Solutions to Classical Mechanics by Golstein, Poole, and Safko (Third Edition)

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Contents

Chapter 1

Survey of the Elementary Principles

1.1 Kinetic Energy Equation of Motion

Show that for a single particle with constant mass the equation of motion implies the following differential equation for the kinetic energy:

$$\frac{dT}{dt} = \vec{F} \cdot \vec{v}$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \vec{F} \cdot \vec{p}$$

We start with the definition of kinetic energy (1.9). If we have a constant mass, we only have to derivatify the velocity terms when we take the time derivative.

$$T = {}^{1}\!/_{2}\, m\vec{v} \cdot \vec{v}$$

$$\frac{dT}{dt} = \frac{m}{2}\left(2\dot{\vec{v}} \cdot \vec{v}\right)$$

$$= m\vec{a} \cdot \vec{v}$$

Using the undergraduate definition of force,

$$\frac{dT}{dt} = \vec{F} \cdot \vec{v}$$

If however the mass is time dependent,

$$\frac{d(mT)}{dt} = \frac{d}{dt}\left[\frac{1}{2}(m\vec{v})^2\right]$$

$$= (m\vec{v}) \cdot \frac{d(m\vec{v})}{dt}$$

We recognize the first term as momentum (1.2) and the second as force (1.5),

$$\frac{d(mT)}{dt} = \vec{F} \cdot \vec{p}$$

1.2 Center of Mass

Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation

$$M^2R^2 = M\sum_i m_i r_i^2 - {}^1\!/\!_2 \sum_{i\neq j} m_i m_j r_{ij}^2$$

We start with the definition of the center of mass (1.14),

$$M\vec{R} = \sum_i m_i \vec{r}_i$$

Squaring this,

$$M^2R^2 = \sum_{ij} m_i m_j (\vec{r}_i \cdot \vec{r}_j)$$

If we look at the distance between two arbitrary points,

$$r_{ij}^2 = (\vec{r}_i - \vec{r}_j)^2$$

$$= r_i^2 + r_j^2 - 2\vec{r}_i \cdot \vec{r}_j$$

$$\vec{r}_i \cdot \vec{r}_j = {}^1\!/\!_2(r_i^2 + r_j^2) - {}^1\!/\!_2 r_{ij}^2$$

Substituting this back in,

$$M^2R^2 = {}^1\!/\!_2 \sum_{ij} m_i m_j r_i^2 + {}^1\!/\!_2 \sum_{ij} m_i m_j r_j^2 - {}^1\!/\!_2 \sum_{ij} m_i m_j r_{ij}^2$$

We note the first two terms are identical since they sum over the same points, so we can combine those two terms. We also note that we don't need to sum over repeated indices in the last term since the distance between a point and itself is 0,

$$M^2R^2 = M\sum_i m_i r_i^2 - {}^1\!/\!_2 \sum_{i\neq j} m_i m_j r_{ij}^2$$

1.3 Newton's Third Law

Newton's a system of two particles is know to obey the equations of motion, Eqs. (1.22) and (1.26). From the equations of the motion of the individual particles show that the internal forces between particles satisfy both the weak and the strong laws of action and reaction. The argument may be generalized to a system with arbitrary number of particles, thus proving the converse of the arguments leading to Eqs. (1.22) and (1.26).

Equation (1.22),

$$M\frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^{(e)} = \vec{F}^{(e)}$$

Equation (1.26),

$$\frac{d\vec{L}}{dt} = \vec{N}^{(e)}$$

The force on each particle is the external force on that particle and the interaction force,

$$\begin{cases} \dot{\vec{p}}_1 = \vec{F}_1^{(e)} + \vec{F}_{21} \\ \dot{\vec{p}}_2 = \vec{F}_2^{(e)} + \vec{F}_{12} \end{cases}$$

We get the total equation of motion by adding these two,

$$\dot{\vec{p}}_1 + \dot{\vec{p}}_2 = \vec{F}_1^{(e)} + \vec{F}_{21} + \vec{F}_2^{(e)} + \vec{F}_{12}$$

Using Newton's Third Law, the internal forces die,

$$\dot{\vec{p}}_1 + \dot{\vec{p}}_2 = \vec{F}_1^{(e)} + \vec{F}_2^{(e)} = \vec{F}^{(e)}$$

If we want to look at the torque,

$$\begin{cases} \dot{\vec{L}}_1 = \vec{r}_1 \times \vec{F}_1^{(e)} + \vec{r}_1 \times \vec{F}_{21} \\ \dot{\vec{L}}_2 = \vec{r}_2 \times \vec{F}_2^{(e)} + \vec{r}_2 \times \vec{F}_{12} \end{cases}$$

Looking at the internal component,

$$\vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{12} = \vec{r}_1 \times \vec{F}_{21} - \vec{r}_2 \times \vec{F}_{12}$$

$$= (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{21} = \vec{r}_{12} \times \vec{F}_{21}$$

This goes to zero since the vector pointing from particle 1 to particle 2 is in the same direction as the force between the two.

1.4 Rolling Disk Constraint

The equations of constraint for the rolling disk, Eqs. (1.39), are special cases of general linear differential equations of constraint of the form

$$\sum_{i=1}^{N} g_i(x_1, ..., x_n) dx_i = 0$$

A constraint condition of this type is holonomic only if an integrating function $f(x_1, ...x_n)$ can be found that turns it into an exact differential. Clearly the function must be such that

$$\frac{\partial(fg_i)}{\partial x_j} = \frac{\partial(fg_j)}{\partial x_i}$$

for all $i \neq j$. Show that no such integrating factor can be found for either of Eqs. (1.39)

Equation (1.39),

$$\begin{cases} dx - a\sin(\theta)d\phi = 0 \\ dy + a\cos(\theta)d\phi = 0 \end{cases}$$

Let's start with the first constraint equation,

$$dx - a\sin(\theta)d\phi = 0$$

Writing the g_i,

$$\begin{cases} g_x = 1 \\ g_\theta = 0 \\ g_\phi = -a\sin(\theta) \end{cases}$$

Let's start by choosing $i = x$ and $j = \phi$,

$$\frac{\partial f}{\partial \phi} = \frac{\partial(-af\sin(\theta))}{\partial x}$$

Using separation of variables, we expect our solution to take the form $f = X(x)Q(\phi)$. Substituting this in,

$$Q'X = -aX'Q\sin(\theta)$$

There is no solution for Q and X that satisfy this equation since we have that factor of $\sin(\theta)$. If we choose θ as one of our indices, one side of the equation will go to zero,

$$\frac{\partial f}{\partial \theta} = 0$$

$$\frac{\partial(-af\sin(\theta))}{\partial\theta} = 0$$

From this, we see that our equation cannot depend on θ, which only reinforces the non-existence of a possible $f(\vec{x})$.

Now for the second constraint equation,

$$dy + a\cos(\theta)d\phi = 0$$

$$\begin{cases} g_y = 1 \\ g_\theta = 0 \\ g_\phi = a\cos(\theta) \end{cases}$$

This is very similar to the first constraint equation, so we can convince ourselves that there is no integrating function here either.

1.5 Constraint Equations

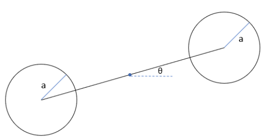
Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a plane. Show that there are two nonholonomic equations of constraint,

$$\begin{cases} \cos(\theta)dx + \sin(\theta)dy = 0 \\ \sin(\theta)dx - \cos(\theta)dy = {}^1\!/_2\; a(d\phi + d\phi') \end{cases}$$

(where θ, ϕ, and ϕ' have meanings similar to those in the problem of a single vertical disk, and (x, y) are the coordinates of a point on the axle midway between the two wheels) and one holonomic equation of constraint,

$$\theta = C - \frac{a}{b}(\phi - \phi')$$

where C is a constant.



Let's refer to the left side as x_1 and y_1 while the right side is x_2 and y_2. As in the previous question, we have the following constraints on each wheel,

$$\begin{cases} dx_1 - a\sin(\theta)d\phi_1 = 0 \\ dy_1 + a\cos(\theta)d\phi_1 = 0 \end{cases}$$

$$\begin{cases} dx_2 - a\sin(\theta)d\phi_2 = 0 \\ dy_2 + a\sin(\theta)d\phi_2 = 0 \end{cases}$$

We also have the coordinates of the axle,

$$\begin{cases} x = {}^1\!/_2\; (x_1 + x_2) \\ y = {}^1\!/_2\; (y_1 + y_2) \end{cases}$$

Substituting in the wheel constraints,

$$dx = {}^1\!/_2\; (a\sin(\theta)d\phi_1 + a\sin(\theta)d\phi_2) = 0$$

$$dx = {}^a\!/_2 \sin(\theta)(d\phi_1 + d\phi_2)$$

$$dy = -{}^a\!/_2 \cos(\theta)(d\phi_1 + d\phi_2)$$

If we now look at the given constraints,

$$\cos(\theta)dx + \sin(\theta)dy = {}^a\!/_2 \cos(\theta)\sin(\theta)(d\phi_1 + d\phi_2) - {}^a\!/_2 \cos(\theta)\sin(\theta)(d\phi_1 + d\phi_2)$$

$$\cos(\theta)dx + \sin(\theta)dy = 0$$

$$\sin(\theta)dx - \cos(\theta)dy = {}^{a}\!/\!_{2}\sin^2(\theta)(d\phi_1 + d\phi_2) + {}^{a}\!/\!_{2}\cos^2(\theta)(d\phi_1 + d\phi_2)$$

$$\sin(\theta)dx - \cos(\theta)dy = {}^{a}\!/\!_{2}\ (d\phi_1 + d\phi_2)$$

For the holonomic constraint, we start by writing the position of the wheels in terms of the axle,

$$\begin{cases} x_1 = -{}^{b}\!/\!_{2}\cos(\theta) \\ x_2 = {}^{b}\!/\!_{2}\cos(\theta) \end{cases}$$

$$dx_2 - dx_1 = -{}^{b}\!/\!_{2}\ d\theta\sin(\theta) - {}^{b}\!/\!_{2}\ d\theta\sin(\theta)$$

$$dx_2 - dx_1 = -b\sin(\theta)d\theta$$

Alternatively,

$$dx_2 - dx_1 = a\sin(\theta)d\phi_2 - a\sin(\theta)d\phi_1$$

$$dx_2 - dx_1 = a\sin(\theta)(d\phi_2 - d\phi_1)$$

Setting these two equal to each other,

$$-bd\theta = a(d\phi_2 - d\phi_1)$$

$$d\theta = -\frac{a}{b}(d\phi_2 - d\phi_1)$$

Integrating, we have to throw in a constant,

$$\theta = C - \frac{a}{b}(\phi_2 - \phi_1)$$

1.6 Non-Holonomic Constraint

A particle moves in the xy plane under the constraint that its velocity vector is always directed towards a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ differentiable, but otherwise arbitrary, the constraint is non-holonomic.

The ratio of the velocity vector components of the particle must be the same as the ratio of the vector that points from the particle to the point on the x-axis. The vector pointing from the point on the x-axis to the particle has components,

$$\begin{cases} V_x = x(t) - f(t) \\ V_y = y(t) \end{cases}$$

For the condition to be true,

$$\frac{dy}{dx} = \frac{y(t)}{x(t) - f(t)}$$

$$ydx + [f(t) - x]dy = 0$$

Since $f(t)$ is arbitrary, this cannot be integrated, so our constraint is non-holonomic.

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1.8 1.8

1.9 1.9

1.10 1.10

1.11 Conservative Forces

Check whether the force $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ is conservative or not.

To show that this force is conservative, we would need to try out every possible path to make sure that equation (1.11) holds regardless of which path the particle travels along. Alternatively, we can use Stokes's theorem,

$$\oint_C \vec{A} \cdot d\vec{l} = \int_S (\nabla \times \vec{A}) \cdot \hat{n}\ da$$

Applying this to equation (1.11), we have

$$\int_S (\nabla \times \vec{F}) \cdot \hat{n}\ da = 0$$

This is easier if instead of choosing some closed surface, we instead find the curl of the force,

$$\nabla \times \vec{F} = \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx), \frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(xy), \frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz) \right)$$

$$\nabla \times \vec{F} = 0$$

Because of this, we satisfy equation (1.11), and the given force is conservative.

1.12 Satellite Orbital Motion

Compute the orbital period and orbital angular velocity of a satellite revolving around the Earth at an altitude of 720km. [Given: radius of Earth $R = 6000km$ and $g = 9.83m/s^2$.]

From undergraduate mechanics, we remember,

$$F = \frac{mv^2}{r} = \frac{m\omega^2 r^2}{r}$$

$$ma = m\omega^2 r$$

Solving for the angular velocity, ω and inputting the given values for gravitational acceleration and distance (remember that the distance of the satellite from earth is going to be 6720km),

$$\omega^2 = \frac{a}{r} = \frac{9.83\ m/s^2}{6.72 \times 10^6\ m}$$

$$\omega = 1.2 \times 10^{-3}\ s^{-1}$$

The orbital period is given by $T = 2\pi/\omega$,

$$T = 5200\ s$$

Or about three and a half hours.

1.13 Rocket Propulsion

Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric friction, is

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg$$

where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m, assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to $2.1km/s$ and a mass loss per second equal to $1/60th$ of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300!

We want to use conservation of linear momentum here for a small time change. Initially, we have just the rocket, the momentum of which is written mv. At some dt later, it has expelled dm fuel and is now moving at $v + dv$. From this, we have

$$mv = (m - dm)(v + dv) + (v + v') \cdot dm$$

$$mv = mv - v \cdot dm + m \cdot dv + (v + v') \cdot dm$$

$$m \cdot dv = -v' \cdot dm$$

Note that we ignore $dm \cdot dv$ because it is very small. Also remember that we are in a uniform gravitational field, so we have to add in a non-conservative change in momentum due to that,

$$m \cdot dv = -v' \cdot dm - mg \cdot dt$$

Rearranging, we get the desired equation of motion. To integrate,

$$dv = -v'\frac{dm}{m} - g \cdot dt$$

Integrating from initial to final mass of the rocket,

$$v_f - v_0 = -v' \ln(m)|_{m_0}^{m_f} - gt = -v' \ln\left(\frac{m_f}{m_0}\right) - gt$$

Looking up the escape velocity (11.2km/s) and setting the initial mass m_0 to the mass of the rocket plus mass of the fuel, $m + m_s$ (s stands for sugar). Also using a gravitational acceleration of $10m/s^2$ and realizing that the time is 60 seconds,

$$v_e = -v' \ln\left(\frac{m}{m + m_s}\right) - gt$$

$$\frac{m + m_s}{m} = \exp\left(\frac{v_e + gt}{v'}\right)$$

$$\frac{m_s}{m} = \exp\left(\frac{v_e + gt}{v'}\right) - 1$$

$$= \exp\left(\frac{11200m/s + 600m/s}{2100m/s}\right) - 1$$

$$\frac{m_s}{m} \approx 274$$

Which is about equal to 300.

1.14 Generalized Coordinates

Two points of mass m are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius a. Express the kinetic energy in generalized coordinates.

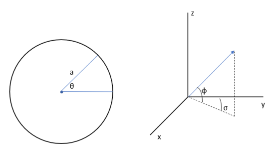

Figure 1.1: Generalized Coordinates

For this problem, we have three degrees of freedom. The rod provides one (θ) while the two masses have two (ϕ and σ) as seen in figure (1.1). From this, we can define the position of the two masses,

$$\begin{cases} \vec{x}_1 = (a\cos(\theta) + {}^{l}\!/\!{}_{2}\cos(\phi)\sin(\sigma), a\sin(\theta) + {}^{l}\!/\!{}_{2}\cos(\phi)\cos(\sigma), {}^{l}\!/\!{}_{2}\sin(\sigma)) \\ \vec{x}_2 = (a\cos(\theta) - {}^{l}\!/\!{}_{2}\cos(\phi)\sin(\sigma), a\sin(\theta) - {}^{l}\!/\!{}_{2}\cos(\phi)\cos(\sigma), -{}^{l}\!/\!{}_{2}\sin(\sigma)) \end{cases}$$

The kinetic energy (1.9),

$$T = {}^{1}\!/\!{}_{2}\, m(\dot{x}_1^2 + \dot{x}_2^2)$$

Let's find the component pieces. First, we take the time derivative of the mass positions,

$$\begin{cases} \dot{x}_1\hat{i} = -a\dot{\theta}\sin(\theta) - \dfrac{l\dot{\theta}}{2}\sin(\theta)\sin(\sigma) + \dfrac{l\dot{\sigma}}{2}\cos(\phi)\cos(\sigma) \\ \dot{x}_1\hat{j} = a\dot{\theta}\cos(\theta) - \dfrac{l\dot{\phi}}{2}\sin(\phi)\cos(\sigma) - \dfrac{l\dot{\sigma}}{2}\cos(\phi)\sin(\sigma) \\ \dot{x}_1\hat{k} = \dfrac{l\dot{\phi}}{2}\cos(\phi) \end{cases}$$

Squaring this,

$$\dot{x}_1^2 = a^2\dot{\theta}^2 + \frac{l^2\dot{\phi}^2}{4}\sin^2(\phi) + \frac{l^2\dot{\sigma}^2}{4}\cos^2(\phi) + \frac{l^2\dot{\phi}^2}{4}\cos^2(\phi)$$

We actually end up getting the same thing when we look for $\dot{x}_2^2$, so the kinetic energy,

$$T = m\left(a^2\dot{\theta}^2 + \frac{l^2\dot{\phi}^2}{4}\sin^2(\phi) + \frac{l^2\dot{\sigma}^2}{4}\cos^2(\phi) + \frac{l^2\dot{\phi}^2}{4}\cos^2(\phi)\right)$$

$$T = m\left(a^2\dot{\theta}^2 + \frac{l^2\dot{\phi}^2}{4} + \frac{l^2\dot{\sigma}^2}{4}\cos^2(\phi)\right)$$

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1.16 1.16

1.17 Conservation of Momentum

A nucleus, originally at rest, decays radioactively by emitting an electron of momentum 1.73MeV/c**, and at right angles to the direction of the electron a neutrino with momentum 1.00** MeV/c**. (The MeV, million electron volt, is a unit of energy used in modern physics, equal to 1.60**$\times 10^{-13} J$**. Correspondingly,** MeV/c **is a unit of linear momentum equal to** $5.34 \times 10^{-22}\ kg \cdot m/s$**.) In what direction does the nucleus recoil? What is its momentum in** MeV/c**? If the mass of the residual nucleus is** $3.90 \times 10^{-25} kg$ **what is its kinetic energy, in electron volts?**

We pretend these are classical objects, ignoring relativistic and quantum effects. Since the electron and neutrino go off in orthogonal directions, we can set those as our axes (with the neutrino traveling in the x-direction and the electron traveling in the y-direction). Since we want to conserve momentum,

$$\vec{p}_n = (-p_\nu, -p_e)$$

From this, we can see the direction of the neutron,

$$\theta = \tan^{-1}\left(\frac{p_e}{p_\nu}\right) = \tan^{-1}\left(\frac{1.73}{1.00}\right) = 60^\circ$$

To find the magnitude of the momentum, we must use Pythagoras,

$$p_n^2 = p_\nu^2 + p_e^2 = 1.73^2 + 1$$

$$p_n = 2MeV/c$$

The kinetic energy,

$$T = \frac{p_n^2}{m_n} = 1.8 \times 10^{-5} MeV$$

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1.19 1.19

1.20 1.20

1.21 1.21

1.22 1.22

1.23 Atwood Machine

Two masses $2kg$ and $3kg$, respectively, are tied to the two ends of a massless, inextensible string passing over a smooth pulley. When the system is released, calculate the acceleration of the masses and the tension in the string.

Atwood's machine is a problem that we cannot solve using Lagrangian formalism. We say that the lighter mass is on the left while the heavier mass is on the right. By drawing the force diagrams, we get the equations of motion,

$$\begin{cases} -m_1 g + T = m_1 a \\ -m_2 g + T = -m_2 a \end{cases}$$

Solving for tension,

$$\begin{cases} T = m_1(a+g) \\ T = m_2(-a+g)) \end{cases}$$

$$m_1 a + m_1 g = -m_2 a + m_2 g$$

$$a = \frac{m_2 g - m_1 g}{m_1 + m_2}$$

$$T = m_1 \left(\frac{m_2 g - m_1 g}{m_1 + m_2} + g \right)$$

$$T = \frac{2 m_1 m_2 g}{m_1 + m_2}$$

Note to self: return to this when you get to Lagrangian