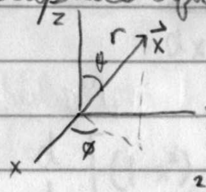
Chapter 3: Boundary-Value Problems in Electrostatics: II

Section 21. Laplace Equation in Spherical Coordinates



For a potential in spherical coordinates, Laplace equation is

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2\Phi}{\partial\phi^2} = 0 \qquad (3.1)$$

Note that we can rewrite the first term as

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)$$

If we use the separation of variables we learned in the previous chapter, we can write

$$\Phi(r,\theta,\phi) = \frac{U(r)}{r}P(\theta)Q(\phi) \qquad (3.2)$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(U(r)P(\theta)Q(\phi)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\cdot\frac{\partial UPQ}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta\cdot r}\frac{\partial^2 UPQ}{\partial\phi^2} = 0$$

$$\frac{PQ}{r}\cdot\frac{d^2U}{dr^2} + \frac{UQ}{r\cdot r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{UP}{r\cdot r^2\sin^2\theta}\cdot\frac{d^2Q}{d\phi^2} = 0$$

If we multiply through by $r^3\sin^2\theta/UPQ$

$$r^2\sin^2\theta\left[\frac{1}{U}\cdot\frac{d^2U}{dr^2} + \frac{1}{Pr^2\sin\theta}\cdot\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right] + \frac{1}{Q}\cdot\frac{d^2Q}{d\phi^2} = 0 \qquad (3.3)$$

$$\frac{1}{Q}\cdot\frac{d^2Q}{d\phi^2} = -m^2 \qquad (3.4)$$

$$Q = \exp(\pm im\phi) \qquad (3.5)$$

In order for Q to be single valued, we must let m be an integer. Now let's try to find separate equations for $P(\theta)$ + $U(r)$.

- Multiply by r^3/UPQ:

$$\frac{r^2}{U}\frac{d^2U}{dr^2} + \frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{1}{Q\sin^2\theta}\cdot\frac{d^2Q}{d\phi^2} = 0$$

$$\frac{r^2}{U}\cdot\frac{d^2U}{dr^2} + \frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = 0$$

$$\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=-\ell(\ell+1)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]P=0 \quad (3.6)$$

$$\frac{r^2}{U}\cdot\frac{d^2U}{dr^2}-\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]-\frac{m^2}{\sin^2\theta}=0$$

$$\frac{r^2}{U}\cdot\frac{d^2U}{dr^2}=\ell(\ell+1)$$

$$\frac{d^2U}{dr^2}-\frac{\ell(\ell+1)}{r^2}U=0 \quad (3.7)$$

$$\frac{d^2U}{dr^2}=\frac{\ell(\ell+1)}{r^2}U$$

Let $U=Ar^{\ell+1}+Br^{-\ell}$ (3.8)

$$(\ell+1)\ell Ar^{\ell-1}+(-\ell)(-\ell-1)Br^{-\ell-2}=(\ell+1)\ell Ar^{\ell-1}+\ell(\ell+1)Br^{-\ell-2}$$

$$(\ell+1)\ell Ar^{\ell-1}+(\ell+1)\ell Br^{-\ell-2}=(\ell+1)\ell Ar^{\ell-1}+(\ell+1)\ell Br^{-\ell-2} \quad \checkmark$$

Section 2. Legendre Equations and Legendre Polynomials

You may have noticed we didn't give a solution for $P(\theta)$. $P(\theta)$ is solved by the Legendre polynomials, i.e. there are different polynomials depending on the value of ℓ. If we perform the substitution $x=\cos\theta$, (3.6) becomes

$$dx=-\sin\theta\, d\theta$$

$$\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]P=0$$

$$\frac{-d}{dx}\left(-\sin^2\theta\cdot\frac{dP}{dx}\right)+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P=0$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P=0 \quad (3.9)$$

Hm, it looks like I lied. We don't actually solve the general Legendre equation here. Instead, we assume azimuthal symmetry (no ϕ dependance), which means $m=0$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\ell(\ell+1)P=0 \quad (3.10)$$

$$P(x)=x^\alpha\sum_{j=0}^{\infty}a_jx^j \quad (3.11)$$

$$\frac{d}{dx}\left[(1-x^2)\sum_{j=0}^{\infty}a_j(j+\alpha)x^{\alpha+j-1}\right]+\ell(\ell+1)\sum_{j=0}^{\infty}a_jx^{\alpha+j}=0$$

$$\left(\frac{d}{dx}\sum_{j=0}^{\infty}a_j(j+\alpha)\left(x^{\alpha+j-1}-x^{\alpha+j+1}\right)\right)+\ell(\ell+1)\sum_{j=0}^{\infty}a_jx^{\alpha+j}=0$$

$$\sum_{j=0}^{\infty}a_j(j+\alpha)(j+\alpha-1)x^{\alpha+j-2}-\left[(\alpha+j)(\alpha+j+1)-\ell(\ell+1)\right]a_jx^{\alpha+j}=0 \quad (3.12)$$

In order to satisfy this equation, the coefficient a_j must vanish separately for each power of x. For $j=0$,

$$\alpha(\alpha-1)a_0x^{\alpha-2}-\left[\alpha(\alpha+1)-\ell(\ell+1)\right]a_0x^\alpha=0$$

if $a_0\neq 0$, then $\alpha(\alpha-1)=0$

$a_1\neq 0$, then $\alpha(\alpha+1)=0$ (3.13)

$$a_{j+2}=\left[\frac{(\alpha+j)(\alpha+j+1)-\ell(\ell+1)}{(\alpha+j+1)(\alpha+j+2)}\right]a_j \quad (3.14)$$

Either a_0 or a_1 must be different from zero but not both. Thus, we get back either the even powers or the odd powers of x but not both. We also see that ℓ must be 0 or a positive integer. Knowing this, we can write the Legendre polynomials

$P_0(x)=1$

$P_1(x)=x$

$P_2(x)=\tfrac{1}{2}(3x^2-1)$

$P_3(x)=\tfrac{1}{2}(5x^3-3x)$ (3.15)

$P_\ell(x)=\tfrac{1}{2^\ell \ell!}\cdot\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$ (3.16)

We claim that the Legendre Polynomials form a complete orthogonal set of functions between $-1\leq x\leq 1$.

$$\int_{-1}^{1}P_{\ell'}(x)\cdot\left[\frac{d}{dx}\left((1-x^2)\frac{dP_\ell}{dx}\right)+\ell(\ell+1)P_\ell\right]dx=0 \quad (3.17)$$

$$=\int_{-1}^{1}P_{\ell'}\cdot\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell}{dx}\right]dx+\int_{-1}^{1}\ell(\ell+1)P_\ell P_{\ell'}\,dx=0$$

$u=P_{\ell'}$ $\qquad v=(1-x^2)\frac{dP_\ell}{dx}$

$du=\frac{dP_{\ell'}}{dx}$ $\qquad dv=\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell}{dx}\right]$

$$= \cancel{P_{l'}(1-x^2)\frac{dP_l}{dx}\Big|_{-1}^{1}} - \int_{-1}^{1}(1-x^2)\frac{dP_l}{dx}\cdot\frac{dP_{l'}}{dx}\,dx + \int_{-1}^{1} l(l+1)P_l P_{l'}\,dx$$

$$= \int_{-1}^{1}(x^2-1)\frac{dP_l}{dx}\cdot\frac{dP_{l'}}{dx} + l(l+1)P_{l'}P_l\,dx = 0 \qquad (3.18)$$

Now if we had written (3.10) in terms of l' and multiplied by $P_l(x)$, we would have

$$\int_{-1}^{1}(x^2-1)\frac{dP_l}{dx}\cdot\frac{dP_{l'}}{dx} + l'(l'+1)P_l P_{l'}\,dx = 0$$

Subtracting from (3.18), we get

$$[l(l+1) - l'(l'+1)]\int_{-1}^{1} P_l P_{l'}\,dx = 0 \qquad (3.19)$$

Using orthonormality condition, we need to find $\int_{-1}^{1} P_l^2\,dx$ using (3.16).

$$N_l = \int_{-1}^{1} P_l^2\,dx = \frac{1}{2^{2l}(l!)^2}\int_{-1}^{1}\frac{d^l}{dx^l}(x^2-1)^l\,\frac{d^l}{dx^l}(x^2-1)^l\,dx$$

$$u = \frac{d^l}{dx^l}(x^2-1)^l \qquad v = \frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l$$

$$du = \frac{d^{l+1}}{dx^{l+1}}(x^2-1)^l \qquad dv = \frac{d^l}{dx^l}(x^2-1)^l$$

$$= \frac{1}{2^{2l}(l!)^2}\left[\cancel{uv} - \int_{-1}^{1}\frac{d^{l+1}}{dx^{l+1}}(x^2-1)^l\cdot\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^l\,dx\right]$$

Repeating a total of l times

$$N_l = \frac{(-1)^l}{2^{2l}(l!)^2}\int_{-1}^{1}(x^2-1)^l\frac{d^{2l}}{dx^{2l}}(x^2-1)^l\,dx$$

$$= \frac{(2l)!}{2^{2l}(l!)^2}\int_{-1}^{1}(1-x^2)^l\,dx$$

$$= \left(\frac{2l-1}{2l}\right)N_{l-1} - \frac{1}{2l}N_l$$

$$(2l+1)N_l = (2l-1)N_{l-1} \qquad (3.20)$$

$$N_l = \frac{2}{2l+1}$$

$$\int_{-1}^{1}P_l(x)P_{l'}(x)\,dx = \frac{2}{2l+1}\delta_{ll'} \qquad (3.21)$$

From (3.16), we can derive certain recurrence relation

$$\frac{dP_{l+1}}{dx} = \frac{1}{2^{l+1}(l+1)!}\cdot\frac{d^{l+2}}{dx^{l+2}}(x^2-1)^{l+1}$$

$$= \frac{1}{2^{l+1}(l+1)!}\cdot 2(l+1)\cdot\frac{d^l}{dx^l}\cdot\frac{d}{dx}\cdot x(x^2-1)^l$$

$$= \frac{1}{2^l l!}\cdot\frac{d^l}{dx^l}\left[(x^2-1)^l + l\cdot 2x^2(x^2-1)^{l-1}\right]$$

$$\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} = \frac{d^l}{dx^l}\left[\frac{(x^2-1)^l}{2^l l!} + \frac{x^2(x^2-1)^{l-1}}{2^{l-1}(l-1)!} - \frac{(x^2-1)^{l-1}}{2^{l-1}(l-1)!}\right]$$

$$= \frac{d^l}{dx^l}\left[\frac{(x^2-1)^l}{2^l l!} + \frac{(x^2-1)^l}{2^l (l)!}\cdot 2(l)\right]$$

$$= (2l+1)\frac{d^l}{dx^l}\frac{(x^2-1)^l}{2^l l!} = (2l+1)P_l$$

$$\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} - (2l+1)P_l = 0 \qquad (3.28)$$

$$(l+1)P_{l+1} - (2l+1)xP_l + lP_{l-1} = 0$$

$$\frac{dP_{l+1}}{dx} - x\cdot\frac{dP_l}{dx} - (l+1)P_l = 0$$

$$(x^2-1)\cdot\frac{dP_l}{dx} - lxP_l + lP_{l-1} = 0 \qquad (3.29)$$

As an example, let's try to evaluate

$$I_1 = \int_{-1}^{1} x P_l P_{l'}\,dx \qquad (3.30)$$

$$= \frac{1}{2l+1}\int_{-1}^{1} P_{l'}[(l+1)P_{l+1} + lP_{l-1}]\,dx$$

$$= \frac{2}{(2l+1)(2l+3)} \qquad l' = l+1$$

$$\frac{2}{(2l+1)(2l-1)} \qquad l' = l-1 \qquad (3.31)$$

$$I_2 = \int_{-1}^{1} x^2 P_l P_{l'}\,dx$$

$$= x I_1$$

$$= \frac{1}{2l+1}\int_{-1}^{1} xP_{l+1}P_{l'}(l+1) + xP_{l-1}P_{l'}\,l\,dx$$

$$= \frac{1}{2l+1}\int_{-1}^{1}\frac{l+1}{2l+3}[(l+2)P_{l+2}P_{l'} + (l+1)P_lP_{l'}] + \frac{l}{2l-1}[lP_lP_{l'} + (l-1)P_{l-2}P_{l'}]\,dx$$

$$= \frac{2(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)};\quad l' = l+2$$

$$\frac{2[(l+1)^2(2l-1) + l^2(2l+3)]}{(2l+1)(2l+3)(2l-1)(2l+1)} = \frac{2(2l^2+l^2-1)}{(2l+1)(2l-1)(2l+3)} \qquad l' = l \qquad (3.32)$$

$$l' \geq l \text{ so } P_{l-2}P_{l'} = 0$$

Section 3 Boundary-Value Problems with Azimuthal Symmetry

If a problem possesses azimuthal symmetry i.e. no ϕ dependance, we can set $m=0$, thus our problem simplifies to

$$\Phi(r,\theta)=\frac{U(r)}{r}P(\theta)$$

$$=\sum_{\ell=0}^{\infty}\left[A_\ell r^\ell+B_\ell r^{-(\ell+1)}\right]P_\ell(\cos\theta) \qquad (3.33)$$

For the potential at $\vec{x}$ due to a unit point charge at $\vec{x}'$, we use

$$\frac{1}{|\vec{x}-\vec{x}'|}=\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\gamma) \qquad (3.38)$$

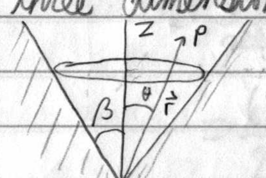
which becomes

$$\frac{1}{|\vec{x}-\vec{x}'|}=\frac{1}{r_>}\sum_{\ell=0}^{\infty}\left(\frac{r_<}{r_>}\right)^\ell$$

on the axis.

Section 4. Behavior of Fields in a Conical Hole or Near a Sharp Point

Here is an example of a boundary-value problem with azimuthal symmetry. In section 2.11, we looked at the field near a corner in two-dimensions. Here, we'll be looking at the three-dimensional analog.



We limit our region to the cone traced out by $0\leq\theta\leq\beta$, $0\leq\phi\leq 2\pi$.

(3.10) gives the angular solution, but we require the solution to be finite and single-valued between $\cos\beta\leq\cos\theta\leq 1$. In addition, the solution must vanish at $\theta=\beta$.

$$\xi=\tfrac{1}{2}(1-x)$$

$$\frac{d}{d\xi}\left[\xi(1-\xi)\frac{dP}{d\xi}\right]+\nu(\nu+1)P=0 \qquad (3.39)$$

$$P_\nu(\xi)=1+\frac{(-\nu)(\nu+1)}{1!1!}\xi+\frac{(-\nu)(-\nu+1)(\nu+1)(\nu+2)}{2!2!}\xi^2 \qquad (3.41)$$

(3.41) is an example of a hypergeometric function

$${}_2F_1(a,b;c;z)=1+\frac{ab}{c}\cdot\frac{z}{1!}+\frac{(a)(a+1)b(b+1)}{c(c+1)}\cdot\frac{z^2}{2!}$$

$$P_\nu(x)={}_2F_1\left(-\nu,\nu+1;1;\frac{1-x}{2}\right) \qquad (3.42)$$

All this is to say that the basic potential is given by

$$\Phi(r,\theta)=\sum_{k=1}^{\infty}A_k r^{\nu_k}P_{\nu_k}(\cos\theta) \qquad (3.44)$$

$$\approx A r^\nu P_\nu(\cos\theta) \qquad (3.45)$$

$$E_r=-\frac{\partial\Phi}{\partial r}=-\nu A r^{\nu-1}P_\nu(\cos\theta)$$

$$E_\theta=-\frac{1}{r}\cdot\frac{\partial\Phi}{\partial\theta}=A r^{\nu-1}\sin\theta\, P_\nu'(\cos\theta)$$

$$\sigma(r)=-\frac{1}{4\pi}E_\theta\big|_{\theta=\beta}=-\frac{A}{4\pi}r^{\nu-1}\sin\beta\, P_\nu'(\cos\beta) \qquad (3.46)$$

Section 5. Associated Legendre Functions and the Spherical Harmonics $Y_{\ell m}(\theta,\phi)$

As much fun as it is to only solve problems with azimuthal symmetry, let's look at problems where the values of ℓ and m are arbitrary. Well, not completely arbitrary, ℓ must be zero or a positive integer. m can take on integer values between $-\ell$ and ℓ. Thus, we have an analogue to (3.16),

$$P_\ell^m(x)=\frac{(-1)^m}{2^\ell\ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell \qquad (3.50)$$

$$P_\ell^{-m}(x)=(-1)^m\cdot\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x) \qquad (3.51)$$

$$\int_{-1}^{1}(P_\ell^m)^2dx=\frac{1}{2^{2\ell}(\ell!)^2}\int_{-1}^{1}(1-x^2)^m\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell dx$$

$$=\frac{2}{2\ell+1}\cdot\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'} \qquad (3.52)$$

If we combine the angular parts of (3.2), we call those spherical harmonics

$$Y_{\ell m}(\theta,\phi)=\sqrt{\frac{(2\ell+1)}{4\pi}\cdot\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\exp(im\phi) \qquad (3.53)$$

$$Y_{\ell,-m}(\theta,\phi)=(-1)^m Y_{\ell,-m}^*(\theta,\phi) \qquad (3.54)$$

$$\int_0^{2\pi}\int_0^{\pi}Y_{\ell'm'}^*Y_{\ell m}\sin\theta\, d\theta\, d\phi=\delta_{\ell'\ell}\delta_{m'm} \qquad (3.55)$$

An arbitrary function can be expanded in spherical harmonics as

$$g(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta,\phi) \qquad (3.58)$$

$$A_{lm} = \int Y^*_{lm}(\theta,\phi)\, g(\theta,\phi)\, d\Omega$$

(3.2) can now be rewritten as

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^{l} + B_{lm} r^{-(l+1)}\right] Y_{lm}(\theta,\phi) \qquad (3.61)$$

Section 6. Addition Theorem for Spherical Harmonics

If we have two vectors $\vec{x}$ & $\vec{x}'$ separated by an angle γ,

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi) \qquad (3.62)$$

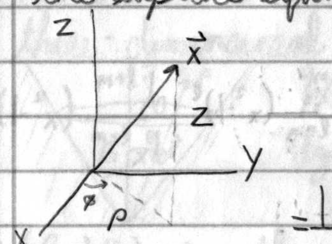
$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

If γ goes to 0,

$$\sum_{m=-l}^{l} |Y_{lm}(\theta,\phi)|^2 = \frac{2l+1}{4\pi} \qquad (3.69)$$

Section 7. Laplace Equation in Cylindrical Coordinates; Bessel Functions

We've solved the Laplace equation in rectangular coordinates (2.71) and in spherical coordinates (3.61), so let's solve the Laplace equation in the remaining coordinate system.



$$\nabla^2\Phi = 0$$

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\cdot\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \qquad (3.71)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\cdot\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

Using separation of variables:

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z) \qquad (3.72)$$

$$QZ\cdot\frac{d^2R}{d\rho^2} + \frac{QZ}{\rho}\cdot\frac{dR}{d\rho} + \frac{RZ}{\rho^2}\cdot\frac{d^2Q}{d\phi^2} + RQ\cdot\frac{d^2Z}{dz^2} = 0$$

Multiply by $1/RQZ$

$$\left[\frac{1}{R}\cdot\frac{d^2R}{d\rho^2} + \frac{1}{R\rho}\cdot\frac{dR}{d\rho} + \frac{1}{Q\rho^2}\cdot\frac{d^2Q}{d\phi^2}\right] + \frac{1}{Z}\cdot\frac{d^2Z}{dz^2} = 0$$

$$\frac{1}{Z}\cdot\frac{d^2Z}{dz^2} = k^2$$

$$\frac{d^2Z}{dz^2} - k^2 Z = 0 \qquad (3.73)$$

Multiply by ρ^2/RQZ to isolate ϕ term

$$\frac{1}{Q}\cdot\frac{d^2Q}{d\phi^2} = -\nu^2$$

$$\frac{d^2Q}{d\phi^2} + \nu^2 Q = 0 \qquad (3.74)$$

$$\frac{1}{R}\cdot\frac{d^2R}{d\rho^2} + \frac{1}{R\rho}\cdot\frac{dR}{d\rho} + \frac{1}{\rho^2}\left(\frac{1}{Q}\cdot\frac{d^2Q}{d\phi^2}\right) + \left(\frac{1}{Z}\cdot\frac{d^2Z}{dz^2}\right) = 0$$

$$\frac{1}{R}\cdot\frac{d^2R}{d\rho^2} + \frac{1}{R\rho}\cdot\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) = 0$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0 \qquad (3.75)$$

The solutions for (3.73) and (3.74) can be gotten easily.

$$Z(z) = \exp(\pm kz)$$

$$Q(\phi) = \exp(\pm i\nu\phi) \qquad (3.76)$$

By letting $x = k\rho$, $dx = k\,d\rho$

$$k^2\cdot\frac{d^2R}{dx^2} + \frac{k^2}{x}\cdot\frac{dR}{dx} + \left(k^2 - \frac{k^2\nu^2}{x^2}\right)R = 0$$

$$\frac{d^2R}{dx^2} + \frac{1}{x}\cdot\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0 \qquad (3.77)$$

The solutions to this equation (the Bessel equation) are the Bessel functions. Assume a solution of the form:

$$R(x) = x^{\alpha}\sum_{j=0}^{\infty} a_j x^j \qquad (3.78)$$

$$\sum_{j=0}^{\infty} (\alpha+j)(\alpha+j-1)a_j x^{\alpha+j-2} + (\alpha+j)a_j x^{\alpha+j-2} + a_j x^{\alpha+j} - a_j\nu^2 x^{\alpha+j-2} = 0$$

We want each power to disappear independently

$$(\alpha+j)(\alpha+j-1) + (\alpha+j) - \nu^2 = 0$$

$$\alpha^2 + \alpha j - \cancel{\alpha} + \alpha j + j^2 - \cancel{j} + \cancel{\alpha} + \cancel{j} = \nu^2$$

$$(\alpha+j)^2 = \nu^2$$

Must satisfy for all values of $j \Rightarrow$

$$\alpha = \pm\nu \qquad (3.79)$$

Only the even powers of x^j survive, so

$$a_{2j} = \frac{(-1)^j\,\Gamma(\alpha+1)}{2^{2j}\,j!\,\Gamma(\alpha+j+1)}\,a_0 \qquad (3.81)$$

Letting $a_0 = [2^\alpha\,\Gamma(\alpha+1)]^{-1}$, the Bessel functions of the first kind

$$J_{\pm\nu} = \left(\frac{x}{2}\right)^{\pm\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j\pm\nu+1)}\left(\frac{x}{2}\right)^{2j} \qquad (3.83)$$

If ν is an integer, this leads to the Bessel functions of the second kind, known as Neumann function

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \qquad (3.85)$$

The Bessel functions of the third kind, Hankel functions

$$H^{(1)}_\nu(x) = J_\nu(x) + i N_\nu(x)$$
$$H^{(2)}_\nu(x) = J_\nu(x) - i N_\nu(x) \qquad (3.86)$$

If we had, instead of (3.75),

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\cdot\frac{dR}{d\rho} - \left(k^2 + \frac{\nu^2}{\rho^2}\right)R = 0 \qquad (3.98)$$

$$\frac{d^2R}{dx^2} + \frac{1}{x}\cdot\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)R = 0 \qquad (3.99)$$

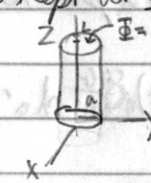
The Bessel functions become purely imaginary

$$I_\nu(x) = i^{-\nu} J_\nu(ix) \qquad (3.100)$$
$$K_\nu(x) = \tfrac{\pi}{2}\cdot i^{\nu+1} H^{(1)}_\nu(ix) \qquad (3.101)$$

I'm not entirely sure what first order and so on mean, it has something to do with the value of α. In any case, we will mostly be working with Bessel functions of the first order.

Section 8. Boundary-Value Problems in Cylindrical Coordinates

Imagine we have a cylinder of radius a and height L. The cylinder is kept at $\Phi = 0$ save the top which is at $\Phi = V(\rho,\phi)$.



$$Q(\phi) = A\sin(m\phi) + B\cos(m\phi)$$
$$Z(z) = \sinh(kz)$$
$$R(\rho) = C J_m(k\rho) + D N_m(k\rho)$$

Since $\Phi = 0$ at $\rho = a$, $k_{mn} = \frac{x_{mn}}{a}$ where $J_m(x_{mn}) = 0$

If we want the potential to be finite at $\rho = 0$, $D = 0$

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)\left(A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)\right) \qquad (3.105a)$$

At $z = L$: $V(\rho,\phi) = \sum_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}L)\left(A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)\right)$

$$A_{mn} = \frac{2\,\mathrm{cosech}(k_{mn}L)}{\pi a^2 J^2_{m+1}(k_{mn}a)}\int_0^{2\pi}\int_0^a \rho V(\rho,\phi) J_m(k_{mn}\rho)\sin(m\phi)\,d\rho\,d\phi$$

$$B_{mn} = \frac{2\,\mathrm{cosech}(k_{mn}L)}{\pi a^2 J^2_{m+1}(k_{mn}a)}\int_0^{2\pi}\int_0^a \rho V(\rho,\phi) J_m(k_{mn}\rho)\cos(m\phi)\,d\rho\,d\phi \qquad (3.105b)$$

Section 9. Expansion of Green Functions in Spherical Coordinates

If there are no boundary conditions

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \qquad (3.70)$$

If we have at spherical boundary at $r = a$

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'\left|\vec{x} - \frac{a^2}{x'^2}\cdot\vec{x}'\right|}$$

$$= 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right]Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \qquad (3.114)$$

Section 10. Solution of Potential Problems with the Spherical Green Function Expansion

- The general solution to the Poisson equation with a potential surface on the boundary surface:

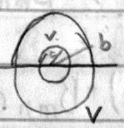
$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')G(\vec{x},\vec{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\cdot\frac{\partial G}{\partial n'}da' \qquad (3.126)$$

$$\frac{\partial G}{\partial n'} = -\frac{4\pi}{b^2}\sum_{l,m}\left(\frac{r}{b}\right)^l Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \qquad (3.127)$$

$$\Phi(\vec{x}) = \sum_{l,m}\left[\int V(\theta',\phi')Y^*_{lm}(\theta',\phi')d\Omega'\right]\left(\frac{r}{b}\right)^l Y_{lm}(\theta,\phi) \qquad (3.128)$$

Problems

1.



Here, we can use the azimuthally symmetric solution

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos\theta) \qquad (3.33)$$

$$V(\theta) = \sum_{l=0}^{\infty}\left[A_l a^l + B_l a^{-(l+1)}\right]P_l(\cos\theta)$$

$$V(\theta) = \begin{cases} V, & 0 \le \theta < \pi/2 \\ 0, & \pi/2 < \theta \le \pi \end{cases}$$

$$\int_0^\pi V(\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta = \int_0^\pi \sum_{l=0}^{\infty}\left[A_l a^l + B_l a^{-(l+1)}\right]P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$$\cos\theta = x$$

$$-\sin\theta\, d\theta = dx$$

$$\int_{-1}^1 V(x)P_{l'}(x)dx = \sum_{l=0}^{\infty}\left[A_l a^l + B_l a^{-(l+1)}\right]\int_{-1}^1 P_l(x)P_{l'}(x)dx$$

$$= \sum_{l=0}^{\infty}\left[A_l a^l + B_l a^{-(l+1)}\right]\cdot\frac{2}{2l+1}\delta_{ll'} \qquad (3.21)$$

$$\int_{-1}^1 V(x)P_l(x)dx = \left(A_l a^l + B_l a^{-(l+1)}\right)\frac{2}{2l+1}$$

$$A_l a^l + B_l a^{-(l+1)} = \frac{2l+1}{2}\cdot V\int_0^1 P_l(x)dx$$

Following the same steps for the boundary at $r=b$

$$A_l b^l + B_l b^{-(l+1)} = \frac{2l+1}{2}\cdot V\int_{-1}^0 P_l(x)dx$$

$$= \frac{2l+1}{2}V(-1)^l\int_0^1 P_l(x)dx$$

$$A_l = \frac{(2l+1)V\int_0^1 P_l(x)dx}{2a^l} - B_l a^{-2l-1}$$

$$\frac{b^l(2l+1)V\int_0^1 P_l(x)dx}{2a^l} - B_l a^{-2l-1}b^l + B_l b^{-l-1} = \frac{(2l+1)V(-1)^l\int_0^1 P_l(x)dx}{2}$$

$$B_l\left(b^{-l-1} - a^{-2l-1}b^l\right) = \frac{(2l+1)V\int_0^1 P_l(x)dx}{2}\left((-1)^l - \frac{b^l}{a^l}\right)$$

$$B_l = \left(\frac{2l+1}{2}\right)V\int_0^1 P_l(x)dx\cdot\left[(-1)^l - \frac{b^l}{a^l}\right]\cdot\frac{1}{b^{-l-1} - b^l a^{-2l-1}}$$

$$= \left(\frac{2l+1}{2}\right)V\int_0^1 P_l(x)dx\cdot\left[(-1)^l - \frac{b^l}{a^l}\right]\cdot\frac{b^{l+1}a^{2l+1}}{a^{2l+1} - b^{2l+1}}$$

$$= \frac{(-1)^l b^{l+1}a^{2l+1} - a^{l+1}b^{2l+1}}{a^{2l+1} - b^{2l+1}}\left(\frac{2l+1}{2}\right)V\int_0^1 P_l(x)dx$$

$$A_l = \frac{(2l+1)V\int_0^1 P_l(x)dx}{2}\cdot\left[\frac{1}{a^l} - \frac{1}{a^{2l+1}}\cdot\frac{(-1)^l b^{l+1}a^{2l+1} - a^{l+1}b^{2l+1}}{a^{2l+1} - b^{2l+1}}\right]$$

$$= \frac{a^{l+1} - (-1)^l b^{l+1}}{a^{2l+1} - b^{2l+1}}\left(\frac{2l+1}{2}\right)V\int_0^1 P_l(x)dx$$

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(\frac{2l+1}{2}\right)V\left(\frac{a^{l+1} - (-1)^l b^{l+1}}{a^{2l+1} - b^{2l+1}}r^l - \frac{a^{l+1}b^{2l+1} - (-1)^l b^{l+1}a^{2l+1}}{a^{2l+1} - b^{2l+1}}r^{-(l+1)}\right)P_l(\cos\theta)\int_0^1 P_l(x)dx$$

$$\int_0^1 P_l(x)dx$$

$$\int_0^1 P_0(x)dx = 1$$

$$\int_0^1 P_l(x)dx = \frac{1}{2l+1}\int_0^1 \frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx}\,dx \qquad (3.28)$$

$$= \frac{1}{2l+1}\left(P_{l+1}(x) - P_{l-1}(x)\right)\Big|_0^1$$

$$= \frac{1}{2l+1}\left(P_{l-1}(0) - P_{l+1}(0)\right)$$

$= 0$ when l is even

Up to $l=4$

$$\Phi(r,\theta) = \frac{V}{2}\left[1 + \frac{3}{2}\left(\frac{(a^2+b^2)r}{a^3-b^3} - \frac{(a^2b^3+a^3b^2)r^{-2}}{a^3-b^3}\right)\cos\theta - \frac{7}{16}\left(\frac{(a^4+b^4)r^3}{a^7-b^7} - \frac{(a^4b^7+a^7b^4)r^{-4}}{a^7-b^7}\right)(5\cos^3\theta - 3\cos\theta) + \dots\right]$$

As $b\to\infty$, $\Phi \approx \frac{V}{2}\left[1+\sum_{l=0}^{\infty}[P_{l-1}(0)-P_{l+1}(0)]\left(\frac{a}{r}\right)^{l+1}P_l(\cos\theta)\right]$

$$\approx \frac{V}{2}\left[1+\left(\frac{3}{2}\right)\cdot\left(\frac{a}{r}\right)^2\cos\theta-\frac{7}{16}\left(\frac{a}{r}\right)^4(5\cos^3\theta-3\cos\theta)+\dots\right]$$

Since r is unbound

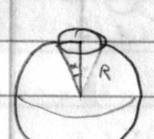
$$V(\theta)=\sum_{l=0}^{\infty} B_l a^{-(l+1)}P_l(\cos\theta)$$

$$B_l=\frac{(2l+1)}{2}a^{l+1}\int_0^1 VP_l(x)\,dx$$

$$=\frac{V}{2}a^{l+1}[P_{l-1}(0)-P_{l+1}(0)]$$

$$\approx\frac{V}{2}\left[1+\sum_{l=0}^{\infty}[P_{l-1}(0)-P_{l+1}(0)]\left(\frac{a}{r}\right)^{l+1}P_l(\cos\theta)\right]$$

2.



Again, we have azimuthal symmetry

a. Since we don't have the boundary conditions,

$$\sigma=\varepsilon_0\left[-\frac{\partial\Phi_{out}}{\partial r}+\frac{\partial\Phi_{in}}{\partial r}\right]\Big|_{r=R}$$

$$\sigma=Q/4\pi R^2,\quad \alpha<\theta\le\pi$$
$$0,\quad 0\le\theta<\alpha$$

$$\Phi_{in}=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)$$

$$\Phi_{out}=A_0+\sum_{l=0}^{\infty}B_l r^{-(l+1)}P_l(\cos\theta) \qquad (3.33)$$

At $r=\infty$, $\Phi_{out}=Q_{tot}/4\pi\varepsilon_0 r$

$$Q_{tot}=2\pi R^2\cdot\frac{Q}{4\pi R^2}\int_\alpha^\pi\sin\theta\,d\theta$$

$$=Q/2\,(-\cos\theta)\big|_\alpha^\pi=Q/2\,(1+\cos\alpha)$$

$$\frac{Q(1+\cos\alpha)}{8\pi\varepsilon_0 r}=A_0+\frac{B_0}{r}$$

$$A_0=0$$
$$B_0=Q(1+\cos\alpha)/8\pi\varepsilon_0$$

Φ must be continuous

$$A_0+\sum_{l=1}^{\infty}A_l R^l P_l(\cos\theta)=Q(1+\cos\alpha)/8\pi\varepsilon_0 R+\sum_{l=1}^{\infty}B_l R^{-(l+1)}P_l(\cos\theta)$$

$$A_0=Q(1+\cos\alpha)/8\pi\varepsilon_0 R$$
$$B_l=A_l R^{2l+1}$$

$$\Phi_{in}=Q(1+\cos\alpha)/8\pi\varepsilon_0 R+\sum_{l=1}^{\infty}A_l r^l P_l(\cos\theta)$$

$$\frac{\partial\Phi_{in}}{\partial r}=\sum_{l=1}^{\infty}A_l\, l\, r^{l-1}P_l(\cos\theta)$$

$$\Phi_{out}=Q(1+\cos\alpha)/8\pi\varepsilon_0 r+\sum_{l=1}^{\infty}A_l R^{2l+1} r^{-(l+1)}P_l(\cos\theta)$$

$$\frac{\partial\Phi_{out}}{\partial r}=-Q(1+\cos\alpha)/8\pi\varepsilon_0 r^2-\sum_{l=1}^{\infty}A_l R^{2l+1}(l+1)r^{-l-2}P_l(\cos\theta)$$

$$\sigma=\varepsilon_0\left[\frac{+Q(1+\cos\alpha)}{8\pi\varepsilon_0 R^2}+\sum_{l=1}^{\infty}A_l P_l(\cos\theta)R^{l-1}(2l+1)\right]$$

Want to find $l\ge 1$, so we ignore the first term since that corresponds to $l=0$

$$\int_0^\pi\sigma\sin\theta P_{l'}(\cos\theta)\,d\theta=\sum_{l=1}^{\infty}\varepsilon_0 A_l R^{l-1}(2l+1)\int_0^\pi\sin\theta P_l(\cos\theta)P_{l'}(\cos\theta)\,d\theta$$

$$=\sum_{l=1}^{\infty}\varepsilon_0 A_l R^{l-1}\cdot 2\delta_{ll'} \qquad (3.21)$$

$$\int_\alpha^\pi\frac{Q}{4\pi R^2}\sin\theta\cdot P_l(\cos\theta)\,d\theta=2\varepsilon_0 A_l R^{l-1}$$

$$A_l=\frac{1}{2\varepsilon_0 R^{l-1}}\cdot\frac{Q}{4\pi R^2}\int_{-1}^{\cos\alpha}P_l(x)\,dx$$

$$=\frac{Q}{8\pi\varepsilon_0 R^{l+1}(2l+1)}\int_{-1}^{\cos\alpha}\frac{dP_{l+1}(x)}{dx}-\frac{dP_{l-1}(x)}{dx}\,dx \qquad (3.28)$$

$$=\frac{Q[P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)]}{8\pi\varepsilon_0 R^{l+1}(2l+1)}$$

$$\Phi_{in}=\frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\cdot[P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)]\cdot\frac{r^l}{R^{l+1}}\cdot P_l(\cos\theta)$$

$$\Phi_{out}=\frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\cdot[P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)]\cdot\frac{R^l}{r^{l+1}}P_l(\cos\theta)$$

b. $E_{in} = -\nabla \Phi_{in}$

$$= -\frac{\partial \Phi}{\partial r}\hat{r} - \frac{1}{r}\cdot\frac{\partial \Phi}{\partial \theta}\hat{\theta}$$

$$= -\frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{l\,[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)]}{2l+1}\cdot\frac{r^{l-1}}{R^{l+1}}P_l(\cos\theta)\cdot\hat{r}$$

$$- \frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)]}{2l+1}\cdot\frac{r^{l-1}}{R^{l+1}}\frac{dP_l(\cos\theta)}{d\theta}\hat{\theta}$$

We see that at the origin, only $l=1$ survives:

$$E_{in}(0) = \frac{-Q(P_2(\cos\alpha) - P_0(\cos\alpha))}{8\pi\varepsilon_0}\left(\frac{\cos\theta\,\hat{r}}{3R^2} - \frac{\sin\theta\,\hat{\theta}}{3R^2}\right)$$

$$= \frac{-Q\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2} - 1\right)(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})}{24\pi\varepsilon_0 R^2}$$

$$= \frac{-Q\sin^2\alpha\,(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})}{16\pi\varepsilon_0 R^2}$$

c. As $\alpha \to 0$, $P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha) = P_{l+1}(1) - P_{l-1}(1) = 0$

Only $l=0$ survives

$$\Phi_{in} = \frac{Q}{8\pi\varepsilon_0}\cdot\frac{2}{R} = \frac{Q}{4\pi\varepsilon_0 R}$$

$$\Phi_{out} = \frac{Q}{4\pi\varepsilon_0 r}$$

As $\alpha \to \pi$, $P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha) = P_{l+1}(-1) - P_{l-1}(-1) = 0$

even for $l=0$

$$\Phi = 0$$

3.

a. Since we have a charge distribution and a boundary, we must determine Green function as in section 3.9

$$G(\vec{x},\vec{x}') = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\cdot\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

$$\Phi(\vec{x}) = -\frac{1}{\varepsilon_0}\int G(\vec{x},\vec{x}')\rho(\vec{x}')\,d^3x'$$

$$\int \rho(\vec{x})\,d^3x = Q$$

$$\int_0^{2\pi}\int_{-1}^{1}\int_0^R a\,\delta(\cos\theta)\,\Theta(R-r)\,(R^2-r^2)^{-1/2}\cdot r^2\,d\phi\,d\cos\theta\,dr = Q$$

$$2\pi a\int_0^R \frac{r^2}{(R^2-r^2)^{1/2}}\,dr = Q$$

$$\rho(\vec{x})\,d^3x = \frac{Q}{2\pi R}\,\delta(\cos\theta)\,\Theta(R-r)\,\frac{r}{\sqrt{R^2-r^2}}\,d\cos\theta\,d\phi\,dr$$

Since we have azimuthal symmetry, $m=0$

$$\Phi = \frac{Q}{2\pi\varepsilon_0 R}\int_{-1}^{1}\int_0^{2\pi}\int_0^R\frac{\Theta(R-r)\,r'\,\delta(\cos\theta)}{\sqrt{R^2-r^2}}\sum_{l=0}^{\infty}\frac{1}{2l+1}\cdot\frac{r_<^l}{r_>^{l+1}}Y_{l0}^*(\theta',\phi')Y_{l0}(\theta,\phi)\,d\cos\theta'\,d\phi'\,dr'$$

$$= \frac{Q}{4\pi\varepsilon_0 R}\sum_{l=0}^{\infty}I_{2n}(r)P_{2n}(0)P_{2n}(\cos\theta) = \frac{2V}{\pi}\sum_{n=0}^{\infty}I_{2n}(r)P_{2n}(0)P_{2n}(\cos\theta)$$

using $Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\theta)$ (3.57)

$$I_l(r) = \int_0^R\frac{r_<^l}{r_>^{l+1}}\cdot\frac{r'\,dr'}{\sqrt{R^2-r'^2}}$$

for $r > R$

$$I_l(r) = \int_0^R\frac{r'^{l+1}}{r^{l+1}\sqrt{R^2-r'^2}}$$

$$\Phi_{out} = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{l+1}P_{2l}(\cos\theta)$$

b. $I_\ell(r) = \int_0^r \frac{r'^{\ell+1}\,dr'}{r^{\ell+1}\sqrt{R^2-r'^2}} + \int_r^R \frac{r^\ell\,dr'}{r'^\ell\sqrt{R^2-r'^2}}$

c. $C = Q/V$

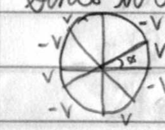
$\Phi = Q/8\varepsilon_0 R$

$C = 8\varepsilon_0 R$

4.

a. $\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[A_{\ell m}r^\ell + B_{\ell m}r^{-(\ell+1)}\right]Y_{\ell m}(\theta,\phi)$ (3.61)

Since we're looking at the inside, $B_{\ell m} = 0$



$$V(\phi) = \begin{cases} +V & \frac{2\pi}{n}\cdot j < \phi < \pi(2j+1)/n \\ -V & (2j+1)\pi/n < \phi < (2j+2)\pi/n \end{cases}$$

$$V(\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m}a^\ell Y_{\ell m}(\theta,\phi)$$

$$\int_0^{2\pi}\int_0^{\pi} V(\phi)Y^*_{\ell' m'}(\theta,\phi)\sin\theta\,d\theta\,d\phi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_0^{2\pi}\int_0^{\pi} A_{\ell m}a^\ell Y_{\ell m}(\theta,\phi)Y^*_{\ell' m'}(\theta,\phi)\sin\theta\,d\theta\,d\phi$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m}a^\ell \delta_{\ell\ell'}\delta_{mm'} \qquad (3.55)$$

$$A_{\ell m}a^\ell = \int_0^{2\pi}\int_0^{\pi} V(\phi)Y^*_{\ell m}(\theta,\phi)\sin\theta\,d\theta\,d\phi$$

$$A_{\ell m} = \frac{1}{a^\ell}\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\int_0^{2\pi}\int_0^{\pi} V(\phi)P_\ell^m(\cos\theta)\exp(-im\phi)\sin\theta\,d\theta\,d\phi \qquad (3.53)$$

$$= \frac{V}{a^\ell}\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\sum_{j=0}^{n-1}\int_{-1}^{1}\left[\int_{2j\pi/n}^{\pi(2j+1)/n} P_\ell^m(x)\exp(-im\phi)\,d\phi\,dx - \int_{(2j+1)\pi/n}^{(2j+2)\pi/n} P_\ell^m(x)\exp(-im\phi)\,d\phi\,dx\right]$$

$$\sum_{j=0}^{n-1}\int_{2j\pi/n}^{\pi(2j+1)/n}\exp(-im\phi)\,d\phi - \int_{(2j+1)\pi/n}^{(2j+2)\pi/n}\exp(-im\phi)\,d\phi$$

for $m=0$: $= \frac{(2j+1)\pi}{n} - \frac{2j\pi}{n} - \frac{(2j+2)\pi}{n} + \frac{(2j+1)\pi}{n} = 0$

for $m \neq 0$: $\frac{i}{m}\sum_{j=0}^{n-1}\left[\exp\left(-im\pi(2j+1)/n\right) - \exp\left(-im2\pi j/n\right) - \exp\left(-im(2j+2)\pi/n\right) + \exp\left(-im\pi(2j+1)/n\right)\right]$

$$= \frac{i}{m}\sum_{j=0}^{n-1}\exp\left(-2im\pi j/n\right)\left[2\exp\left(-im\pi/n\right) - 1 - \exp\left(-2im\pi/n\right)\right]$$

$$= i/m\sum_{j=0}^{\infty}\exp\left(-2im\pi j/n\right)\left(\exp\left(-im\pi/n\right) - 1\right)^2$$

for $m/2n = k$, $= 0$

$$\Phi(r,\theta,\phi) = -V\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\cdot\frac{i}{m}\left(\exp\left(-im\pi/n\right)-1\right)^2\sum_{j=0}^{n-1}\exp\left(-2imj\pi/n\right)\int_{-1}^{1}P_\ell^m(x)\,dx \cdot \left(\frac{r}{a}\right)^\ell Y_{\ell m}(\theta,\phi)$$

b. $n=1$

$$\Phi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell,\,\text{odd}}^{\ell} -V\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\cdot\frac{4i}{m}\left(\frac{r}{a}\right)^\ell Y_{\ell m}(\theta,\phi)\int_{-1}^{1}P_\ell^m(x)\,dx$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell,\,\text{odd}}^{\ell} A_{\ell m}(r/a)^\ell Y_{\ell m}(\theta,\phi)$$

$$A_{\ell m} = -V\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\cdot\frac{4i}{m}\int_{-1}^{1}P_\ell^m(x)\,dx$$

$$A_{1,1} = -V\sqrt{\frac{3}{8\pi}}\cdot\frac{4i}{1}\int_{-1}^{1} -(1-x^2)^{1/2}\,dx$$

$$= Vi\sqrt{3/8\pi}\cdot 4i\cdot\pi/2 = iV\sqrt{3\pi/2}$$

$$A_{1,-1} = iV\sqrt{3\pi/2}$$

$$A_{2,1} = -V\sqrt{\frac{5}{4\pi\cdot 6}}\cdot 4i\int_{-1}^{1} -(1-x^2)^{1/2}\cdot 3x\,dx$$

$$= 0$$

$$A_{2,-1} = 0$$

$$A_{3,3} = -V\sqrt{\frac{7}{4\pi\cdot 6!}}\cdot\frac{4i}{3}\int_{-1}^{1} -(1-x^2)^{3/2}\cdot 15\,dx$$

$$= iV\sqrt{7/4\pi\cdot 6!}\cdot\frac{4i}{3}\cdot 45\pi/2 = iV\sqrt{\frac{35\pi}{256}}$$

$$A_{3,-3} = -iV\sqrt{35\pi/256}$$

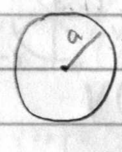
$$A_{3,1} = -V\sqrt{\frac{7\cdot 2!}{4\pi\cdot 4!}}\cdot\frac{4i}{1}\int_{-1}^{1} -(1-x^2)^{1/2}\cdot(15/2\,x^2-3)\,dx$$

$$= iV\sqrt{21\pi/256}$$

$$A_{3,-1} = -iV\sqrt{21\pi/256}$$

$$\Phi = \left(\frac{r}{a}\right) iV\sqrt{\frac{3\pi}{2}}\left(Y_{1,1}(\theta,\phi)+Y_{1,-1}(\theta,\phi)\right)$$
$$+\left(\frac{r}{a}\right)^3 iV\left(\sqrt{\frac{35\pi}{256}}\left(Y_{3,3}(\theta,\phi)+Y_{3,-3}(\theta,\phi)\right) + \sqrt{\frac{21\pi}{256}}\left(Y_{3,1}(\theta,\phi)+Y_{3,-1}(\theta,\phi)\right)\right)$$
$$= \frac{3}{2}V(r/a)\sin\theta\sin\phi + (r/a)^3 V\left(\frac{35}{64}\sin^3\theta\sin^3\phi + \frac{21}{64}\sin\theta(5\cos^2\theta-1)\sin\phi\right)$$
$\cos\theta' = \sin\theta\sin\phi$
$$\Phi = \frac{3}{2} V(r/a)\cos\theta' - \frac{7}{8}(r/a)^3 V\left(\frac{5}{2}\cos^3\theta' - \frac{3}{2}\cos\theta'\right)$$
compare to (2.27)

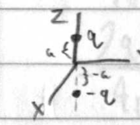
5.

a. 

$$G(\vec x,\vec x') = \frac{1}{|\vec x-\vec x'|} - \frac{1}{\left|\frac{x'}{a}\vec x - \frac{a}{x'}\vec x'\right|} \quad (2.16)$$
$$= \frac{1}{(x^2+x'^2-2xx'\cos\gamma)^{1/2}} - \frac{1}{\left(\frac{x^2x'^2}{a^2}+a^2-2xx'\cos\gamma\right)^{1/2}} \quad (2.17)$$
$$\Phi(\vec x) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec x')G_D(\vec x,\vec x')\,d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec x')\frac{\partial G_D}{\partial n'}\,da' \quad (1.44)$$
$$-\frac{\partial G}{\partial x'} = -\frac{2x'-2x\cos\gamma}{2(x^2+x'^2-2xx'\cos\gamma)^{3/2}} + \frac{\left(2x^2x'/a^2 - 2x\cos\gamma\right)}{2\left(x^2x'^2/a^2+a^2-2xx'\cos\gamma\right)^{3/2}}$$
$$-\frac{\partial G}{\partial x'}\Big|_{x'=a} = \frac{a - x^2/a}{(x^2+a^2-2xa\cos\gamma)^{3/2}} = \frac{a^2-x^2}{a(x^2+a^2-2ax\cos\gamma)^{3/2}}$$
$$\Phi = \frac{1}{4\pi}\int \frac{V(\theta',\phi')(a^2-r^2)}{a(r^2+a^2-2ar\cos\gamma)^{3/2}}\cdot a^2\, d\Omega'$$
$$= \frac{a(a^2-r^2)}{4\pi}\int\frac{V(\theta',\phi')}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\,d\Omega'$$

b. $$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^l + B_{lm}r^{-(l+1)}\right]Y_{lm}(\theta,\phi) \quad (3.61)$$
Since we're looking at inside the sphere
$$V(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}a^l Y_{lm}(\theta,\phi)$$
$$\int \sin\theta\, V(\theta,\phi)\, Y^*_{l'm'}(\theta,\phi)\,d\Omega = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}a^l\int\sin\theta\, Y_{lm}(\theta,\phi)Y^*_{l'm'}(\theta,\phi)\,d\Omega$$
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}a^l\,\delta_{ll'}\delta_{mm'} \quad (3.55)$$
$$A_{lm} = \int Y^*_{lm}(\theta',\phi')V(\theta',\phi')\,d\Omega'\cdot 1/a^l$$

$$\Phi(\vec x) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}(r/a)^l Y_{lm}(\theta,\phi)$$
$$A_{lm} = \int Y^*_{lm}(\theta',\phi')V(\theta',\phi')\,d\Omega'$$

6. 

a. $$\Phi = \frac{q}{4\pi\epsilon_0|\vec x - a\hat k|} - \frac{q}{4\pi\epsilon_0|\vec x + a\hat k|}$$
$$\frac{1}{|\vec x-\vec x'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\cdot\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \quad (3.70)$$
$$\Phi = \frac{q}{\epsilon_0}\left[\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\cdot\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)\left[Y^*_{lm}(0,0)-Y^*_{lm}(\pi,0)\right]\right]$$

$r<a$	$r>a$
$r_<=r$	$r_<=a$
$r_>=a$	$r_>=r$

b. As $a\to 0$, we will be in the $r>a$
$$\Phi = \frac{q}{\epsilon_0}\left[\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\cdot\frac{a^l}{r^{l+1}}Y_{lm}(\theta,\phi)\left[Y^*_{lm}(0,0)-Y^*_{lm}(\pi,0)\right]\right]$$
Since we have azimuthal symmetry, $m=0$
$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\sum_{l=0}^{\infty}\frac{a^l}{r^{l+1}}P_l(\cos\theta)\left[P_l(1)-P_l(-1)\right]\right]$$
$$= \frac{q}{4\pi\epsilon_0}\sum_{l=0,\,odd}\frac{2a^l P_l(\cos\theta)}{r^{l+1}}$$
$$= \frac{p}{4\pi\epsilon_0}\sum_{l=0,\,odd}\frac{a^{l-1}}{r^{l+1}}P_l(\cos\theta)$$
$$= \frac{p}{4\pi\epsilon_0}\left[\frac{1}{r^2}P_1(\cos\theta) + \frac{a^2}{r^4}P_3(\cos\theta)+\dots\right]$$
$$= \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

c. We have the potential found in the previous section in addition to the potential due to a grounded sphere

$$\Phi = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos\theta) \qquad (3.33)$$

Since inside

$$= \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Since it is grounded

$$0 = \frac{p\cos\theta}{4\pi\varepsilon_0 b^2} + A_l b^l P_l(\cos\theta)$$

$$\Rightarrow l = 1$$

$$0 = \frac{p\cos\theta}{4\pi\varepsilon_0 b^2} + A_1 b\cos\theta$$

$$A_1 = -\frac{p}{4\pi\varepsilon_0 b^3}$$

$$\Phi = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} - \frac{pr\cos\theta}{4\pi\varepsilon_0 b^3} = \frac{p\cos\theta}{4\pi\varepsilon_0 b^2}\left[\frac{b^2}{r^2} - \frac{r}{b}\right]$$

7.

a. $$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x} - a\hat{k}|} + \frac{1}{|\vec{x} + a\hat{k}|} - \frac{2}{|\vec{x}|}\right]$$

Since we have azimuthal symmetry

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta) \qquad (3.38)$$

$$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\sum_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}} + \frac{(-1)^l r_<^l}{r_>^{l+1}}\right)P_l(\cos\theta) - \frac{2}{r}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\sum_{l=0,\text{even}} \frac{2r_<^l}{r_>^{l+1}} P_l(\cos\theta) - \frac{2}{r}\right]$$

As $a \to 0$, $r > a$

$$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\sum_{l=0,\text{even}} \frac{2a^l}{r^{l+1}} P_l(\cos\theta) - \frac{2}{r}\right]$$

$$= \frac{2q}{4\pi\varepsilon_0}\left[\sum_{l\,\text{even}} \frac{a^l}{r^{l+1}} P_l(\cos\theta)\right]$$

$$= \frac{2Q}{4\pi\varepsilon_0}\sum_{\text{even}} \frac{a^{l-2}}{r^{l+1}} P_l(\cos\theta)$$

$$= \frac{2Q}{4\pi\varepsilon_0}\left[\frac{a^0}{r^3} P_2(\cos\theta)\right] = \frac{Q(3\cos^2\theta - 1)}{4\pi\varepsilon_0 r^3}$$

b. $$\Phi = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos\theta) \qquad (3.33)$$

$$= \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$0 = \frac{q}{4\pi\varepsilon_0}\left[\sum_{l=0,\text{even}} \frac{2a^l}{b^{l+1}} P_l(\cos\theta) - \frac{2}{r}\right] + A_l b^l P_l(\cos\theta)$$

since at the boundary, $r > a$

$$0 = \frac{q}{4\pi\varepsilon_0} \cdot \frac{2a^l}{b^{l+1}} + A_l b^l$$

$$A_l = \frac{-qa^l}{2\pi\varepsilon_0 b^{2l+1}}$$

Since orthogonality, this only holds for l even. Other terms $= 0$

$$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\sum_{l=0}^{\text{even}} \frac{2r_<^l}{r_>^{l+1}} P_l(\cos\theta) - \frac{2}{r}\right] - \sum_{l\,\text{even}} \frac{qa^l r^l P_l(\cos\theta)}{2\pi\varepsilon_0 b^{2l+1}}$$

As $a \to 0$, only $l = 2$ survives

$$\Phi = \frac{Q(3\cos^2\theta - 1)}{4\pi\varepsilon_0 r^3} - \frac{2qa^2 r^2 \cdot \frac{1}{2}(3\cos^2\theta - 1)}{4\pi\varepsilon_0 b^5}$$

$$= \frac{Q(3\cos^2\theta - 1)}{4\pi\varepsilon_0 r^3}\left[1 - \frac{r^5}{b^5}\right]$$

$$= \frac{Q}{2\pi\varepsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right) P_2(\cos\theta)$$

8.

a. $$\Phi(\vec{x}) = \frac{Q}{4\pi\varepsilon_0 b}\left\{ \ln(b/r) + \sum_{j=1}^{\infty} \frac{(4j+1)}{2j(2j+1)}\left[1-\left(\frac{r}{b}\right)^{2j}\right] P_{2j}(\cos\theta)\right\} \quad (3.136)$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + l(l+1)P = 0 \quad (3.10)$$

$$\ln(\csc\theta) = \ln(1/\sin\theta) = \ln(1/\sqrt{1-\cos^2\theta}) = -\tfrac{1}{2}\ln(1-\cos^2\theta)$$

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \quad (3.23)$$

$$A_l = \frac{2l+1}{2}\int_{-1}^{1} f(x) P_l(x)\,dx \quad (3.24)$$

$$A_0 = \tfrac{1}{2}\int_{-1}^{1} -\tfrac{1}{2}\ln(1-x^2) P_0(x)\,dx$$

$$= -\tfrac{1}{4}\int_{-1}^{1}\ln(1-x^2)\,dx$$

$$= -\tfrac{1}{4}(4\ln 2 - 4) = 1-\ln 2$$

$$A_l = \frac{2l+1}{2}\int_{-1}^{1} \frac{-\ln(1-x^2)}{2}\cdot\frac{-1}{l(l+1)}\cdot\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]dx$$

$$= \frac{2l+1}{4l(l+1)}\int_{-1}^{1}\ln(1-x^2)\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]dx$$

$u = \ln(1-x^2)$ $\quad v = (1-x^2)\frac{dP}{dx}$

$du = (1/1-x^2)(-2x)$ $\quad dv = \frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]$

$$= \frac{2l+1}{4l(l+1)}\left[\cancel{\ln(1-x^2)(1-x^2)\frac{dP}{dx}\Big|_{-1}^{1}} + \int_{-1}^{1}\frac{2x(1-x^2)}{(1-x^2)}\frac{dP}{dx}dx\right]$$

$$= \frac{2l+1}{2l(l+1)}\int_{-1}^{1} x\frac{dP}{dx}dx$$

$u = x$ $\quad v = P_l(x)$

$du = 1$ $\quad dv = \frac{dP}{dx}$

$$= \frac{2l+1}{2l(l+1)}\left[xP_l(x)\Big|_{-1}^{1} - \int_{-1}^{1}P_l(x)\,dx\right]$$

$= \frac{2l+1}{l(l+1)}$ $\quad l$ even

0 $\quad l$ odd

$$\ln(\csc\theta) = 1-\ln 2 + \sum_{l\,\text{even}} \frac{2l+1}{l(l+1)}P_l(\cos\theta)$$

$$\Phi(x) = \frac{Q}{4\pi\varepsilon_0 b}\left[\ln\left(\frac{2b}{r\sin\theta}\right) - 1 - \sum_{j=1}^{\infty}\frac{4j+1}{2j(2j+1)}\left(\frac{r}{b}\right)^{2j}P_{2j}(\cos\theta)\right]$$

b. $$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma) \quad (3.38)$$

$$\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$$

$$\frac{1}{\sqrt{2-2\cos\theta}} = \sum_{l=0}^{\infty}P_l(\cos\theta)$$

$$\frac{1}{2}\sqrt{\frac{2}{1+\cos\theta}} = \sum_{l=0}^{\infty}P_l(\cos\theta)\cdot(-1)^l$$

$\sin\theta/2 = \sqrt{1-\cos\theta/2}$

$\cos\theta/2 = \sqrt{1+\cos\theta/2}$

$$\frac{1}{\sin\theta/2} + \frac{1}{\cos\theta/2} = \sqrt{\frac{2}{1-\cos\theta}} + \sqrt{\frac{2}{1+\cos\theta}} = 2\sum_{l=0}^{\infty}P_l(\cos\theta) + (-1)^l P_l(\cos\theta)$$

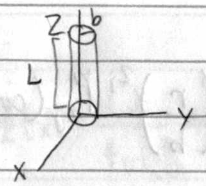
$$\frac{1}{2}\left(\frac{1}{\sin\theta/2} + \frac{1}{\cos\theta/2}\right) = 2\sum_{j=0}^{\infty}P_{2j}(\cos\theta)$$

$$\sigma = \frac{-Q}{4\pi b^2}\sum_{j=0}^{\infty}\frac{4j+1}{2j+1}P_{2j}(\cos\theta) \quad (3.137)$$

$$= \frac{-Q}{4\pi b^2}\left[2P_{2j}(\cos\theta) + \sum_{j=1}^{\infty}\frac{4j+1}{2j+1}P_{2j}(\cos\theta)\right]$$

$$= \frac{-Q}{4\pi b^2}\left[\frac{1}{2}\left(\frac{1}{\sin\theta/2} + \frac{1}{\cos\theta/2}\right) + \sum_{j=0}^{\infty}\frac{1}{2j+1}P_{2j}(\cos\theta)\right]$$

9.



$$\Phi(\rho,\phi,z)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)\left(A_{mn}\sin(m\phi)+B_{mn}\cos(m\phi)\right) \quad (3.105a)$$

$$\sinh(k_{mn}z)=\exp(x)-\exp(-x)/2$$

$$0=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}L)\left(A_{mn}\sin(m\phi)+B_{mn}\cos(m\phi)\right)$$

$$\sinh(k_{mn}L)=0$$

$$\exp(k_{mn}L)-\exp(-k_{mn}L)=0$$

$$\cos(k_{mn}L)+i\sin(k_{mn}L)=\cos(k_{mn}L)-i\sin(k_{mn}L)$$

$$\sin(k_{mn}L)=0$$

$$k_{mn}=in\pi/L$$

$$\Phi=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(in\pi\rho/L)\sinh(in\pi z/L)\left(A_{mn}\sin(m\phi)+B_{mn}\cos(m\phi)\right)$$

$$V=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(in\pi\rho/L)\, i\sin(n\pi z/L)\left(A_{mn}\sin(m\phi)+B_{mn}\cos(m\phi)\right)$$

$$V=\sum_{m=0}^{\infty}\sum_{n=1} i^{m+1} I_m(n\pi b/L)\sin(n\pi z/L)\left[A_{mn}\sin(m\phi)+B_{mn}\cos(m\phi)\right)$$

$$\Phi=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{I_m(n\pi\rho/L)}{I_m(n\pi b/L)}\sin(n\pi z/L)\left[A_{mn}\sin(m\phi)+B_{mn}\cos(m\phi)\right]$$

$$A_{mn}=2\int_0^L\int_0^{2\pi}V(\phi',z')\sin(m\phi')\sin(n\pi z'/L)\,d\phi'\,dz'$$

$$B_{mn}=2\int_0^L\int_0^{2\pi}V(\phi',z')\cos(m\phi')\sin(n\pi z'/L)\,d\phi'\,dz'$$

10. We can start with the solution from the previous section

a. $$A_{mn}=2\left[\int_0^L\int_{-\pi/2}^{\pi/2}V\sin(m\phi')\sin(n\pi z'/L)\,d\phi'\,dz' - \int_0^L\int_{\pi/2}^{3\pi/2}V\sin(m\phi')\sin(n\pi z'/L)\,d\phi'\,dz'\right]$$

$$=2\left[\int_{-\pi/2}^{\pi/2}V\sin(m\phi')\,d\phi' - \int_{\pi/2}^{3\pi/2}V\sin(m\phi')\,d\phi'\right]\left(-L/n\pi\cos(n\pi z'/L)\right)\Big|_0^L$$

$$=4L/n\pi\cdot V\left[-1/m\cos(m\phi')\Big|_{-\pi/2}^{\pi/2}+1/m\cos(m\phi')\Big|_{\pi/2}^{3\pi/2}\right]=0$$

$$B_{mn}=4L/n\pi\; V\left[1/m\sin(m\phi')\Big|_{-\pi/2}^{\pi/2}-1/m\sin(m\phi')\Big|_{\pi/2}^{3\pi/2}\right]$$

$$=16LV/n\pi \quad \text{for } n \text{ odd and } m \text{ odd}$$

$$\Phi=\sum_{m\,\text{odd}}\sum_{n\,\text{odd}}\frac{16LV}{n\pi}\sin(n\pi z/L)\cdot\frac{I_m(n\pi\rho/L)}{I_m(n\pi b/L)}\cos(m\phi)$$

b. at $z=L/2$

$$\Phi=\sum_{m,n\,\text{odd}}\frac{16LV(-1)^{n+1}}{n\pi}\cdot\frac{I_m(n\pi\rho/L)}{I_m(n\pi b/L)}\cos(m\phi)$$

13. $$G(\vec{x},\vec{x}')=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left(1-(a/b)^{2l+1}\right)}\left(r_<^l-\frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right) \quad (3.125)$$

$$\Phi(\vec{x})=-\frac{1}{4\pi}\oint_S\Phi(\vec{x}')\frac{\partial G}{\partial n'}\,da' \quad (3.126)$$

for $r'=a$, $r_<=r'$

$r_>=r$

$$\frac{\partial G}{\partial n'}=-\frac{\partial G}{\partial r'}=-\frac{\partial}{\partial r'}\left(4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left(1-(a/b)^{2l+1}\right)}\left(r'^l-\frac{a^{2l+1}}{r'^{l+1}}\right)\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)\right)$$

$$=-4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left(1-(a/b)^{2l+1}\right)}\left(la^{l-1}+(l+1)a^{l-1}\right)\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)$$

$$=-\frac{4\pi}{a^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{1-(a/b)^{2l+1}}\left[\left(\frac{a}{r}\right)^{l+1}-\frac{a^{l+1}r^l}{b^{2l+1}}\right]$$

for $r'=b$, $r_<=r$

$r_>=r'$

$$\frac{\partial G}{\partial r'}=\frac{-4\pi}{b^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{1-(a/b)^{2l+1}}\left[\left(\frac{r}{b}\right)^l-\frac{a^{2l+1}}{b^l r^{l+1}}\right]$$

Since we have azimuthal symmetry, we say $m=0$ ($Y_{lm}\to\sqrt{\frac{2l+1}{4\pi}}\cdot P_l$),

$$\Phi(\vec{x})=\sum_{l=0}^{\infty}\left[\int_0^1\frac{VP_l(\cos\theta')P_l(\cos\theta)}{1-(a/b)^{2l+1}}\,\frac{2l+1}{4\pi}\left[\left(\frac{a}{r}\right)^{l+1}-\left(\frac{a}{b}\right)^{l+1}\left(\frac{r}{b}\right)^l\right]d\cos\theta'\right.$$

$$\left.+\int_{-1}^{0}\frac{VP_l(\cos\theta')P_l(\cos\theta)}{1-(a/b)^{2l+1}}\cdot\frac{2l+1}{4\pi}\left[\left(\frac{r}{b}\right)^l-\left(\frac{a}{b}\right)^l\left(\frac{a}{r}\right)^{l+1}\right]d\cos\theta'\right]$$

$$=\sum_{l\,\text{odd}}\Big[$$