}{dt} = \frac{\partial T}{\partial q}\cdot\dot{q} + \frac{\partial T}{\partial \dot{q}}\ddot{q} + \frac{\partial V}{\partial q}\dot{q} + \frac{\partial T}{\partial t} + \frac{\partial V}{\partial t}$

$\frac{d}{dT}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -\frac{\partial V}{\partial q} + \tilde{Q}$

$\frac{dE}{dt} = \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} + \frac{\partial V}{\partial q} - \tilde{Q}\right)\dot{q} + \frac{\partial T}{\partial \dot{q}}\ddot{q} + \frac{\partial T}{\partial t} + \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial t}$

$= \frac{d}{dT}\frac{\partial T}{\partial \dot{q}} + \frac{2\partial V}{\partial q}\dot{q} + \frac{\partial T}{\partial t} + \frac{\partial V}{\partial t} - \tilde{Q}\dot{q}$

23.

a. 

$x_1 = a\cos\theta_1$ $\quad x_2 = b\cos\theta_2$

$y_1 = a\sin\theta_1$ $\quad y_2 = b\sin\theta_2$

$z_1 = 0$ $\quad z_2 = c$

$\dot{x}_1 = -a\dot{\theta}_1\sin\theta_1$ $\quad \dot{x}_2 = -b\dot{\theta}_2\sin\theta_2$

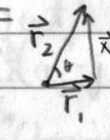
$\dot{y}_1 = a\dot{\theta}_1\cos\theta_1$ $\quad \dot{y}_2 = b\dot{\theta}_2\cos\theta_2$

$\dot{z}_1 = 0$ $\quad \dot{z}_2 = 0$

$\mathcal{L} = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) - \frac{k}{2}[(x_2 - x_1)^2 + (y_2 - y_1)^2 + c^2] - m_2 g c$

Let's also find $\mathcal{L}$ in cylindrical coordinates. Note that we also assume the equilibrium position $= 0$ (the math gets a bit more complicated than I would like).

$(x_1 - x_2)^2 + (y_1 - y_2)^2 =$



$\vec{x} = \vec{r}_2 - \vec{r}_1$

$|\vec{x}|^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + c^2$

$\mathcal{L} = \frac{m_1}{2}(\dot{r}_1^2 + r_1^2\dot{\theta}_1^2) + \frac{m_2}{2}(\dot{r}_2^2 + r_2^2\dot{\theta}_2^2) - \frac{k}{2}[r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + c^2] - m_2 g c$

b. $a = x_1^2 + y_1^2$ $\qquad f_1 = x_1^2 + y_1^2 - a = 0$,

$b = x_2^2 + y_2^2$ $\qquad = r_1 - a = 0$

$$f_2 = x_2^2 + y_2^2 - b = 0$$
$$= r_2 - b = 0$$

$$m_1 \ddot{x}_1 + k(x_1 - x_2) + \lambda_1 \cdot 2x_1 = 0$$
$$m_2 \ddot{x}_2 - k(x_1 - x_2) + \lambda_2 \cdot 2x_2 = 0$$
$$m_1 \ddot{y}_1 + k(y_1 - y_2) + \lambda_1 \cdot 2y_1 = 0$$
$$m_2 \ddot{y}_2 - k(y_1 - y_2) + \lambda_2 \cdot 2y_2 = 0$$
$$x_1 = \frac{a}{2} \exp\left(i\sqrt{2\lambda/m_1}\, t\right) + \frac{a}{2} \exp\left(-i\sqrt{2\lambda_1/m_1}\, t\right)$$

and so on. Now it turns out, this problem is much easier to solve in polar coordinates.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} + \lambda \frac{\partial f}{\partial q} = 0$$
$$m_1 \ddot{r}_1 - \left(m_1 r_1 \dot{\theta}_1^2 - \tfrac{k}{2}\left(2r_1 - 2r_2 \cos(\theta_2 - \theta_1)\right)\right) + \lambda_1 = 0$$
$$m_1 \ddot{r}_1 - \left(m_1 r_1 \dot{\theta}_1^2 - k\left(r_1 - r_2 \cos(\theta_2 - \theta_1)\right)\right) + \lambda_1 = 0$$
$$m_2 \ddot{r}_2 - \left(m_2 r_2 \dot{\theta}_2^2 - k\left(r_2 - r_1 \cos(\theta_1 - \theta_2)\right)\right) + \lambda_2 = 0$$
$$m_1 r_1^2 \ddot{\theta}_1 + k r_1 r_2 \sin(\theta_1 - \theta_2) = 0$$
$$m_2 r_2^2 \ddot{\theta}_2 - k r_1 r_2 \sin(\theta_1 - \theta_2) = 0$$

$$\dot{r}_1 = \dot{r}_2 = \ddot{r}_1 = \ddot{r}_2 = 0$$
$$-m_1 r_1 \dot{\theta}_1^2 + k r_1 - k r_2 \cos(\theta_1 - \theta_2) + \lambda_1 = 0$$
$$-m_2 r_2 \dot{\theta}_2^2 + k r_2 - k r_1 \cos(\theta_1 - \theta_2) + \lambda_2 = 0$$
$$m_1 r_1^2 \ddot{\theta}_1 + k r_1 r_2 \sin(\theta_1 - \theta_2) = 0$$
$$m_2 r_2^2 \ddot{\theta}_2 - k r_1 r_2 \sin(\theta_1 - \theta_2) = 0$$
$$m_1 r_1^2 \ddot{\theta}_1 + m_2 r_2^2 \ddot{\theta}_2 = 0$$
$$m_1 r_1^2 \dot{\theta}_1 + m_2 r_2^2 \dot{\theta}_2 = 0$$
$$\dot{\theta}_2 = -\dot{\theta}_1 \left(\frac{m_1 r_1^2}{m_2 r_2^2}\right)$$

$$-m_1 r_1 \dot{\theta}_1^2 + m_2 r_2 \dot{\theta}_2^2 + \lambda_1 - \lambda_2 = 0$$
$$\lambda_1 = \lambda_2 + m_1 r_1 \dot{\theta}_1^2 - m_2 r_2 \dot{\theta}_2^2$$

$$\sigma(m_1 r_1^2 \ddot{\theta}_1) = \sigma(-k r_1 r_2 \sin(\theta_1 - \theta_2))$$
$$\int m_1 r_1^2 \ddot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2)\, dt = -k r_1 r_2 \int \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\, dt$$
$$\lambda_1 = k\left[b \cos(\theta_2 - \theta_1)\left(1 + \frac{2}{1 + \frac{m_1 a^2}{m_2 b^2}}\right) - a\right]$$
$$\lambda_2 = k\left[a \cos(\theta_1 - \theta_2)\left(1 + \frac{2}{1 + \frac{m_2 b^2}{m_1 a^2}}\right) - b\right]$$

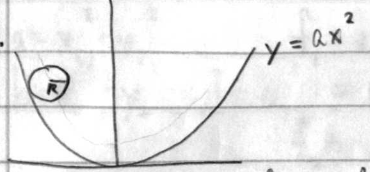
$$x_1 = \frac{a}{2} \exp\left(i\sqrt{2\lambda_1/m_1}\, t\right) + \frac{a}{2} \exp\left(-i\sqrt{2\lambda_1/m_1}\, t\right)$$
$$x_2 = \frac{b}{2} \exp\left(i\sqrt{2\lambda_2/m_2}\, t\right) + \frac{b}{2} \exp\left(-i\sqrt{2\lambda_2/m_2}\, t\right)$$
$$y_1 = \frac{a}{2} \exp\left(i\sqrt{2\lambda_1/m_1}\, t\right) - \frac{a}{2} \exp\left(-i\sqrt{2\lambda_1/m_1}\, t\right)$$
$$y_2 = \frac{b}{2} \exp\left(i\sqrt{2\lambda_2/m_2}\, t\right) - \frac{b}{2} \exp\left(-i\sqrt{2\lambda_2/m_2}\, t\right)$$

Sorry, will elaborate in the next edition. It's getting late, and my hand is getting tired

24. $\mathcal{L} = m\dot{x}^2/2 - kx^2/2$

$$x = \sum_{j=0} a_j \cos(j\omega t)$$
$$\dot{x} = \sum_{j=0} -a_j j\omega \sin(j\omega t)$$
$$I = \int_{t_1}^{t_2} \mathcal{L}\, dt = \sum_{j=0} \int_{t_1}^{t_2} \frac{m}{2}\left(a_j^2 j^2 \omega^2 \sin^2(j\omega t)\right) - \frac{k}{2}\left(a_j^2 \cos^2(j\omega t)\right) dt$$
$$= \frac{m a_j^2 j^2 \omega^2}{2} \int_0^{2\pi/\omega} (1 - \cos(2j\omega t))\, dt - \frac{k a_j^2}{2} \int_0^{2\pi/\omega} (1 + \cos(2j\omega t))\, dt$$
$$= \frac{m a_j^2 j^2 \omega^2}{2} \cdot \frac{2\pi}{\omega} - \frac{k a_j^2}{2} \cdot \frac{2\pi}{\omega} = a_j^2 \left(m j^2 \omega \pi - k\pi/\omega\right)$$
$$\frac{dI}{da_j} = 2a_j \left(m j^2 \omega \pi - k\pi/\omega\right) = 0$$
$$m j^2 \omega \pi - k\pi/\omega = 0$$
$$(j\omega)^2 = k/m$$

25.



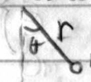
The condition for rolling without slipping:

$$R\,d\theta = 2a\dot{x}\,dx$$

For the ball to always be in contact, the center of the disk follows the path $y = ax^2 + R$

$$y - ax^2 - R = 0$$

26. Theres a typo here. It should say string of length L



$$\mathcal{L} = \tfrac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta$$

$$f = L - r = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} + \lambda\frac{\partial f}{\partial r} = 0$$

$$\tfrac{d}{dt}(m\dot{r}) - (mr\dot{\theta}^2 + mg\cos\theta) + \lambda(-1) = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta - \lambda = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} + \lambda\frac{\partial f}{\partial \theta} = 0$$

$$\tfrac{d}{dt}(mr^2\dot{\theta}) - (-mgr\sin\theta) = 0$$

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + mgr\sin\theta = 0$$

Using the condition $L = r$ as well as $\theta = \omega t$,

$$\Rightarrow \dot{r} = 0 \,\&\, \ddot{r} = 0$$

$$-mr\dot{\theta}^2 - mg\cos\theta - \lambda = 0$$

$$mr^2\ddot{\theta} + mgr\sin\theta = 0$$

$$mr^2\dot{\theta}\ddot{\theta} + mgr\dot{\theta}\sin\theta = 0$$

$$r\dot{\theta}\ddot{\theta} + g\dot{\theta}\sin\theta = 0 \quad \text{integrate}$$

$$\tfrac{r}{2}\dot{\theta}^2 - g\cos\theta = 0$$

$$r\dot{\theta}^2 = 2g\cos\theta$$

$$-m(2g\cos\theta) - mg\cos\theta - \lambda = 0$$

$$\lambda = -3mg\cos(\omega t)$$

$$-mr\dot{\theta}^2 = 4mg\cos(\omega t)$$

$$\dot{\theta}^2 = -\tfrac{4g}{r}\cos(\omega t)$$

27. $f(x, \dot{x}, y, \dot{y}, z) = \dot{x}y^2 + x\dot{y} + kz = 0$ (2.29)

a. $f(x, y, z, t) = 0$ (1.37)

$$\frac{dx}{dt}y^2 + \frac{dy}{dt}x + kz = 0$$

$$y^2dx + x\,dy = -kz\,dt$$

I think theres something about z being time-dependant

b. $F_x = \mu(t)\frac{\partial f}{\partial x} = \mu(t)\dot{y}$

$F_y = \mu(t)\frac{\partial f}{\partial y} = \mu(t)\cdot 2\dot{x}y$

$F_z = \mu(t)\frac{\partial f}{\partial z} = \mu(t)\cdot k$

$F = \int \mu(t)\cdot\dot{x}$

Im sorry