Solutions to Modern Quantum Mechanics by J. J. Sakurai, 1ed.

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Acknowledgements

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Contents

Chapter 1

Fundamental Concepts

1.1 Commutation Relations

Prove

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB$$

It's a little easier if we start with the right side of this equation. Looking at the individual components,

$$AC\{D, B\} = ACDB + ACBD \tag{1}$$

$$A\{C, B\}D = ACBD + ABCD \tag{2}$$

$$C\{D, A\}B = CDAB + CADB \tag{3}$$

$$\{C, A\}DB = CADB + ACDB \tag{4}$$

Inserting (1), (2), (3), (4) into our initial equation and killing terms, we get

$$ABCD - CDAB$$

which is what we expect if we expand the left side of the equation.

1.2 Pauli Matrices

Suppose a 2×2 matrix X(not necessarily Hermitian, nor unitary) is written as

$$X = a_0 + \vec{\sigma} \cdot \vec{a}$$

where a_0 and $a_{1,2,3}$ are numbers.

1.2.a How are a_0 and a_k $(k = 1, 2, 3)$ related to $\mathrm{Tr}(X)$ and $\mathrm{Tr}(\sigma_k X)$?

We start by taking the trace of X,

$$\mathrm{Tr}(X) = \mathrm{Tr}(a_0 I) + \mathrm{Tr}(\vec{\sigma} \cdot \vec{a}) \tag{1}$$

By definition, the Pauli matrices(σ_k) are traceless, so re-scaling them by a constant factor does nothing to the trace,

$$\mathrm{Tr}(X) = Tr(a_0 I) = 2a_0 \tag{2}$$

If we multiply X by one of the Pauli matrices, we write out explicitly,

$$\sigma_k X = a_0 \sigma_k + a_1 \sigma_k \sigma_1 + a_2 \sigma_k \sigma_2 + a_3 \sigma_k \sigma_3 \tag{3}$$

We should also remember the following relation,

$$\sigma_a \sigma_b = \delta_{ab} I + i\epsilon_{abc} \sigma_c \tag{4}$$

where ϵ_{abc} is the Levi-Civita tensor. When we take the trace of (3), the first term dies since that is just a re-scaled Pauli matrix. Looking at (4), we can convince ourselves that only $a = b$ terms survive since we just get another Pauli matrix otherwise. Thus, only the k term survives

$$\mathrm{Tr}(\sigma_k X) = 2a_k \tag{5}$$

Rewriting for convenience,

$$a_0 = {}^1\!/\!_2\, \mathrm{Tr}(X) \tag{6}$$

$$a_k = {}^1\!/\!_2\, \mathrm{Tr}(\sigma_k X) \tag{7}$$

1.2.b Obtain a_0 and a_k in terms of the matrix elements X_{ij}.

As a reminder, the Pauli matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

From this, we can write X,

$$X = \begin{bmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{bmatrix} \tag{9}$$

Multiplying each Pauli matrix by X, i.e., $\sigma_k X$,

$$\begin{cases} \sigma_1 X = \begin{bmatrix} X_{21} & X_{22} \\ X_{11} & X_{12} \end{bmatrix} \\ \\ \sigma_2 X = \begin{bmatrix} -iX_{21} & -iX_{22} \\ iX_{11} & iX_{12} \end{bmatrix} \\ \\ \sigma_3 X = \begin{bmatrix} X_{11} & X_{12} \\ -X_{12} & -X_{22} \end{bmatrix} \end{cases} \tag{10}$$

Using (6) and (7),

$$\begin{cases} a_0 = {}^1\!/\!_2 \operatorname{Tr}(X) = {}^1\!/\!_2(X_{11} + X_{22}) \\ a_1 = {}^1\!/\!_2 \operatorname{Tr}(\sigma_1 X) = {}^1\!/\!_2(X_{21} + X_{12}) \\ a_2 = {}^1\!/\!_2 \operatorname{Tr}(\sigma_2 X) = {}^1\!/\!_2(-iX_{21} + iX_{12}) \\ a_3 = {}^1\!/\!_2 \operatorname{Tr}(\sigma_3 X) = {}^1\!/\!_2(X_{11} - X_{22}) \end{cases} \tag{11}$$

1.3 Invariant Determinant

Show that the determinant of a 2×2 matrix $\vec{\sigma} \cdot \vec{a}$ is invariant under

$$\vec{\sigma} \cdot \vec{a} \to \vec{\sigma} \cdot \vec{a}' \equiv \exp\left(\frac{i\vec{\sigma} \cdot \hat{n}\phi}{2}\right) \vec{\sigma} \cdot \vec{a} \exp\left(\frac{-i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)$$

Find a'_k in terms of a_k when $\hat{n}$ is in the positive z-direction and interpret your result.

Let's go ahead and take the determinant of both sides. We know that the determinant of a matrix product is equal to the product of the determinant of the individual matrices, so we can break up the right side of this equation,

$$\det(\vec{\sigma} \cdot \vec{a}') = \det\left(\exp\left(\frac{i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right) \det(\vec{\sigma} \cdot \vec{a}') \det\left(\exp\left(\frac{-i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right) \tag{1}$$

Each determinant is just a scalar, so we can rearrange them for free,

$$= \det\left(\exp\left(\frac{i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right) \det\left(\exp\left(\frac{-i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right) \det(\vec{\sigma} \cdot \vec{a})$$

And then recombine,

$$= \det\left(\exp\left(\frac{i\vec{\sigma} \cdot \hat{n}\phi}{2}\right) \exp\left(\frac{i\vec{\sigma} \cdot \hat{n}\phi}{2}\right)\right) \det(\vec{\sigma} \cdot \vec{a})$$

$$\det(\vec{\sigma} \cdot \vec{a}') = \det(\vec{\sigma} \cdot \vec{a}) \tag{2}$$

Setting $\hat{n}$ in the z-direction,

$$\hat{n} = \hat{z} = (0, 0, 1) \tag{3}$$

Substituting this in, we pick out the σ_z Pauli matrix,

$$\sigma \cdot \hat{a}' = \exp\left(\frac{i\sigma_z\phi}{2}\right) \sigma \cdot \vec{a} \exp\left(\frac{-i\sigma_z\phi}{2}\right) \tag{4}$$

In matrix form,

$$= \begin{pmatrix} \exp(^{i\phi}/_2) & 0 \\ 0 & \exp(-^{i\phi}/_2) \end{pmatrix} \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} \begin{pmatrix} \exp(-^{i\phi}/_2) & 0 \\ 0 & \exp(^{i\phi}/_2) \end{pmatrix}$$

$$\vec{\sigma} \cdot \vec{a}' = \begin{pmatrix} a_3 & (a_1 - ia_2)\exp(i\phi) \\ (a_1 + ia_2)\exp(-i\phi) & -a_3 \end{pmatrix} \tag{5}$$

We notice that $\vec{\sigma} \cdot \vec{a}'$ can be written as X from problem 1.2,

$$\vec{\sigma} \cdot \vec{a}' = a_0 + \sum_{k=1}^{3} \sigma_k a_k \tag{6}$$

We already solved for how to find a_k in terms of the matrix elements of X, see (6) and (7) of problem 1.2.

$$\begin{cases} a_0' = \dfrac{a_3 - a_3}{2} = 0 \\ a_1' = {}^1\!/\!{}_2[(a_1 + ia_2)\exp(-i\phi) + (a_1 - ia_2)\exp(i\phi)] = a_1\cos(\phi) + a_2\sin(\phi) \\ a_2' = {}^1\!/\!{}_2[-i(a_1 + ia_2\exp(-i\phi) + i(a_1 - ia_2)\exp(i\phi)] = -a_1\sin(\phi) + a_2\cos(\phi) \\ a_3' = {}^1\!/\!{}_2(a_3 + a_3) = a_3 \end{cases} \tag{7}$$

This transformation represents a rotation about the z-axis.

1.4 Bra-Ket Algebra

Using the rules of bra-ket algebra, prove or evaluate the following:

1.4.a $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$, where X and Y are operators;

From (1.5.14), the trace of XY is:

$$\mathrm{Tr}(XY) = \sum_{a'} \langle a'|XY|a'\rangle \tag{1}$$

We can insert identity,

$$= \sum_{a'} \sum_{b'} \langle a'|X|b'\rangle \langle b'|Y|a'\rangle \tag{2}$$

Since both terms are scalars, we can rearrange them freely,

$$= \sum_{a'} \sum_{b'} \langle b'|Y|a'\rangle \langle a'|X|b'\rangle \tag{3}$$

$$= \sum_{b'} \langle b'|YX|b'\rangle \tag{4}$$

Since the trace is independent of representation (1.5.15), we can convert b' back to a',

$$= \sum_{a'} \langle a'|YX|a'\rangle \tag{5}$$

$$\mathrm{Tr}(XY) = \mathrm{Tr}(YX) \tag{6}$$

1.4.b $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators;

Let's act XY on some unsuspecting ket, $|\alpha\rangle$,

$$(XY)\,|\alpha\rangle \tag{7}$$

The dual-correspondence (1.2.10),

$$\langle\alpha|\,(XY)^\dagger \tag{8}$$

Alternatively, we can write,

$$XY\,|\alpha\rangle = X(Y\,|\alpha\rangle) \tag{9}$$

The dual-correspondence,

$$\langle\alpha|\,Y^\dagger X^\dagger \tag{10}$$

Comparing (8) and (10), it follow,

$$(XY)^\dagger = Y^\dagger X^\dagger \tag{11}$$

1.4.c $\exp[if(A)] =?$ in ket-bra form, where A is a Hermitian operator whose eigenvalues are known;

Let's act the function on a vector,

$$\exp(if(A))\,|\alpha\rangle = [\cos(f(A)) + i\sin(f(A))]\,|\alpha\rangle \tag{12}$$

From (1.7.9), if we know the eigenvalues of A, we can replace the operator with the eigenvalues,

$$= [\cos(f(\alpha)) + i\sin(f(\alpha))]\,|\alpha\rangle \tag{13}$$

Matching solutions,

$$\exp(if(A)) = \exp(if(\alpha)) \tag{14}$$

1.4.d $\sum_{a'} \psi^*_{a'}(\vec{x}')\psi_{a'}(\vec{x}'')$, where $\psi_{a'}(\vec{x}') = \langle \vec{x}'|a'\rangle$

Writing out explicitly,

$$\sum_{a'} \psi^*_{a'}(\vec{x}')\psi_{a'}(\vec{x}'') = \sum_{a'} \langle a'|\vec{x}'\rangle \langle \vec{x}''|a'\rangle \tag{15}$$

We recognize this as the trace (1.5.14), and we see from (1.5.16c) that this is the delta function,

$$= \mathrm{Tr}(|\vec{x}'\rangle \langle \vec{x}''|) = \delta_{\vec{x}',\vec{x}''} \tag{16}$$

1.5 Matrix Representation

1.5.a Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Suppose $\langle a'|\alpha\rangle$, $\langle a''|\alpha\rangle$, ... and $\langle a'|\beta\rangle$, $\langle a''|\beta\rangle$,... are all known, where $|a'\rangle$, $|a''\rangle$,... form a complete set of base kets. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$ in that basis.

The answer is given in the text (1.3.31),

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} \langle a'|\alpha\rangle\langle a'|\beta\rangle^* & \langle a'|\alpha\rangle\langle a''|\beta\rangle^* & \ldots \\ \langle a''|\alpha\rangle\langle a'|\beta\rangle^* & \langle a''|\alpha\rangle\langle a''|\beta\rangle^* & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{1}$$

1.5.b We now consider a spin 1/2 system and let $|\alpha\rangle$ and $|\beta\rangle$ be $|s_z = \hbar/2\rangle$ and $|s_x = \hbar/2\rangle$, respectively. Write down explicitly the square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the usual (s_z diagonal) basis.

We expect a 2×2 matrix. We can get $|s_x = \hbar/2\rangle$ from (1.1.9a).

$$|s_z = \hbar/2\rangle\langle s_x = \hbar/2| = |+\rangle \cdot 1/2(\langle +| + \langle -|) \tag{2}$$

In the s_z basis, we define,

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3}$$

From this,

$$|s_z = \hbar/2\rangle\langle s_x = \hbar/2| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \tag{4}$$

1.6 Adding Eigenkets

Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator A. Under what condition can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of A? Justify your answer.

Acting A on our eigenkets,

$$\begin{cases} A\,|i\rangle = a\,|i\rangle \\ A\,|j\rangle = a'\,|j\rangle \end{cases} \tag{1}$$

In order for $|i\rangle + |j\rangle$ to be an eigenket of A,

$$A(|i\rangle + |j\rangle) = a''(|i\rangle + |j\rangle) \tag{2}$$

Alternatively, we could distribute the operator,

$$A(|i\rangle + |j\rangle) = A\,|i\rangle + A\,|j\rangle \tag{3}$$

$$A(|i\rangle + |j\rangle) = a\,|i\rangle + a'|\,|j\rangle \tag{4}$$

Comparing (2) and (4), they are only equal if either $|i\rangle = |j\rangle$ (trivial) or $a = a'$, i.e., the eigenvalues are degenerate.

1.7 Operators in Ket Space

Consider a ket space spanned by the eigenkets $\{|a'\rangle\}$ of a Hermitian operator A. There is no degeneracy.

1.7.a Prove that

$$\prod_{a'}(A - a')$$

is the null operator.

Let's act A on an eigenvector,

$$A\,|\Psi\rangle = a'\,|\Psi\rangle \tag{1}$$

where a' is the corresponding eigenvalue of $|\Psi\rangle$.

$$A\,|\Psi\rangle - a'\,|\Psi\rangle = |0\rangle \tag{2}$$

$$(A - a'I)\,|\Psi\rangle = |0\rangle \tag{3}$$

$A - a' = 0$ for at least one eigenvector. Since we product over all eigenvalues, if $A - a' = 0$ for one case, then the total must be 0.

1.7.b What is the significance of

$$\prod_{a''\neq a'} \frac{(A - a'')}{(a' - a'')}?$$

Acting the given on $|a'\rangle$, we can use (1.7.9) to replace the operator with the eigenvalue,

$$\prod_{a''\neq a'} \frac{(A - a'')}{(a' - a'')}\,|a'\rangle = \prod_{a''\neq a'} \frac{(a' - a'')}{(a' - a'')}\,|a'\rangle \tag{4}$$

$$\prod_{a''\neq a'} \frac{(A - a'')}{(a' - a'')}\,|a'\rangle = |a'\rangle \tag{5}$$

Acting the operation on another vector,

$$\prod_{a''\neq a'} \frac{(A - a'')}{(a' - a'')}\,|\Psi\rangle \tag{6}$$

We can insert identity and then use (5),

$$\prod_{a''\neq a'} \frac{(A - a'')}{(a' - a'')}\,|a'\rangle\,\langle a'|\Psi\rangle = |a'\rangle\,\langle a'|\Psi\rangle \tag{7}$$

We recognize the first part as the projection operator (1.3.15), so we conclude that this operation is the projection operator of a'.

1.7.c Illustrate (a) and (b) using A set equal to S_z of a spin $1/2$ system.

As a reminder,

$$S_z = \begin{bmatrix} \hbar/2 & 0 \\ 0 & \hbar/2 \end{bmatrix} \tag{8}$$

with eigenvalues $\omega = \pm\hbar/2$. Showing (a), we substitute in $A = S_z$ and a' as the eigenvalues,

$$\prod_{a'}(A - a') = (S_z - \hbar/2)(S_z + \hbar/2) \tag{9}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & -\hbar \end{bmatrix} \begin{bmatrix} \hbar & 0 \\ 0 & 0 \end{bmatrix} = 0$$

For (b), we have $a' = \hbar/2$ and $a'' = -\hbar/2$,

$$\prod_{a'' \neq a'} \frac{(A - a'')}{(a' - a'')} = \frac{S_z + \hbar/2}{\hbar} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{10}$$

Acting this on a general vector,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} \tag{11}$$

we pick out the spin-up component.

1.8 Orthonormality

Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove

$$[S_i, S_j] = i\epsilon_{ijk}\hbar S_k, \quad \{S_i, S_j\} = \left(\frac{\hbar^2}{2}\right)\delta_{ij},$$

where

$$\begin{cases} S_x = \dfrac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|) \\ S_y = \dfrac{i\hbar}{2}(-|+\rangle\langle -| + |-\rangle\langle +|) \\ S_z = \dfrac{\hbar}{2}(|+\rangle\langle +| - |-\rangle\langle -|) \end{cases}$$

As an example, let's set $i = x$ and $j = y$ and brute force. Remember the orthonormality rules(1.2.14),

$$\begin{cases} \langle +|+\rangle = \langle -|-\rangle = 1 \\ \langle +|-\rangle = \langle -|+\rangle = 0 \end{cases} \tag{1}$$

The commutation relation,

$$\begin{aligned}[S_x, S_y] = S_xS_y - S_yS_x = \frac{i\hbar^2}{4}&(|+\rangle\langle -| + |-\rangle\langle +|)(-|+\rangle\langle -| + |-\rangle\langle +| \\ -\frac{i\hbar^2}{4}&(-|+\rangle\langle -| + |-\rangle\langle +|)(|+\rangle\langle -| + |-\rangle\langle +|)\end{aligned} \tag{2}$$

$$\begin{aligned}= \frac{i\hbar^2}{4}(&-|+\rangle\langle -|+\rangle\langle -| + |+\rangle\langle -|-\rangle\langle +| - |-\rangle\langle +|+\rangle\langle -| + |-\rangle\langle +|-\rangle\langle +| \\ &+|+\rangle\langle -|+\rangle\langle -| + |+\rangle\langle -|-\rangle\langle +| - |-\rangle\langle +|+\rangle\langle -| - |-\rangle\langle +|-\rangle\langle +|)\end{aligned}$$

$$= \frac{i\hbar^2}{2}(|+\rangle\langle +| - |-\rangle\langle -|) = i\hbar S_z \tag{3}$$

We do the same thing with the anti-commutation relation,

$$\begin{aligned}\{S_x, S_y\} = S_xS_y + S_yS_x = \frac{i\hbar^2}{4}&(|+\rangle\langle -| + |-\rangle\langle +|)(-|+\rangle\langle -| + |-\rangle\langle +| \\ +\frac{i\hbar^2}{4}&(-|+\rangle\langle -| + |-\rangle\langle +|)(|+\rangle\langle -| + |-\rangle\langle +|)\end{aligned} \tag{4}$$

$$\begin{aligned}= \frac{i\hbar^2}{4}(&-|+\rangle\langle -|+\rangle\langle -| + |+\rangle\langle -|-\rangle\langle +| - |-\rangle\langle +|+\rangle\langle -| + |-\rangle\langle +|-\rangle\langle +| \\ &-|+\rangle\langle -|+\rangle\langle -| - |+\rangle\langle -|-\rangle\langle +| + |-\rangle\langle +|+\rangle\langle -| + |-\rangle\langle +|-\rangle\langle +|)\end{aligned}$$

Everything cancels, and we are left with 0. We can repeat this for all other combinations to prove the desired relations.

1.9 Rotation Operators

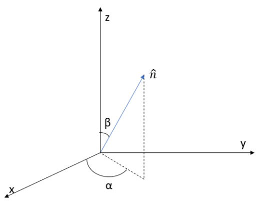

Figure 1.1: Angles

Construct $|\vec{S}\cdot\hat{n};+\rangle$ such that

$$\vec{S}\cdot\hat{n}\,|\vec{S}\cdot\hat{n};+\rangle = \left(\frac{\hbar}{2}\right)|\vec{S}\cdot\hat{n};+\rangle$$

where $\hat{n}$ is characterized by the angles shown in (Figure 1.1). Express your answer as a linear combination of $|+\rangle$ and $|-\rangle$. [Note: The answer is

$$\cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)\exp(i\alpha)\,|-\rangle$$

But do not just verify that this answer satisfies the above eigenvalue equation. Rather, treat the problem as a straightforward eigenvalue problem. Also do not use rotation operators, which we will introduce later in this book.]

The first thing we want to do is write $\vec{S}\cdot\hat{n}$ in matrix form,

$$\begin{cases}\vec{S} = \hbar/2(\sigma_x,\sigma_y,\sigma_z)\\ \hat{n} = (\cos(\alpha)\sin(\beta),\sin(\alpha)\sin(\beta),\cos(\beta))\end{cases} \tag{1}$$

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\left[\begin{pmatrix}0 & \cos(\alpha)\sin(\beta)\\ \cos(\alpha)\sin(\beta) & 0\end{pmatrix} + \begin{pmatrix}0 & -i\sin(\alpha)\sin(\beta)\\ i\sin(\alpha)\sin(\beta) & 0\end{pmatrix} + \begin{pmatrix}\cos(\beta) & 0\\ 0 & \cos(\beta)\end{pmatrix}\right]$$

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\begin{bmatrix}\cos(\beta) & \sin(\beta)\exp(-i\alpha)\\ \sin(\beta)\exp(i\alpha) & \cos(\beta)\end{bmatrix} \tag{2}$$

We can now solve the eigenvalue problem,

$$\vec{S}\cdot\hat{n}\,|\vec{S}\cdot\hat{n};+\rangle = \left(\frac{\hbar}{2}\right)|\vec{S}\cdot\hat{n};+\rangle \tag{3}$$

$$\begin{bmatrix}\cos(\beta) & \sin(\beta)\exp(-i\alpha)\\ \sin(\beta)\exp(i\alpha) & -\cos(\beta)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x\\ y\end{bmatrix}$$

which gives us two equations,

$$\begin{cases}x\cos(\beta) + y\sin(\beta)\exp(-i\alpha) = x\\ x\sin(\beta)\exp(i\alpha) - y\cos(\beta) = y\end{cases} \tag{4}$$

In addition, we have the normalization condition,

$$|x|^2 + |y|^2 = 1 \tag{5}$$

Looking at the first equation in (4), we can solve for y,

$$y = \frac{(1-\cos(\beta))x}{\sin(\beta)\exp(-i\alpha)} \tag{6}$$

$$|y|^2 = \frac{(1-\cos(\beta))^2|x|^2}{\sin^2(\beta)} \tag{7}$$

Inserting into the normalization condition (5),

$$|x|^2 + \frac{|x|^2 - 2|x|^2\cos(\beta) + |x|^2\cos^2(\beta)}{\sin^2(\beta)} = 1 \tag{8}$$

$$\frac{2|x|^2 - 2|x|^2\cos(\beta)}{\sin^2(\beta)} = 1$$

$$|x|^2 = \frac{1+\cos(\beta)}{2} \tag{9}$$

From half-angle formulas,

$$x = \cos({}^{\beta}\!/\!{}_{2}) \tag{10}$$

Plugging into the second line of (4),

$$\cos({}^{\beta}\!/\!{}_{2})\sin(\beta)\exp(i\alpha) - y\cos(\beta) = y \tag{11}$$

$$y = \sin({}^{\beta}\!/\!{}_{2})\exp(i\alpha) \tag{12}$$

Combining (10) and (12),

$$|\vec{S}\cdot\hat{n};+\rangle = \begin{bmatrix} \cos({}^{\beta}\!/\!{}_{2}) \\ \sin({}^{\beta}\!/\!{}_{2})\exp(i\alpha) \end{bmatrix} \tag{13}$$

When writing using $|+\rangle$ and $|-\rangle$, this gives the solution provided by Sakurai.

1.10 Energy Eigenvalues

The Hamiltonian operator for a two-state system is given by

$$\mathscr{H} = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)$$

where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$).

To find the energy eigenvalues, we must solve,

$$\mathscr{H}|\Psi\rangle = E|\Psi\rangle \tag{1}$$

It is probably easiest to do this in matrix representation. Setting

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2}$$

In this basis,

$$\mathscr{H} = \begin{bmatrix} a & a \\ a & -a \end{bmatrix} \tag{3}$$

Solving the characteristic equation, our eigenvalues are $\lambda = \pm a\sqrt{2}$. The associated eigenvectors,

$$|a\sqrt{2}\rangle = \frac{1}{4+2\sqrt{2}} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} \tag{4}$$

$$|-a\sqrt{2}\rangle = \frac{1}{4-2\sqrt{2}} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} \tag{5}$$

In the $|1\rangle$, $|2\rangle$ basis,

$$\begin{cases} |a\sqrt{2}\rangle = \dfrac{1}{4+2\sqrt{2}}[(1+\sqrt{2})|1\rangle + |2\rangle] \\ \\ |-a\sqrt{2}\rangle = \dfrac{1}{4-2\sqrt{2}}[(1-\sqrt{2})|1\rangle + |2\rangle] \end{cases} \tag{6}$$

1.11 Energy Eigenvalues

A two-state system is characterized by the Hamiltonian

$$H = H_{11} |1\rangle \langle 1| + H_{22} |2\rangle \langle 2| + H_{12}[|1\rangle \langle 2| + |2\rangle \langle 1|]$$

where H_{11}, H_{22}, and H_{12} are real numbers with the dimension of energy, and $|1\rangle$ and $|2\rangle$ are eigenkets of some observable($\neq H$). Find the energy eigenkets and corresponding eigenvalues. Make sure that your answer makes good sense for $H_{12} = 0$. (You need not solve this problem from scratch. The following fact may be used without proof:

$$(\vec{S} \cdot \hat{n}) |\hat{n}; +\rangle = {}^{\hbar}/_{2} |\hat{n}; +\rangle ;$$

with $|\hat{n}; +\rangle$ given by

$$|\hat{n}; +\rangle = \cos({}^{\beta}/_{2}) |+\rangle + \exp(i\alpha) \sin({}^{\beta}/_{2}) |-\rangle ,$$

where β and α are the polar and azimuthal angles, respectively, that characterize $\hat{n}$.

The easiest way to solve this is using matrices. In this case, the Hamiltonian is given by

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix} \tag{1}$$

Solving the characteristic equation gives two eigenvalues,

$$\begin{cases} \lambda_1 = \dfrac{(H_{11} + H_{22}) + \sqrt{(H_{11} + H_{22})^2 - 4(H_{11}H_{22} - H_{12}^2)}}{2} \\ \\ \lambda_2 = \dfrac{(H_{11} + H_{22}) - \sqrt{(H_{11} + H_{22})^2 - 4(H_{11}H_{22} - H_{12}^2)}}{2} \end{cases} \tag{2}$$

To find the eigenkets, we need to satisfy

$$\begin{cases} (H_{11} - \lambda)a + H_{12}b = 0 \\ H_{12}a + (H_{22} - \lambda)b = 0 \end{cases} \tag{3}$$

Let's start by setting $a = 1$ in the first equation,

$$|\lambda_1\rangle = \begin{bmatrix} 1 \\ -\dfrac{H_{11} - \lambda}{H_{12}} \end{bmatrix} \tag{4}$$

Setting $b = 1$ in the second equation,

$$|\lambda_2\rangle = \begin{bmatrix} -\dfrac{H_{22} - \lambda}{H_{12}} \end{bmatrix} \tag{5}$$

1.12 Measurement of Spin

A spin $^1/_2$ system is known to be in an eigenstate of $\vec{S}\cdot\hat{n}$ with eigenvalue $\hbar/2$, where $\hat{n}$ is a unit vector lying the xz-plane that makes an angle γ with the positive z-axis.

As a reminder, (1.1.9a) tells us what $|S_x\rangle$ looks like in the S_z basis,

$$|S_x;+\rangle = {}^1/\sqrt{2}(|+\rangle + |-\rangle) \tag{1}$$

Furthermore, from question (1.9), we know a generic ket in this basis can be written as

$$|\vec{S}\cdot\hat{n};+\rangle = \cos(\gamma/2)\,|+\rangle + \sin(\gamma/2)\,|-\rangle \tag{2}$$

Note that because we are in the xz-plane, we set $\alpha = 0$.

1.12.a Suppose S_x is measured. What is the probability of getting $+\hbar/2$?

Using (1.4.4), starting in $|\vec{S}\cdot\hat{n};+\rangle$, the probability of going to the state $|S_x;+\rangle$ is given by

$$P(\hbar/2) = |\langle S_x;+|\vec{S}\cdot\hat{n};+\rangle|^2 \tag{3}$$

$$= \frac{1}{2}\left|(\langle+| + \langle-|)(\cos(\gamma/2)\,|+\rangle + \sin(\gamma/2)\,|-\rangle)\right|^2$$

$$= \frac{1}{2}|\cos(\gamma/2) + \sin(\gamma/2)|^2$$

$$P(\hbar/2) = \frac{1}{2}(1 + \sin(\gamma)) \tag{4}$$

Let's check for some easy cases. If $\hat{n}$ is aligned orthogonal to the x-axis ($\gamma = 0$ or $\gamma = \pi$) we are starting in the $|+\rangle$ or $|-\rangle$ state, and we expect half of the particles in the $+\hbar/2$ state. If $\hat{n}$ is aligned orthogonal to the x-axis ($\gamma = \pi/2$) we are starting in the $|S_x;+\rangle$ state, and all the particles should remain in that state.

1.12.b Evaluate the dispersion in S_x, that is,

$$\langle(S_x - \langle S_x\rangle)^2\rangle$$

From (1.4.11),

$$S_x = \frac{\hbar}{2}[(|+\rangle\langle-|) + (|-\rangle\langle+|)] \tag{5}$$

$$S_x^2 = \frac{\hbar^2}{4}[(|+\rangle\langle+|) + (|-\rangle\langle-|)] \tag{6}$$

Using (2), we can use (1.4.5) to find the expected values,

$$\langle S_x\rangle = \frac{\hbar}{2}[\cos(\gamma/2)\,\langle+| + \sin(\gamma/2)\,\langle-|][(|+\rangle\langle-|) + (|-\rangle\langle+|)][\cos(\gamma/2)\,|+\rangle + \sin(\gamma/2)\,|-\rangle] \tag{7}$$

$$\langle S_x \rangle = \frac{\hbar}{2} \sin(\gamma) \tag{8}$$

$$\langle S_x^2 \rangle = \frac{\hbar^2}{4} [\cos(\gamma/2) \langle + | + \sin(\gamma/2) \langle - |][(|+\rangle \langle + |) + (|-\rangle \langle - |)][\cos(\gamma/2) |+\rangle + \sin(\gamma/2) |-\rangle] \tag{9}$$

$$\langle S_x^2 \rangle = \frac{\hbar^2}{4} \tag{10}$$

Combining (8) and (10),

$$\langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} \cos^2(\gamma) \tag{11}$$

1.13 Stern-Gerlach

A beam of spin $^1\!/_2$-atoms goes through a series of Stern-Gerlach-type measurements as follows:

a. The first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms.

b. The second measurement accepts $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms, where s_n is the eigenvalue of the operator $\vec{S}\cdot\hat{n}$, with $\hat{n}$ making an angle β in the xz-plane with respect to the z-axis.

c. The third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms.

What is the intensity of the final $s_z = -\hbar/2$ beam when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity? How must we orient the second measuring apparatus if we are to maximize the intensity of the final $s_z = -\hbar/2$ beam?

In matrix form, we can write the first measurement as $A = |+\rangle\langle+|$ since only the $|+\rangle$ state survives. The second measurement can be written as $B = |\hat{n};+\rangle\langle\hat{n};+|$ where $|\hat{n};+\rangle = \cos(^\beta\!/_2)|+\rangle + \sin(^\beta\!/_2)|+\rangle$.The third measurement can be written as $C = |-\rangle\langle-|$. Thus, the total will be CBA,

$$CBA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos^2(^\beta\!/_2) & \cos(^\beta\!/_2)\sin(^\beta\!/_2) \\ \cos(^\beta\!/_2)\sin(^\beta\!/_2) & \sin^2(^\beta\!/_2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \cos(^\beta\!/_2)\sin(^\beta\!/_2) & 0 \end{pmatrix} \tag{1}$$

In bra-ket notation, this can be rewritten

$$T = CBA = \cos(^\beta\!/_2)\sin(^\beta\!/_2)|-\rangle\langle+| \tag{2}$$

Acting this on a generic beam,

$$T(|+\rangle + |-\rangle) = \cos(^\beta\!/_2)\sin(^\beta\!/_2)|-\rangle \tag{3}$$

Intensity is related to the beam squared,

$$I = \cos^2(^\beta\!/_2)\sin^2(^\beta\!/_2) = \frac{\sin^2(\beta)}{4} \tag{4}$$

which is maximized when $\beta = ^\pi\!/_2$, which gives an intensity a quarter the initial surviving beam.

1.14 Eigenvalues

A certain observable in quantum mechanics has a 3×3 matrix representation as follows:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

1.14.a Find the normalized eigenvectors of this observable and the corresponding eigenvalues. Is there any degeneracy?

Solving the characteristic equation gives the eigenvalues $\lambda = 0, \pm 1$. There is no degeneracy since we have three eigenvalues for a 3×3 matrix. Solving for the eigenvectors,

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} ; \quad |1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} ; \quad |-1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \tag{1}$$

1.14.b Give a physical example where all this is relevant.

Looking this up, these are the eigenvalues and eigenvectors for the spin-1 particle. This is further explained in chapter 3 of Sakurai.

1.15 Simultaneous Eigenkets

Let A and B be observables. Suppose the simultaneous eigenkets of A and B $\{|a',b'\rangle\}$ form a complete orthonormal set of base kets. Can we always conclude that

$$[A,B]=0?$$

If your answer is yes, prove the assertion. If your answer is no, give a counterexample.

We start by writing $[A,B]$ out and inserting identity on both sides,

$$[A,B]=\sum_{a',b'}\sum_{a'',b''}|a'',b''\rangle\langle a'',b''|(AB-BA)|a',b'\rangle\langle a',b'| \tag{1}$$

If we act the operators on our ket, we use the relation,

$$AB|a',b'\rangle=a'b'|a',b'\rangle \tag{2}$$

Inserting this into (1),

$$[A,B]=\sum_{a'',b''}\sum_{a',b'}|a'',b''\rangle\langle a'',b''|(a'b'-b'a')|a',b'\rangle\langle a',b'| \tag{3}$$

We know that $a'b'-b'a'=0$ since these are not operators, so the order does not matter. $[A,B]=0$ if the simultaneous eigenkets of A and B form a complete orthonormal set of base kets.

1.16 Simultaneous Eigenkets

Two Hermitian operators anticommute:

$$\{A, B\} = AB + BA = 0$$

Is it possible to have a simultaneous (that is, common) eigenket of A and B? Prove or illustrate your assertion.

Let's act eigenkets of A on the anti-commutator,

$$\langle a''|AB|a'\rangle + \langle a''|BA|a'\rangle$$

Using eigenvalue relations,

$$= a'' \langle a''|B|a'\rangle + a' \langle a''|B|a'\rangle = (a'' + a') \langle a''|B|a'\rangle$$

We expect this to be equal to 0 if A and B anti-commute. Since $(a'' + a') \neq 0$, this implies $\langle a''|B|a'\rangle = 0$ for both $a'' = a'$ and $a'' \neq a'$, which implies they do not have simultaneous eigenkets.

1.17 Degenerate Observables

Two observables A_1 and A_2, which do not involve time explicitly, are known not to commute,

$$[A_1, A_2] \neq 0,$$

yet we also know that A_1 and A_2 both commute with the Hamiltonian:

$$[A_1, \mathscr{H}] = 0, \quad [A_2, \mathscr{H}] = 0$$

Prove that the energy eigenstates are, in general, degenerate. Are there exceptions? As an example, you may think of the central-force problem $\mathscr{H} = \vec{p}^2/2m + V(r)$, with $A_1 \to L_z$, $A_2 \to L_x$.

We start with the eigenvalue equation for the Hamiltonian,

$$\mathscr{H} |n\rangle = E |n\rangle \tag{1}$$

Acting the commutation relations on the eigenket,

$$[A_1, \mathscr{H}] |n\rangle = 0, \quad [A_2, \mathscr{H}] |n\rangle = 0 \tag{2}$$

Let's look at just the left. Expanding out,

$$A_1 \mathscr{H} |n\rangle - \mathscr{H} A_1 |n\rangle = 0 \tag{3}$$

$$E(A_1 |n\rangle) = \mathscr{H}(A_1 |n\rangle) \tag{4}$$

Since A_1 and $\mathscr{H}$ commute, they must share a complete set of eigenstates.

$$A_1 |n\rangle = a_1 |n\rangle \tag{5}$$

Similarly,

$$A_2 |n\rangle = a_2 |n\rangle \tag{6}$$

Acting the non-commuting relation on $|n\rangle$,

$$\begin{aligned} [A_1, A_2] |n\rangle &= (A_1 A_2 - A_2 A_1) |n\rangle \\ &= (a_1 a_2 - a_2 a_1) |n\rangle \end{aligned} \tag{7}$$

Since a_1 and a_2 are both scalars, they can be rearranged freely, meaning this all goes to 0. However, we know this to not be true since A_1 and A_2 do not commute. Therefore, the energy eigenstates must be degenerate.

1.18 Uncertainty Relations

1.18.a The simplest way to derive the Schwarz inequality goes as follows. First, observe

$$(\langle\alpha| + \lambda^* \langle\beta|) \cdot (|\alpha\rangle + \lambda |\beta\rangle) \geq 0$$

for any complex number λ; then choose λ in such a way that the preceding inequality reduces to the Schwarz inequality.

Nothing doing, let's start by expanding out the given equation,

$$\langle\alpha|\alpha\rangle + \lambda \langle\alpha|\beta\rangle + \lambda^* \langle\beta|\alpha\rangle + \lambda^*\lambda \langle\beta|\beta\rangle \geq 0 \tag{1}$$

We want this to match the Schwarz inequality(1.4.54),

$$\langle\alpha|\alpha\rangle \langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2 \tag{2}$$

For this,

$$\lambda = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} \tag{3}$$

Inserting this into (1),

$$\langle\alpha|\alpha\rangle - \frac{\langle\beta|\alpha\rangle \langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle} - \frac{\langle\alpha|\beta\rangle \langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} + \frac{\langle\alpha|\beta\rangle \langle\beta|\alpha\rangle \langle\beta|\beta\rangle}{\langle\beta|\beta\rangle} \geq 0 \tag{4}$$

$$\langle\alpha|\alpha\rangle \langle\beta|\beta\rangle - |\langle\alpha|\beta\rangle|^2 - |\langle\alpha|\beta\rangle|^2 + |\langle\alpha|\beta\rangle|^2 \geq 0 \tag{5}$$

$$\langle\alpha|\alpha\rangle \langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2 \tag{6}$$

1.18.b Show that the equality sign in the generalized uncertainty relation holds if the state in question satisfies

$$\Delta A |\alpha\rangle = \lambda \Delta B |\alpha\rangle$$

with λ purely imaginary.

The generalized uncertainty relation is given by (1.4.59),

$$\langle(\Delta A)^2\rangle \langle(\Delta B)^2\rangle \geq |\, \langle\Delta A \Delta B\rangle\, |^2 \tag{7}$$

The right-side of this equation, we can use (1.4.63),

$$|\, \langle\Delta A \Delta B\rangle\, |^2 = {}^1\!/\!_4 |\, \langle[A, B]\rangle\, |^2 + {}^1\!/\!_4 |\, \langle\{\Delta A, \Delta B\}\rangle\, |^2 \tag{8}$$

We can use the definition of an operator (1.4.50), to show

$$\begin{cases} [A, B] = [\Delta A, \Delta B]; \\ \{A, B\} = \{\Delta A, \Delta B\} \end{cases} \tag{9}$$

Let's now look at each term in Equation 8,

$$\langle [A, B] \rangle = \langle \alpha | (\Delta A \Delta B - \Delta B \Delta A) | \alpha \rangle \tag{10}$$

$$= \lambda^* \langle \alpha | (\Delta B)^2 | \alpha \rangle - \lambda \langle \alpha | (\Delta B)^2 | \alpha \rangle \tag{11}$$

Since λ is purely imaginary, $\lambda^* = -\lambda$,

$$\langle [A, B] \rangle = -2\lambda \langle (\Delta B)^2 \rangle \tag{12}$$

Similarly, we can show,

$$\langle \{\Delta A, \Delta B\} \rangle = \langle \alpha | (\Delta A \Delta B + \Delta B \Delta A) | \alpha \rangle = 0 \tag{13}$$

Equation 8 becomes,

$$| \langle \Delta A \Delta B \rangle |^2 = \lambda^2 \langle (\Delta B)^2 \rangle^2 \tag{14}$$

Using the relation given in the problem, we can rewrite,

$$= \langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \tag{15}$$

1.18.c Explicit calculations using the usual rules of wave mechanics show that the wave function for a Gaussian wave packet given by

$$\langle x' | \alpha \rangle = (2\pi d^2)^{-1/4} \exp \left[\frac{i \langle p \rangle x'}{\hbar} - \frac{(x' - \langle x \rangle)^2}{4d^2} \right]$$

satisfies the minimum uncertainty relation

$$\sqrt{\langle (\Delta x)^2 \rangle} \sqrt{\langle (\Delta p)^2 \rangle} = \frac{\hbar}{2}$$

Prove that the requirement

$$\langle x' | \Delta x | \alpha \rangle = (imaginary\ number) \langle x' | \Delta p | \alpha \rangle$$

is indeed satisfied for such a Gaussian wave packet, in agreement with (b).
Let's start by finding $\langle x' | \Delta x | \alpha \rangle$ and $\langle x' | \Delta p | \alpha \rangle$ in integral form,

$$\langle x' | \Delta x | \alpha \rangle = \int \langle x' | x'' \rangle \langle x'' | x | \alpha \rangle \ dx'' - \int \langle x' | x'' \rangle \langle x'' | \langle x \rangle | \alpha \rangle \ dx'' \tag{16}$$

$$= \int \delta(x' - x'') x'' \langle x'' | \alpha \rangle \ dx'' - \int \delta(x' - x'') \langle x \rangle \langle x'' | \alpha \rangle \ dx'' \tag{17}$$

Inserting the Gaussian wave packet,

$$= \int \delta(x' - x'')x''(2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x'' - \langle x\rangle)^2}{4d^2}\right) dx'' - \int \delta(x - x'')\langle x\rangle (2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x'' - \langle x\rangle)^2}{4d^2}\right) dx'' \quad (18)$$

Turning now to p,

$$\langle x'|\Delta p|\alpha\rangle = \int \langle x'|x''\rangle \langle x''| - \frac{i\hbar\partial}{\partial x}|\alpha\rangle \; dx'' - \int \langle x'|x''\rangle \langle x''| \langle p\rangle |\alpha\rangle \; dx'' \quad (19)$$

$$= \int \delta(x' - x'')\left(-i\hbar\frac{\partial}{\partial x}\right)\langle x''|\alpha\rangle \; dx'' - \int \delta(x' - x'')\langle p\rangle \langle x''|\alpha\rangle \; dx'' \quad (20)$$

Inserting the Gaussian wave packet,

$$= \int \delta(x' - x'')\left(-i\hbar\frac{\partial}{\partial x}\right)(2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x'' - \langle x\rangle)^2}{4d^2}\right) dx'' - \int \delta(x - x'')\langle p\rangle (2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x'' - \langle x\rangle)^2}{4d^2}\right) dx'' \quad (21)$$

$$= \int \delta(x' - x'')(-i\hbar)\left[(2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x'' - \langle x\rangle)^2}{4d^2}\right)\left(\frac{i\langle p\rangle}{\hbar} - \frac{2(x'' - \langle x\rangle)}{4d^2}\right)\right] dx'' - \int \delta(x - x'')\langle p\rangle (2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x'' - \langle x\rangle)^2}{4d^2}\right) dx'' \quad (22)$$

$$= \int \delta(x' - x'')\left(\frac{i\hbar x''}{2d^2}\right)(2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x'' - \langle x\rangle)^2}{4d^2}\right) dx'' - \int \delta(x - x'')\left(\frac{i\hbar\langle x\rangle}{2d^2}\right)(2\pi d^2)^{-1/4} \exp\left(\frac{i\langle p\rangle x''}{\hbar} - \frac{(x'' - \langle x\rangle)^2}{4d^2}\right) dx'' \quad (23)$$

Comparing the two,

$$\langle x'|\Delta x|\alpha\rangle = -\frac{2id^2}{\hbar}\langle x'|\Delta p|\alpha\rangle \quad (24)$$

1.19 Expectation Value of Spin States

1.19.a Compute

$$\langle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$$

where the expectation value is taken for the $|S_z;+\rangle$ state. Using your result, check the generalized uncertainty relation

$$\langle(\Delta A)^2\rangle\,\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\,\langle[A,B]\rangle\,|^2$$

with $A \rightarrow S_x$, $B \rightarrow S_y$.

Let's check the dispersion relation by looking at each component separately using (1.4.18a),

$$\langle S_x^2\rangle = \langle S_z;+|S_x^2|S_z;+\rangle \tag{1}$$

$$= \frac{\hbar^2}{4}\langle+|[(|+\rangle\langle+|) + (|-\rangle\langle-|)]|+\rangle = \frac{\hbar^2}{4}$$

$$\langle S_x\rangle = \frac{\hbar}{2}\langle+|[(|+\rangle\langle-|) + (|-\rangle\langle+|)]|+\rangle = 0 \tag{2}$$

What we find agrees with (1.4.52),

$$\langle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4}$$

Similarly, we can convince ourselves,

$$\langle(\Delta S_y)^2\rangle = \frac{\hbar^2}{4} \tag{3}$$

Let's now turn to the uncertainty relation,

$$\langle(\Delta S_x)^2\rangle\,\langle(\Delta S_y)^2\rangle \geq \frac{1}{4}|\,\langle[S_x,S_y]\rangle\,|^2 \tag{4}$$

On the right side,

$$\langle[S_x,S_y]\rangle = \langle+|\frac{i\hbar^2}{2}[(|+\rangle\langle+|) - (|-\rangle\langle-|)]|+\rangle = \frac{i\hbar^2}{2} \tag{5}$$

Both sides of Equation 4 give $\hbar^4/16$.

1.19.b

Check the uncertainty relation with $A \to S_x$, $B \to S_y$ for the $|S_x; +\rangle$ state

Reminder that from (1.4.9),

$$|S_x; +\rangle = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \tag{6}$$

Let's calculate the dispersion,

$$\langle S_x^2 \rangle = \frac{\hbar^2}{8} [\langle +| + \langle -|][(|+\rangle \langle +|) + (|-\rangle \langle -|)][|+\rangle + |-\rangle] \tag{7}$$

$$= \frac{\hbar^2}{8} [\langle +| + \langle -|][|+\rangle + |-\rangle] = \frac{\hbar^2}{4}$$

$$\langle S_x \rangle = \frac{\hbar}{4} [\langle +| + \langle -|][(|+\rangle \langle -|) + (|-\rangle \langle +|)][|+\rangle + |-\rangle] \tag{8}$$

$$= \frac{\hbar}{4} [\langle +| + \langle -|][|+\rangle + |-\rangle] = \frac{\hbar}{2}$$

Combining this,

$$\langle (\Delta S_x)^2 \rangle = 0 \tag{9}$$

which means we don't need to calculate $\langle (\Delta S_y)^2 \rangle$. Let's now turn to the uncertainty relation and check to make sure $\langle [S_x, S_y] \rangle \geq 0$.

$$\langle [S_x, S_y] \rangle = \frac{i\hbar^2}{4} [\langle +| + \langle -|][(|+\rangle \langle +|) - (|-\rangle \langle -|)][|+\rangle + |-\rangle] \tag{10}$$

$$= \frac{i\hbar^2}{4} [\langle +| + \langle -|][|+\rangle - |-\rangle] = 0$$

1.20 Uncertainty Relation

Find the linear combination of $|+\rangle$ and $|-\rangle$ kets that maximizes the uncertainty product

$$\langle(\Delta S_x)^2\rangle \langle(\Delta S_y)^2\rangle$$

Verify explicitly that for the linear combination you found, the uncertainty relation for S_x and S_y is not violated.

As a reminder,

$$\begin{cases} \langle(\Delta S_x)^2\rangle = \langle\psi|\frac{\hbar^2}{4}[(|+\rangle\langle+|) + (|-\rangle\langle-|)]|\psi\rangle - \left(\langle\psi|\frac{\hbar}{2}[(|+\rangle\langle-|) + (|-\rangle\langle+|)]|\psi\rangle\right)^2 \\ \langle(\Delta S_y)^2\rangle = \langle\psi|\frac{\hbar^2}{4}[(|+\rangle\langle+|) + (|-\rangle\langle-|)]|\psi\rangle - \left(\langle\psi|\frac{-i\hbar}{2}[(|+\rangle\langle-|) + (|-\rangle\langle+|)]|\psi\rangle\right)^2 \end{cases} \tag{1}$$

For a generalized wavefunction, let's use $|\vec{S}\cdot\hat{n};+\rangle$,

$$|\psi\rangle = a|+\rangle + (1-a^2)^{1/2}\exp(i\beta)|-\rangle \tag{2}$$

Substituting in, we find the expectation values,

$$\begin{cases} \langle(\Delta S_x)^2\rangle = \frac{\hbar^2}{4}[1 - 4a^2(1-a^2)\cos^2(\beta)] \\ \langle(\Delta S_y)^2\rangle = \frac{\hbar^2}{4}[1 - 4a^2(1-a^2)\sin^2(\beta)] \end{cases} \tag{3}$$

$$\langle(\Delta S_x)^2\rangle \langle(\Delta S_y)^2\rangle = \frac{\hbar^4}{16}[1 - 4a^2(1-a^2) + 4a^4(1-a^2)^2\sin^2(2\beta)] \tag{4}$$

This value is maximized when $\beta = \pi/4$,

$$= \frac{\hbar^4}{16}[1 - 4a^2(1-a^2) + 4a^4(1-a^2)^2] \tag{5}$$

This is maximized when $a^2 = 0$ or 1. The linear combinations are either

$$\begin{cases} \pm|+\rangle \\ \exp\left(\frac{i\pi}{4}\right)|-\rangle \end{cases} \tag{6}$$

We've already shown $\pm|+\rangle$ obeys the uncertainty relation in the previous problem, so let's look at the other one,

$$\langle[S_x, S_y]\rangle \exp\left(-\frac{i\pi}{4}\right)\langle-|\frac{i\hbar^2}{2}(|+\rangle\langle+|) - \frac{i\hbar^2}{2}(|-\rangle\langle-|)]|-\rangle \exp\left(\frac{i\pi}{4}\right) \tag{7}$$

$$= -\frac{i\hbar^2}{2}$$

1.21 Uncertainty Relation, Particle in a Box

Evaluate the $x - p$ uncertainty product $\langle(\Delta x)^2\rangle \langle(\Delta p)^2\rangle$ for a one-dimensional particle confined between two rigid walls

$$V = \begin{cases} 0; & for\ 0 < x < a \\ \infty; & otherwise \end{cases}$$

Do this for both the ground and excited states

We recognize this as a particle in a box, so we can use the solution found in (A.2.4), Appendix 2,

$$\psi = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

Ground state at $n = 1$, excited states for $n >= 2$.
We can now go ahead and calculate the uncertainty product. Using (1.4.51),

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^* x^2 \psi \, dx = \int_0^a \frac{2}{a} x^2 \sin\left(\frac{n\pi x}{a}\right) \, dx \tag{1}$$

Using the double-angle formulas,

$$= \frac{1}{a} \int_0^a x^2 - x^2 \cos\left(\frac{2n\pi x}{a}\right) \, dx = \frac{a^2}{3} - \frac{1}{a} \int_0^a x^2 \cos\left(\frac{2n\pi x}{a}\right) \, dx \tag{2}$$

Integrating by parts with $u = x^2$ and $v' = \cos\left(\frac{2n\pi x}{a}\right)$,

$$= \frac{a^2}{3} - \frac{1}{a} \left[\frac{ax^2}{2n\pi} \sin\left(\frac{2n\pi x}{a}\right) \Big|_0^a - \int_0^a \frac{ax}{n\pi} \sin\left(\frac{2n\pi x}{a}\right) \, dx\right] \tag{3}$$

$$= \frac{a^2}{3} + \frac{1}{a} \int_0^a \frac{ax}{n\pi} \sin\left(\frac{2n\pi x}{a}\right) \, dx$$

Integrating by parts again, $u = x$ and $v' = \sin\left(\frac{2n\pi x}{a}\right)$,

$$= \frac{a^2}{3} + \frac{1}{n\pi} \left[-\frac{ax}{2n\pi} \cos\left(\frac{2n\pi x}{a}\right) \Big|_0^a + \frac{a}{2n\pi} \int_0^a \cos\left(\frac{2n\pi x}{a}\right) \, dx\right] \tag{4}$$

$$= a^2 \left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) \tag{5}$$

$$\langle x \rangle = \frac{2}{a} \int_0^a x \sin^2\left(\frac{n\pi x}{a}\right) \, dx \tag{6}$$

Using the double-angle formulas,

$$= \frac{a}{2} - \int_0^a x \cos\left(\frac{2n\pi x}{a}\right) \, dx \tag{7}$$

Integrating by parts, $u = x$, $v' = \cos\left(\frac{2n\pi x}{a}\right)$,

$$= \frac{a}{2} - \left[\frac{ax}{2n\pi} \sin\left(\frac{2n\pi x}{a}\right)\Big|_0^a - \int_0^a \frac{a}{2n\pi} \sin\left(\frac{2n\pi x}{a}\right) \, dx\right] = \frac{a}{2} \tag{8}$$

$$\langle(\Delta x)^2\rangle = a^2 \left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right) \tag{9}$$

Let's look at momentum (1.7.28),

$$p\,|\psi\rangle = -i\hbar\sqrt{\frac{2}{a}}\frac{n\pi}{a} \cos\left(\frac{n\pi x}{a}\right) \tag{10}$$

$$p^2\,|\psi\rangle = \hbar^2\sqrt{\frac{2}{a}}\frac{n^2\pi^2}{a^2} \sin\left(\frac{n\pi x}{a}\right) \tag{11}$$

$$\langle p^2\rangle = \frac{\hbar^2 n^2\pi^2}{a^2}\frac{2}{a}\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) \, dx \tag{12}$$

Using the double angle formulas,

$$= \frac{\hbar^2 n^2\pi^2}{a^3}\int_0^a 1 - \cos\left(\frac{2n\pi x}{a}\right) \, dx = \frac{\hbar^2 n^2\pi^2}{a^2} \tag{13}$$

$$\langle p\rangle = -\frac{2i\hbar n\pi}{a^2}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi x}{a}\right) \, dx \tag{14}$$

Using double angle formulas,

$$= -\frac{i\hbar n\pi}{a^2}\int_0^a \sin\left(\frac{2n\pi x}{a}\right) \, dx = 0 \tag{15}$$

$$\langle(\Delta p)^2\rangle = \frac{\hbar^2 n^2\pi^2}{a^2} \tag{16}$$

$$\langle(\Delta x)^2\rangle\,\langle(\Delta p)^2\rangle = \frac{\hbar^2}{2}\left(\frac{n^2\pi^2}{6} - 1\right) \tag{17}$$

1.22 Uncertainty Principle, Fermi Question

Estimate the rough order of magnitude of the length of time that an ice pick can be balanced on its point if the only limitation is that set by the Heisenberg uncertainty principle. Assume that the point is sharp and that the point and the surface on which it rests are hard. You may make approximations which do not alter the general order of magnitude of the result. Assume reasonable values for the dimensions and weight of the ice pick. Obtain an approximate numerical result and express it in seconds.

We can approximate this as a point particle with mass m on the end of a massless rod of length l. From classical mechanics, the torque equation,

$$\tau = ml^2 \frac{d^2\theta}{dt^2} = mg\theta l \tag{1}$$

Solving, we get an time-dependant equation for the angle,

$$\theta(t) = a \exp\left(\sqrt{\frac{g}{l}}t\right) + b \exp\left(-\sqrt{\frac{g}{l}}t\right) \tag{2}$$

The uncertainty in position at time $t = 0$,

$$\Delta x = l\theta(0) = (a+b)l \tag{3}$$

The uncertainty in momentum at time $t = 0$,

$$\Delta p = ml\frac{d\theta}{dt} = \sqrt{\frac{g}{l}}(a-b)ml = m\sqrt{gl}(a-b) \tag{4}$$

Using the Heisenberg uncertainty principle,

$$\Delta x \Delta p = \frac{\hbar}{2} \tag{5}$$

$$(a^2 - b^2)m\sqrt{gl^3} = \frac{\hbar}{2} \tag{6}$$

We want to minimize this, so let's start by setting $b = 0$,

$$a^2 = \frac{\hbar}{2m\sqrt{gl^3}} \tag{7}$$

Let's say we notice deviation at around $\theta = 1°$. Solving for time in Equation 2 simplifies to,

$$t = \sqrt{\frac{l}{g}} \ln\left(\sqrt{\frac{2m\sqrt{gl^3}}{\hbar}}\right) \tag{8}$$

Using,

$$\begin{cases} l = 0.1m; \\ g = 10m/s^2; \\ m = 0.1kg; \\ \hbar = 10^{-34}m^2kg/s \end{cases} \tag{9}$$

We end up with something around 2 seconds.

1.23 Simultaneous Eigenkets

Consider a three-dimensional ket space. If a certain set of orthonormal kets-say, $|1\rangle$, $|2\rangle$, and $|3\rangle$-are used as the base kets, the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

with a and b both real.

1.23.a Obviously A exhibits a degenerate spectrum. Does B also exhibit a degenerate spectrum?

Solving the characteristic equation, we get eigenvalues $\lambda = \pm b$. Since there are repeated eigenvalues ($\lambda = b$), there is degeneracy.

1.23.b Show that A and B commute

Nothing doing,

$$AB = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} \tag{1}$$

$$BA = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} \tag{2}$$

1.23.c Find a new set of orthonormal kets which are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

Since A and B commute, they are compatible observables and must have simultaneous eigenkets. We start by finding two such eigenkets based on the eigenvalues of A,

$$|a\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

$$|-a\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \tag{4}$$

Note that we chose Equation 4 because we looked ahead and saw the solution for $|b\rangle$, and we want the eigenvectors to be orthonormal.

We need one more eigenket, so let's use $\lambda = b$,

$$|b\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} \tag{5}$$

Checking that these are simultaneous eigenkets,

$$\begin{cases} A\,|a\rangle = a\,|a\rangle \quad B\,|a\rangle = b\,|a\rangle \\ A\,|-a\rangle = -a\,|-a\rangle \quad B\,|-a\rangle = -b\,|-a\rangle \\ A\,|b\rangle = -a\,|b\rangle \quad B\,|b\rangle = b\,|b\rangle \end{cases} \tag{6}$$

1.24 Rotation

1.24.a Prove that $(1/\sqrt{2})(1 + i\sigma_x)$ acting on a two-component spinor can be regarded as the matrix representation of the rotation operator about the x-axis by angle $-\pi/2$. (The minus sign signifies that the rotation is clockwise.)

We look ahead to (3.2.44) to find that the rotation matrix is given by

$$R = \cos\left(\frac{\phi}{2}\right) - i\vec{\sigma}\cdot\hat{n}\sin\left(\frac{\phi}{2}\right) \tag{1}$$

A clockwise rotation about the x-axis means $\phi = -{}^{\pi}/_{2}$ and $\hat{n} = \hat{x}$

$$R_x = \frac{1}{\sqrt{2}}(1 + i\sigma_x) \tag{2}$$

1.24.b Construct the matrix representation of S_z when the eigenkets of S_y are used as base vectors

Since we know what S_z is in the S_z basis, we just need to use (1.5.12) to change the basis,

$$S_z = \frac{\hbar}{2}\frac{1}{\sqrt{2}}(1 - i\sigma_x)\sigma_z\frac{1}{\sqrt{2}}(1 + i\sigma_x) \tag{3}$$

$$= \frac{\hbar}{4}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} \tag{4}$$

1.25 Real Operators

Some authors define an operator to be real when every member of its matrix elements $\langle b'|A|b''\rangle$ is real in some representation ($\{|b'\rangle\}$ basis in this case). Is this concept representation independent, that is, do the matrix elements remain real even if some basis other than $\{|b'\rangle\}$ is used? Check your assertion using familiar operators such as S_y and S_z (see Problem 24) or x and p_x.

Given a basis, $\{|c\rangle\}$, in the $\{|b'\rangle\}$ basis,

$$|c'\rangle = \sum_{b'} |b'\rangle \langle b'|c'\rangle \tag{1}$$

Performing the change of basis (1.5.12) on an operator A to find the matrix elements in $\{|c\rangle\}$,

$$\langle c'|A|c''\rangle = \sum_{b'} \sum_{b''} \langle c'|b'\rangle \langle b'|A|b''\rangle \langle b''|c''\rangle \tag{2}$$

$$= \sum_{b'} \sum_{b''} \langle c'|b'\rangle \langle b''|c''\rangle \langle b'|A|b''\rangle \tag{3}$$

$\langle c'|b'\rangle \langle b''|c''\rangle$ needs to be real, but the individual components don't need to be.

1.26 Spin Transformation Matrix

Construct the transformation matrix that connects the S_z diagonal basis to the S_x diagonal basis. Show that your result is consistent with the general relation

$$U = \sum_r |b^{(r)}\rangle \langle a^{(r)}|$$

In the S_z basis, based on (1.1.9),

$$\begin{cases} |S_x; +\rangle = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \\ \\ |S_x; -\rangle = \frac{1}{\sqrt{2}} |+\rangle - \frac{1}{\sqrt{2}} |-\rangle \end{cases} \tag{1}$$

In the S_x basis,

$$|S_x; +\rangle' = |+\rangle \ |S_x; -\rangle' = |-\rangle \tag{2}$$

(1.5.5), the transformation matrix takes it from the S_z basis to the S_x basis,

$$\begin{cases} |S_x; +\rangle' = U |S_x; +\rangle \\ |S_x; -\rangle' = U |S_x; -\rangle \end{cases} \tag{3}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \tag{4}$$

Solving,

$$U_{21} = -U_{22} \tag{5}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \tag{6}$$

Solving,

$$U_{11} = U_{12} \tag{7}$$

Combining Equation 5 and Equation 7,

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{8}$$

We can now check this using (1.5.4),

$$U = |+\rangle \left(\frac{1}{\sqrt{2}} \langle +| + \frac{1}{\sqrt{2}} \langle -| \right) + |-\rangle \left(\frac{1}{\sqrt{2}} \langle +| - \frac{1}{\sqrt{2}} \langle -| \right) \tag{9}$$

$$= \frac{1}{\sqrt{2}} \left(|+\rangle \langle +| + |+\rangle \langle -| + |-\rangle \langle +| - |-\rangle \langle -| \right) \tag{10}$$

In matrix representation,

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{11}$$

1.27 Change of Basis

1.27.a Suppose that $f(A)$ is a function of a Hermitian operator A with the property $A\,|a'\rangle = a'\,|a'\rangle$. Evaluate $\langle b''|f(A)|b'\rangle$ when the transformation matrix from the a' basis to the b' basis is known.

Inserting identity,

$$\langle b''|f(A)|b'\rangle = \sum_{a'a''} \langle b''|a''\rangle \langle a''|f(A)|a'\rangle \langle a'|b'\rangle \tag{1}$$

We know $f(A)\,|a'\rangle = f(a')\,|a'\rangle$. Then, since $f(a')$ is a scalar, we can pull that out. The middle section then goes to zero unless $a'' = a'$,

$$\langle b''|f(A)|b'\rangle = \sum_{a'} f(a') \langle b''|a'\rangle \langle a'|b'\rangle \tag{2}$$

1.27.b Using the continuum analogue of the result obtained in (a), evaluate

$$\langle \vec{p}''|F(r)|\vec{p}'\rangle$$

Simplify your expression as far as you can. Note that r is $\sqrt{x^2+y^2+z^2}$, where x, y, and z are operators.

In the continuum analogue, the sum becomes an integral,

$$\langle \vec{p}''|F(r)|\vec{p}'\rangle = \int F(r') \langle \vec{p}''|\vec{r}'\rangle \langle \vec{r}'|\vec{p}'\rangle \; d\vec{r}' \tag{3}$$

From (1.7.32),

$$= \frac{1}{(2\pi\hbar)^3} \int F(\vec{r}') \exp\left(\frac{i(\vec{p}' - \vec{p}'')\vec{r}'}{\hbar}\right) \; d\vec{r}' \tag{4}$$

Since we are integrating over all space, let's write this in polar coordinates instead,

$$= \frac{2\pi}{(2\pi\hbar)^3} \int_{-1}^{1} \int_{0}^{\infty} r'^2 F(r') \exp\left(\frac{i(\vec{p}' - \vec{p}'')r'\cos(\theta)}{\hbar}\right) \; dr' d\cos(\theta) \tag{5}$$

$$= \frac{1}{2\pi^2\hbar^2 q} \int_0^\infty F(r') \sin\left(\frac{qr'}{\hbar}\right) \; dr' \tag{6}$$

where $q = |\vec{p}' - \vec{p}''|$.

1.28 Linear Momentum Commutation

1.28.a Let x and p_x be the coordinate and linear momentum in one dimension. Evaluate the classical Poisson bracket

$$[x, F(p_x)]_{classical}$$

From (1.6.48),

$$[x, F(p_x)]_{cl} = \frac{\partial x}{\partial x} \cdot \frac{\partial F(p_x)}{\partial p_x} - \frac{\partial x}{\partial p_x} \cdot \frac{\partial F(p_x)}{\partial x} = \frac{\partial F(p_x)}{\partial p_x} \tag{1}$$

1.28.b Let x and p_x be the corresponding quantum-mechanical operators this time. Evaluate the commutator

$$\left[x, \exp\left(\frac{ip_x a}{\hbar}\right)\right]$$

We first evaluate the classical Poisson bracket,

$$\left[x, \exp\left(\frac{ip_x a}{\hbar}\right)\right]_{cl} = \frac{\partial \exp\left(\frac{ip_x a}{\hbar}\right)}{\partial p_x} = \frac{ia}{\hbar}\exp\left(\frac{ip_x a}{\hbar}\right) \tag{2}$$

Using (1.6.47),

$$\left[x, \exp\left(\frac{ip_x a}{\hbar}\right)\right] = i\hbar\frac{ia}{\hbar}\exp\left(\frac{ip_x a}{\hbar}\right) = -a\exp\left(\frac{ip_x a}{\hbar}\right) \tag{3}$$

1.28.c Using the result obtained in (b), prove that

$$\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle, \quad (x\,|x'\rangle = x'\,|x'\rangle)$$

is an eigenstate of the coordinate operator x. What is the corresponding eigenvalue?
What this is saying is for

$$|\psi\rangle = \exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle \tag{4}$$

We have the eigenvalue relation,

$$x\,|\psi\rangle = x\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle \tag{5}$$

From the commutation relation in (b),

$$xF(p_x) - F(p_x)x = -aF(p_x) \tag{6}$$

Inserting this into Equation 5,

$$x|\psi\rangle = -a\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle + \exp\left(\frac{ip_x a}{\hbar}\right)x'|x'\rangle \tag{7}$$

$$= (x' - a)\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle \tag{8}$$

We get the eigenvalue relation,

$$x|\psi\rangle = (x' - a)|\psi\rangle \tag{9}$$

1.29 Gottfried

1.29.a On page 247, Gottfried(1966) states that

$$[x_i, G(\vec{p})] = i\hbar\frac{\partial G}{\partial p_i}, \quad [p_i, F(\vec{x})] = -i\hbar\frac{\partial F}{\partial x_i}$$

can be "easily derived" from the fundamental commutation relations for all functions of F and G that can be expressed as power series in their arguments. Verify this statement.

We can use (1.6.47),

$$[x_i, G(\vec{p})]_{cl} = \frac{\partial G}{\partial p_i} = \frac{[x_i, G(\vec{p})]}{i\hbar} \tag{1}$$

$$[x_i, G(\vec{p})] = i\hbar\frac{\partial G}{\partial p_i} \tag{2}$$

A similar argument can be made for $[p_i, F(\vec{x})]$.

1.29.b Evaluate $[x^2, p^2]$. Compare your result with the classical Poisson bracket $[x^2, p^2]_{cl}$

Using (1.7.28),

$$p^2 = -\hbar^2\frac{\partial^2}{\partial x^2}$$

Applying the commutator to an unsuspecting vector,

$$[x^2, p^2]\,|\psi\rangle = -x^2\hbar^2\frac{\partial^2\psi}{\partial x^2} + \hbar^2\frac{\partial^2(x^2\psi)}{\partial x^2} \tag{3}$$

$$= \hbar^2[2\psi + 4x\psi'] \tag{4}$$

$$= 2i\hbar\{x, p\}\,|\psi\rangle \tag{5}$$

The classical Poisson bracket,

$$[x^2, p^2]_{cl} = \frac{\partial x^2}{\partial x}\cdot\frac{\partial p^2}{\partial p} - \frac{\partial x^2}{\partial p}\cdot\frac{\partial p^2}{\partial x} = 4xp \tag{6}$$

1.30 Translation Operator

The translation operator for a finite(spatial) displacement is given by

$$\mathscr{T}(\vec{l}) = \exp\left(-\frac{i\vec{p}\cdot\vec{l}}{\hbar}\right)$$

where $\vec{p}$ is the momentum operator.

1.30.a Evaluate

$$[x_i, \mathscr{T}(\vec{l})$$

Let's first calculate the classical Poisson bracket,

$$[x_i, \mathscr{T}(\vec{l})]_{cl} = \sum_i \frac{\partial}{\partial p_i}\left(\exp\left(\frac{-i\vec{p}\cdot\vec{l}}{\hbar}\right)\right) = -\frac{i\vec{l}}{\hbar}\exp\left(\frac{-i\vec{p}\cdot\vec{l}}{\hbar}\right) \tag{1}$$

Using (1.6.47),

$$[x_i, \mathscr{T}(\vec{l})] = \sum_i l_i \mathscr{T}(\vec{l}) \tag{2}$$

1.30.b Using (a) (or otherwise), demonstrate how the expectation value $\langle\vec{x}\rangle$ changes under translation.

Acting the translation operator on a waveform,

$$|\psi'\rangle = \mathscr{T}(\vec{l})\,|\psi\rangle \tag{3}$$

Calculating the expectation value using $|\psi'\rangle$,

$$\langle\psi'|\vec{x}|\psi'\rangle = \langle\psi|\mathscr{T}^\dagger(\vec{l})\vec{x}\mathscr{T}(\vec{l})|\psi\rangle \tag{4}$$

Inserting the commutation relation Equation 2,

$$= \langle\psi|\mathscr{T}^\dagger(\vec{l})\mathscr{T}(\vec{l})\vec{x}|\psi\rangle + \sum_i \langle\psi|\mathscr{T}^\dagger(\vec{l})l_i\mathscr{T}(\vec{l})|\psi\rangle \tag{5}$$

$$= \langle\psi|\vec{x}|\psi\rangle + \sum_i l_i = \langle\vec{x}\rangle + \vec{l} \tag{6}$$

The expectation value translates

1.31 Translation Operator

In the main text we discuss the effect of $\mathscr{T}(d\vec{x}')$ on the position and momentum eigenkets and on a more general state ket $|\alpha\rangle$. We can also study the behavior of expectation values $\langle\vec{x}\rangle$ and $\langle\vec{p}\rangle$ under infinitesimal translation. Using (1.6.25), (1.6.45), and $|\alpha\rangle \to \mathscr{T}(d\vec{x}')\,|\alpha\rangle$ only, prove $\langle\vec{x}\rangle \to \langle\vec{x}\rangle + d\vec{x}'$, $\vec{p} \to \vec{p}$ under infinitesimal translation.

As a reminder, (1.6.25) is

$$[\vec{x}, \mathscr{T}(d\vec{x}')] = d\vec{x}'$$

(1.6.45) is

$$[\vec{p}, \mathscr{T}(d\vec{x}')] = 0$$

We calculate the expectation value of $\vec{x}$ using $|\alpha\rangle$,

$$\langle\alpha'|\vec{x}|\alpha'\rangle = \langle\alpha|\mathscr{T}^\dagger(d\vec{x}')\vec{x}\mathscr{T}(d\vec{x}')|\alpha\rangle \tag{1}$$

Using (1.6.45),

$$= \langle\alpha|\mathscr{T}^\dagger(d\vec{x}')(\mathscr{T}(d\vec{x}')\vec{x} + d\vec{x}')|\alpha\rangle \tag{2}$$

$$= \langle\alpha|\mathscr{T}^\dagger(d\vec{x}')\mathscr{T}(d\vec{x}')\vec{x} + \mathscr{T}^\dagger(d\vec{x}')d\vec{x}'|\alpha\rangle \tag{3}$$

$$= \langle\vec{x}\rangle + d\vec{x}' \tag{4}$$

Doing something similar for the momentum,

$$\langle\alpha'|\vec{p}|\alpha'\rangle = \langle\alpha|\mathscr{T}^\dagger(d\vec{x}')\vec{p}\mathscr{T}(d\vec{x}')|\alpha\rangle \tag{5}$$

$$= \langle\alpha|\mathscr{T}^\dagger(d\vec{x}')\mathscr{T}(d\vec{x}')\vec{p}|\alpha\rangle \tag{6}$$

$$= \langle\alpha|\vec{p}|\alpha\rangle = \langle\vec{p}\rangle \tag{7}$$

1.32 Gaussian Wave Packet

1.32.a Verify (1.7.39a) and (1.7.39b) for the expectation value of p and p^2 from the Gaussian wave packet (1.7.35)

The Gaussian wave packet (1.7.35),

$$\langle x'|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \exp\left(ikx' - \frac{x'^2}{2d^2}\right)$$

The expectation value of p,

$$\langle p\rangle = \langle\alpha|x'\rangle \left(-i\hbar\frac{\partial}{\partial x}\right) \langle x'|\alpha\rangle \tag{1}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi^{1/2}d} \exp\left(-ikx' - \frac{x'^2}{2d^2}\right) \cdot -i\hbar \left(exp\left(ikx' - \frac{x'^2}{2d^2}\right)\left(ik - \frac{x'}{d^2}\right)\right) dx' \tag{2}$$

$$= \int_{-\infty}^{\infty} -\frac{i\hbar\left(ik - \dfrac{x'}{d^2}\right)}{\pi^{1/2}d} \exp\left(-\frac{x'^2}{d^2}\right)\ dx' \tag{3}$$

We then use the Gaussian integral,

$$\frac{\hbar k}{\pi^{1/2}d}\sqrt{\pi d^2} \tag{4}$$

$$\langle p\rangle = \hbar k \tag{5}$$

As expected, matches up with (1.7.39a). We then turn to the expectation value of p^2,

$$\langle p^2\rangle = -\hbar^2 \langle\alpha|x'\rangle \frac{\partial^2}{\partial x^2} \langle x'|\alpha\rangle \tag{6}$$

$$\frac{\partial^2}{\partial x^2}\langle x'|\alpha\rangle = \frac{ik}{\pi^{1/4}d^{1/2}} \exp\left(ikx' - \frac{x'^2}{2d^2}\right)\left(ik - \frac{x'}{d^2}\right) - \frac{\exp\left(ikx' - \dfrac{x'^2}{2d^2}\right)}{\pi^{1/4}d^{5/2}} - \frac{x'\exp\left(ikx' - \dfrac{x'^2}{2d^2}\right)}{\pi^{1/4}d^{5/2}}\left(ik - \frac{x'}{d^2}\right) \tag{7}$$

Since we know $\int_{-\infty}^{\infty} x\exp(-x^2)\ dx = 0$, we can just pick out the terms that don't die.

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \frac{\hbar^2k^2}{\pi^{1/2}d} \exp\left(-\frac{x'^2}{d^2}\right) + \frac{\hbar^2}{\pi^{1/2}d^3} \exp\left(-\frac{x'^2}{d^2}\right) - \frac{\hbar^2x'^2}{\pi^{1/2}d^5} \exp\left(-\frac{x'^2}{d^2}\right)\ dx' \tag{8}$$

$$= \frac{\hbar^2k^2}{\pi^{1/2}d}\sqrt{\pi d^2} + \frac{\hbar^2}{\pi^{1/2}d^3}\sqrt{\pi d^2} - \frac{\hbar^2}{\pi^{1/2}d^5}\frac{d^2}{2}\sqrt{\pi d^2} \tag{9}$$

Agreeing with (1.7.39b),

$$\langle p^2\rangle = \frac{\hbar^2}{2d^2} + \hbar^2k^2 \tag{10}$$

1.32.b Evaluate the expectation value of p and p^2 using the momentum-space wave function (1.7.42)

As a reminder,

$$\langle p'|\alpha\rangle = \frac{d^{1/2}}{\hbar^{1/2}\pi^{1/4}} \exp\left(-\frac{(p'-\hbar k)^2 d^2}{2\hbar^2}\right)$$

Finding the expectation value of $\langle p\rangle$,

$$\langle p\rangle = \langle\alpha|p\rangle\, p\,\langle p|\alpha\rangle \tag{11}$$

$$= \int_{-\infty}^{\infty} \frac{d}{\hbar\pi^{1/2}} p \exp\left(-\frac{(p'-\hbar k)^2 d^2}{\hbar^2}\right)\, dp \tag{12}$$

Setting $u = p - \hbar k$,

$$= \int_{-\infty}^{\infty} \frac{d\hbar k}{\hbar\pi^{1/2}} \exp\left(-\frac{u^2 d^2}{\hbar^2}\right)\, du \tag{13}$$

$$= \frac{dk}{\pi^{1/2}}\sqrt{\frac{\pi\hbar^2}{d^2}} = \hbar k \tag{14}$$

The expectation value of $\langle p^2\rangle$,

$$\langle p^2\rangle = \int_{\infty}^{\infty} \frac{d}{\hbar}\pi^{1/2} p^2 \exp\left(-\frac{(p-\hbar k)^2 d^2}{\hbar^2}\right)\, dp \tag{15}$$

Setting $u = p - \hbar k$,

$$= \int_{-\infty}^{\infty} \frac{du^2}{\hbar\pi^{1/2}} \exp\left(-\frac{u^2 d^2}{\hbar^2}\right)\, du + \int_{-\infty}^{\infty} \frac{dk^2\hbar^2}{\hbar\pi^{1/2}} \exp\left(-\frac{u^2 d^2}{\hbar^2}\right)\, du \tag{16}$$

$$= \frac{\hbar^2}{2d^2} + \hbar^2 k^2 \tag{17}$$

1.33 Momentum Translation Operator

1.33.a Prove the following:

$$\begin{cases} \langle p'|x|\alpha\rangle = i\hbar\dfrac{\partial}{\partial p'}\langle p'|\alpha\rangle \\ \\ \langle\beta|x|\alpha\rangle = \int dp'\phi_\beta^*(p')i\hbar\dfrac{\partial}{\partial p'}\phi_\alpha(p') \end{cases}$$

where $\phi_\alpha(p') = \langle p'|\alpha\rangle$ and $\phi_\beta(p') = \langle p'|\beta\rangle$ are momentum-space wave functions.
For the first, insert identity

$$\langle p'|x|\alpha\rangle = \langle p'|x|p''\rangle\langle p''|\alpha\rangle = \langle p'|x|p'\rangle\langle p'|\alpha\rangle \tag{1}$$

Using (1.7.28),

$$= i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle \tag{2}$$

For the second, insert identity twice,

$$\langle\beta|x|\alpha\rangle = \langle\beta|p'\rangle\langle p'|x|p'\rangle\langle p'|\alpha\rangle \tag{3}$$

$$= \int \phi_\beta^* i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')\ dp' \tag{4}$$

1.33.b What is the physical significance of

$$\exp\left(\frac{ix\Xi}{\hbar}\right)$$

where x is the position operator and Ξ is some number with the dimension of momentum? Justify your answer.
This is the momentum translation operator.